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ON (2, 2)-MODULAR LAW FOR TERNARY *GD*-GROUPOIDS

Svetozar Milić

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1. Let P, Q, R, S be four non empty sets and

$$A: P \times Q \times R \to S,$$

then we call the ordered fivefold $(P, Q, R, S; A)$ a ternary G-groupoid (a generalized ternary groupoid).

Let us introduce the following denotation

$$L_1^A x = A(x, b, c), \quad L_2^A y = A(a, y, c), \quad L_3^A z = A(a, b, z)$$

where $a \in P$, $b \in Q$, $c \in R$ are some fixed elements. We call the mappings L_k^A $(k = 1, 2, 3)$ G-translations with respect to fixed elements.

If L_k^A $(k = 1, 2, 3)$ are surjections for arbitrary f.xed elements, then a ternary G-groupoid we call a ternary GD-groupoid (a G-groupoid with division). Namely, a ternary G-groupoid we call a ternary GD-groupo:d, if for each $a \in P$, $b \in Q$, $c \in R$, $d \in S$ the equat:ons

$$A(x, b, c) = d, \quad A(a, y, c) = d, \quad A(a, b, z) = d$$

have always solutions at $x \in P$, $y \in Q$, $z \in R$, but not necessarily single-valued. If they are single-valued (or if L_k^A $(k = 1, 2, 3)$ are bijections), then a ternary G-groupoid we call a ternary G-quasigroup.

For G-groupoids, GD-groupoids as well as for G-quasigroup we shall involve the notion of homotopy.

D e f i n i t i o n. For a ternary G-groupoid $(P', Q', R', S'; B)$ we say to be a homotop:c image of a G-groupo.d $(P, Q, R\ S; A)$ if there is a quadruple $H = [\alpha, \beta, \gamma, \delta]$ of surjections

$$\alpha: P \to P', \quad \beta: Q \to Q', \quad \gamma: R \to R', \quad \delta: S \to S'$$

such that the following condition

$$\delta A(x, y, z) = B(\alpha x, \beta y, \gamma z)$$

is fulfilled for every $x \in P$, $y \in Q$, $z \in R$.

If $\alpha, \beta, \gamma, \delta$ are bijections, then the homotopy H we call a isotopy.

2. Here we shall prove some theorems concerning ternary *GD*-groupoids satisfying so called generalized (2, 2)-modular law (1).

Theorem 1. *If four ternary GD-groupoids A, B, C, D, where*

$$B: Q_2 \times Q_3 \times Q_4 \to Q_6 \qquad D: Q_1 \times Q_3 \times Q_5 \to Q_7$$
$$A: Q_1 \times Q_6 \times Q_5 \to Q \qquad C: Q_2 \times Q_7 \times Q_4 \to Q$$

satisfy the equation

$$A(x, B(y, z, u), v) = C(y, D(x, z, v), u) \tag{1}$$

for every $x \in Q_1$, $y \in Q_2$, $z \in Q_3$, $u \in Q_4$, $v \in Q_5$ *and if* $L_2^A: Q_6 \to Q$, $L_2^C: Q_7 \to Q$ *(for arbitrary fixed elements) are bijections, then there exists a group* $(Q, \circ)$ *such that*

$$\begin{aligned} A(x, y, z) &= L_2^A y \circ K(x, z) \\ B(x, y, z) &= (L_2^A)^{-1} (P(x, z) \circ L_2^C L_2^D y) \\ C(x, y, z) &= P(x, z) \circ L_2^C y \\ D(x, y, z) &= (L_2^C)^{-1} (L_2^C L_2^D y \circ K(x, z)) \end{aligned} \tag{2}$$

where $L_2^D: Q_3 \to Q_7$ *is a mapping onto and K arbitrary binary GD-groupoids.*

Proof. From (1) by fixing $x = a \in Q_1$ $v = p \in Q_5$, then $y = b \in Q_3$, $u = d \in Q_4$ and finally $x = a$, $y = b$, $u = d$, $v = p$, we obtain

$$\begin{aligned} L_2^A B(y, z, u) &= C(y, L_2^D z, u) \\ L_2^C D(x, z, v) &= A(x, L_2^B z, v) \\ L_2^A L_2^B &= L_2^C L_2^D \end{aligned} \tag{3}$$

