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NUMERICAL INVARIANTS OF O-DIMENSIONAL SPACES AND THEIR CARTESIAN MULTIPLICATION

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Introduction.

In this paper we study Cartesian multiplication of accumulation orders which we have defined and have made use of them in connection with hyperspaces in one of our previous papers ([3]).

The first section is just a recapitulation of the related main points from [3]. In the second section we define a multiplication of accumulation orders associated with the Cartesian product of two spaces. We also find the explicit formulae giving this multiplication in terms of the usual operations with natural numbers. Then, these formulae reveal easily the structure of the monoid of all natural numbers under this multiplication.

As application, in the third section, we find infinitely many pairs of non-homeomorphic spaces having their squares homeomorphic. All these spaces are compact metric and 0-dimensional. The phenomenon of different spaces with homeomorphic squares, which came into question in the form of a problem of S. Ulam (Fundamenta Mathematicae 20 (1933), p. 285), seems to be more hidden than rare. Namely, R. H. Fox in [1], solving the Ulam problem, has shown that there exists a pair of non-homeomorphic 4-dimensional manifolds having their squares homeomorphic and that this cannot happen if dimension of manifolds is less than three. At the end of the third section we attach an easily described pair of such spaces.

1. Accumulation orders.

In this section, we construct a sequence of spaces from the class of all compact metric 0-dimensional spaces (denoted here by $\mathcal{Z}$) whose members represent the spaces having points with greater and greater accumulation orders. We also repeat the definition of accumulation order from [3] together with some its corollaries that we will be using here.

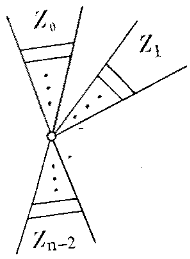
Let Z_0 be an isolated point and Z_2 the space of points $\{1, 1/2, \ldots, 1/n, \ldots\} \cup \{0\}$ having the relative topology of Euclidean line. Let Z_1 be the Cantor discontinuum C. Thus, we already have a sequence Z_0, Z_1, Z_2 and call 0 the leading point of Z_2. Assume the sequence

$$Z_0, Z_1, \ldots, Z_n$$

has already been defined and each of the spaces $Z_2, \ldots, Z_n$ has its leading point. Let Z_{n-1}^k, $k=1, 2, \ldots$ be the sequence of different copies of Z_{n-1} and Z one-point compactification of the disjoint topological sum of spaces Z_{n-1}^k, that is

$$Z=(\infty)\cup\left(\sum\{Z_{n-1}^k \mid k=1, 2, \ldots\}\right),$$

Then Z_{n+1} is the space obtained by the identification of point ∞ and the leading point of Z_n in $Z+Z_n$. The point of Z_{n+1} obtained by this identification is the leading point of Z_{n+1}.



The space $Z_n (n>3)$ can be pictured and realized as the following star space where the segments in each angle carry a copy of the indicated space.

Now we describe a classification of points of a space X from $\mathcal{Z}$ and give the definition of accumulation order.

For a space $X\in\mathcal{Z}$, X_0 stands for the set of all isolated points of X and X_1 for the set of those points of X which have a clopen neighborhood homeomorphic to C. For instance,

$$(Z_1)_0=\varnothing,\quad (Z_1)_1=C,\quad (Z_2)_0=\{1, 1/2, \ldots, 1/n, \ldots\}$$
$$(Z_2)_1=\varnothing,\quad (Z_2)_2=\{0\}.$$

We put $X_{(0)}=X\setminus(X_0\cup X_1)$, then as it can easily be seen,

$$(Z_3)_{(0)}=\{0\},\quad (Z_4)_{(0)}=\{\text{each } \{0\}_k=\{0\}\in Z_2^k \text{ plus } \{0\}\}.$$

The points of $X_{(0)}$ can be divided into those $X_{(0)(\bar{1})}$ not being the accumulation points of X_1 and those $X_{(0)(1)}$ being the accumulation points of this set. For instance,

$$(Z_4)_{(0)(\bar{1})}=\{\text{each } \{0\}_k\in Z_2^k\},\quad (Z_4)_{(0)(1)}=\{0\}.$$

Thus

$$X_{(0)(\bar{1})}=X_{(0)}\setminus\bar{X}_1,\quad X_{(0)(1)}=X_{(0)}\cap\bar{X}_1.$$

Now assume that $X_0, X_1, \ldots, X_n$ and $X_{(0)}, X_{(0)(1)}, \ldots, X_{(0)(1)\ldots(n-1)}$ have already been defined. Then, let

$$X_{n+1}=X_{(0)(1)\ldots(n-1)(\bar{n})}=X_{(0)(1)\ldots(n-1)}\setminus\bar{X}_n$$
$$X_{(0)(1)\ldots(n-1)(n)}=X_{(0)(1)\ldots(n-1)}\cap\bar{X}_n.$$

Thus, for each $X\in\mathcal{Z}$ we have an inductive definition of sequences

$$X_0, X_1, \ldots, X_n, \ldots$$

and

$$X_{(0)}, X_{(0)(1)}, \ldots, X_{(0)(1)\ldots(n)}, \ldots$$

Let

$$X_\omega=\cap\{X_{(0)\ldots(k)} \mid k=0, 1, \ldots\},$$

then, we have the following

1.1. *For each* $X \in \mathcal{Z}$,

$$X = (X_0 \cup X_1 \cup \cdots \cup X_n \cup \cdots) \cup X_\omega,$$

the sets $X_0, X_1, \ldots, X_n, \ldots, X_\omega$ *are mutually disjoint and* $X_0 \cup \cdots \cup X_n$ *is open for each* $n = 0, 1, \ldots$

By 1.1, each point $x \in X$ belongs to exactly one X_n. If $x \in X_n$, then we call x *n-point* of X and the number n *accumulation order* of x and we denote it by ord(x). If X and Y are homeomorphic and $f: X \approx Y$, then, as it is easy to see, $x \in X_n$ implies $fx \in Y_n$. Hence, for a point $x \in X$, the properties of being an n-point or having accumulation order n are topological.

In order to make it easier to deal with the above definitions, we also include some of their corollaries as follows.

1.2. (a) *The closure of* X_n *is*

$$\overline{X}_n = X_n \cup (\cup \{X_k \mid k = n+2, n+3, \ldots, \omega\}).$$

(b) *If* $X_n = \varnothing$, *then* $X_t = \varnothing$ *for* $t = n+2, \ldots, \omega$.

1.3. *If* $X + Y$ *is the disjoint topological sum of two spaces* X *and* Y, *then*

$$(X + Y_n) = X_n + Y_n.$$

Since each point x in X has a clopen neighborhood U, it follows from 1.3 that the accumulation order is a local invariant. If x is an n-point, then by 1.2, $\overline{X}_{n-1} \cap X_n = \varnothing$ and the neighborhood U can be chosen so that $X_{n-1} \cap U = \varnothing$. In this case we will call U *canonical.* Thus, if x is an n-point of X, then there is a canonical neighborhood U of X having for the accumulation orders of its points the numbers $0, \ldots, n-2, n$ and only them.

2. Multiplication of accumulation orders.

Here we define a multiplication of accumulation orders associated with the Cartesian product of two spaces. Let X and Y be two spaces from $\mathcal{