Substituting (3) into (1) we obtain

$$A(x, (L_2^A)^{-1} C(y, L_2^D z, u), v) = C(y, (L_2^C)^{-1} A(x, L_2^B z, v), u).$$

Because of the third equality from (3) we have

$$A(x, (L_2^A)^{-1} C(y, (L_2^C)^{-1} L_2^A L_2^B z, u), v) = C(y, (L_2^C)^{-1} A(x, L_2^B z, v), u)$$

i.e.

$$A(x, (L_2^A)^{-1} C(y, (L_2^C)^{-1} t, u), v) = C(y, (L_2^C)^{-1} A(x, (L_2^A)^{-1} t, v), u) \tag{4}$$

where $L_2^A L_2^B z = t \in Q$.

Let

$$\begin{aligned} A(x, (L_2^A)^{-1} t, v) &= S(x, t, v) \\ C(y, (L_2^C)^{-1} t, u) &= T(y, t, u) \end{aligned} \tag{5}$$

Then we have that the *GD*-groupoids A and S as well as C and T are isotopic.

By using (5) the equation (4) becomes

$$S(x, T(y, t, u), v) = T(y, S(x, t, v), u). \tag{6}$$

If we fix in (6) $u = d \in Q_4$, then we obtain

$$S(x, F(y, t), v) = F(y, S(x, t, v)) \tag{7}$$

where $F(y, t) = T(y, t, d)$, i.e. that F is a binary *GD*-groupoid $F: Q_2 \times Q \to Q$.

Let us regard a homotopic image $(Q, Q, Q, \circ)$ of *GD*-groupoid $(Q_2, Q, Q; F)$ at the homotopy $G = [L_1^F, 1, 1]$ (1 = identity mapping and $L_1^F y = F(y, k)$ where $t = k$ a fixed element from Q), i.e.

$$F(y, t) = L_1^F y \circ t \tag{8}$$

That homotopic image $(Q, Q, Q, \circ) \overset{\text{def}}{=} (Q, \circ)$ is also a *GD*-groupoid (see the Lemma [1]).

If we fix in (7) $t = k \in Q$ we obtain

$$S(x, L_1^F y, v) = F(y, K(x, v)) \tag{9}$$

where $K(x, v) = S(x, k, v)$.

From (9) by using (8) we find

$$S(x, L_1^F y, v) = L_1^F y \circ K(x, v)$$

i.e.

$$S(x, w, v) = w \circ K(x, v) \tag{10}$$

where $L_1^F y = w \in Q$.

If we substitute (10) into (7) we obtain

$$F(y, t) \circ K(x, v) = F(y, t \circ K(x, v))$$

i.e.

$$(L_1^F y \circ K(x, v) = L_1^F y \circ (t \circ K(x, v))$$

or

$$(\xi \circ \eta) \circ \zeta = \xi \circ (\eta \circ \zeta)$$

where $L_1^F y = \xi \in Q$, $\eta = t \in Q$, $K(x, v) = \zeta \in Q$. It means that $(Q, \circ)$ is an associative *D*-groupoid. Hence $(Q, \circ)$ is a group.

Substituting (10) into (6) we obtain

$$T(y, t, u) \circ K(x, v) = T(y, t \circ K(x, v), u)$$

hence by fixing $t = e$, where e is a unit of the group $(Q, \circ)$, we obtain

$$T(y, e, u) \circ K(x, v) = T(y, K(x, v), u)$$

or

$$T(y, s, u) = P(y, u) \circ s \tag{11}$$

where $K(x, v) = s \in Q$ and $P(y, u) = T(y, e, u)$.

Thus, the general solution of the equation (6) is

$$S(x, w, v) = w \circ K(x, v)$$
(12)
$$T(y, s, u) = P(y, u) \circ s.$$

By using (12), from (5) and (3) we obtain (2).

In such a way the theorem is proved.

From (8) we have that $(Q, \circ)$ is a group if we fix $t = k \in Q$ i.e.

$$F(y, t) = L_1^F(k)\, y \circ t$$

where it is emphasized which element is fixed. If we fix $t = m \in Q$ we obtain

$$F(y, t) = L_1^F(m)\, y * t$$

the groups $(Q, \circ)$ and $(Q, *)$ are isomorphic.

Really, from

(13) $$L_1^F(k)\, y \circ t = L_1^F(m)\, y * t$$

for $t = e$, where e is the unit of the group $(Q, \circ)$ we obtain

$$L_1^F(k)\, y = L_1^F(m)\, y * e$$

i.e.

(14) $$L_1^F(m)\, y = L_1^F(k)\, y * e^{-1}.$$

Substituting (14) into (13) we obtain

(15) $$L_1^F(k)\, y \circ t = L_1^F(k)\, y * e^{-1} * t.$$

Let us regard a mapping $f: Q \to Q$ defined in the following manner

$$f(x) = x * e^{-1}.$$

Since f is a translation of the group $(Q, *)$ then f is a bijection. From the definition of the mapping f and (15) we obtain

$$f(x \circ t) = (x \circ t) * e^{-1} = x * e^{-1} * t * e^{-1} = f(x) * f(t).$$

Thus, the groups $(Q, \circ)$ and $(Q, *)$ are isomorphic.

Let us involve in the set of all surjections from a set X into a set Y the relation of equivalence in the following manner

$$\alpha \sim \beta \overset{\text{def}}{\Leftrightarrow} (\exists a, b \in Y)\,(\forall x \in X)\,(\alpha x = a \cdot \beta x \cdot b)$$

where $(Y, \cdot)$ is a group.

If we put

$$\alpha = L_2^A, \quad \beta = L_2^C L_2^D = L_2^A L_2^B, \quad \gamma = L_2^C$$

in the equalities (2) then the following theorem is valid.

Theorem 2. *If four ternary GD-groupoids A, B, C, D, satisfy the conditions of the Theorem 1., then the general solution of the equation* (1) *is*

$$(16)\quad \begin{aligned} A(x, y, z) &= \alpha y \circ K(x, z) \\ B(x, y, z) &= \alpha^{-1}(P(x, z) \circ \beta y) \\ C(x, y, z) &= P(x, z) \circ \gamma y \\ D(x, y, z) &= \gamma^{-1}(\beta y \circ K(x, z)) \end{aligned}$$

where $(Q, \circ)$ *is a group determined up to an isomorphism and mappings* α, β, γ *are determined with accuracy up to an equivalence and* K *and* P *are arbitrary binary GD-groupoids.*

3. Let us consider a functional equation so called the generalized (i, j)-modular law

$$(17)\quad \begin{aligned} &A(x_1, x_2, \ldots, x_{i-1}, B(y_1, \ldots, y_m), x_{i+1}, \ldots, x_n) \\ &= C(y_1, y_2, \ldots, y_{j-1}, D(x_1, \ldots, x_{i-1}, y_{j+1}, \ldots, x_k, x_{i+1}, \ldots, x_n), y_{k+1}, \ldots, y_m) \end{aligned}$$

where A, B, C, D are quasigroups defined on the same non empty set Q of the arities

$$|A| = n, \quad |B| = m, \quad |C| = m+j-k, \quad |D| = n+k-j$$

(The arity of the operation K is denoted by $|K|$).

A special case of the equation (17) for $k=j$, $m=n$ is regarded in the paper [2].

To short notations we involve some abreviations. So the sequence $x_k, x_{k+1}, \ldots, x_s$ we denote by x_k^s. The symbol x_k^s we regard to be empty if $s<k$. The same, an ordered $s-k+1$—tuple $(x_k, x_{k+1}, \ldots, x_s)$ we denote by (x_k^s). If x_k^k, then we write x_k.

Utilizing involved notations the equation (17) becomes

$$(18)\quad A(x_1^{i-1}, B(y_1^m), x_{i+1}^n) = C(y_1^{j-1}, D(x_1^{i-1}, y_j^k, x_{i+1}^n), y_{k+1}^m).$$

Let us consider the case of the equation (18) for $1<i<n$, $1<j \leqslant k<m$. Put

$$(19)\quad \begin{aligned} A(x_1^{i-1}, y, x_{i+1}^n) &\overset{\text{def}}{=} \tilde{A}((x_1^{i-1}), y, (x_{i+1}^n)) \\ B(y_1^m) &\overset{\text{def}}{=} \tilde{B}((y_1^{j-1}), (y_j^k), (y_{j+1}^m)) \\ C(y_1^{j-1}, z, y_{j+1}^m) &\overset{\text{def}}{=} \tilde{C}((y_1^{j-1}), z, (y_{j+1}^m)) \\ D(x_1^{i-1}, y_j^k, x_{i+1}^n) &\overset{\text{def}}{=} \tilde{D}((x_1^{i-1}), (y_j^k), (x_{i+1}^n)) \end{aligned}$$

If, moreover, we involve the following notations

$$X = (x_1^{i-1}), \quad Y = (y_1^{j-1}), \quad Z = (y_j^k), \quad U = (y_{j+1}^m), \quad V = (x_{i+1}^n)$$

then (18) becomes

$$(20)\quad \tilde{A}(X, \tilde{B}(Y, Z, U), V) = \tilde{C}(Y, \tilde{D}(X, Z, V), U)$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, because of (19), are ternary GD-groupoids. Since

$$\tilde{B}: Q^{j-1} \times Q^{k-j+1} \times Q^{m-k} \to Q \qquad \tilde{D}: Q^{i-1} \times Q^{k-j+1} \times Q^{n-i} \to Q$$

$$\tilde{A}: Q^{i-1} \times Q \times Q^{n-i} \to Q \qquad \tilde{C}: Q^{j-1} \times Q \times Q^{m-k} \to Q$$

then taking into account the relation (19) by means of which these GD-groupoids are defined we have that GD-groupoids $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ satisfy the conditions of the Theorem2.; then from (20) we have

$$A(X, y, V) = \alpha y \circ K(X, V)$$
$$B(Y, Z, U) = \alpha^{-1}(P(Y, U) \circ \beta Z)$$
$$C(Y, z, U) = P(Y, U) \circ \gamma z$$
$$D(X, Z, V) = \gamma^{-1}(\beta Z \circ K(X, U))$$

i.e.

$$\begin{aligned} A(x_1^{i-1}, y, x_{i+1}^n) &= \alpha y \circ K(x_1^{i-1}, x_{i+1}^n) \\ B(y_1^m) &= \alpha^{-1}(P(y_1^{j-1}, y_{j+1}^m) \circ \beta(y_j^k)) \\ C(y_1^{j-1}, z, y_{j+1}^m) &= P(y_1^{j-1}, y_{j+1}^m) \circ \gamma z \\ D(x_1^{i-1}, y_j^k, x_{i+1}^n) &= \gamma^{-1}(\beta(y_j^k) \circ K(x_1^{i-1}, x_{i+1}^n)) \end{aligned} \tag{21}$$

where $(Q, \circ)$ is a group, K, P and β are quasigroups of arities $|K| = n-1$, $|P| = m-1$, $|\beta| = k-j+1$, and α and γ are permutations of the set Q (a quasigroup of the lenght one).

Thus, the relations from (21) represent the general solution of the equation (17).

The equation (17), by using the parastrophic of quasigroups, can be reduced to the so-called general $(1, n)$-associative law. Then by applying the Theorem of Hosszú [3] we find the solution.

R e m a r k. In the meantime, while this paper was accepted for publication, the author has learned B. Alimpić' results, communicated at the Symposium "Quasigroups and functional equations" held in Belgrade, Sept. 18-21. IX 1974, in which she considered the balance laws of the form $\omega_1 = \omega_2$ on quasigroups, where ω_1, ω_2, are terms. The equation (17) is a special case of the considered law.

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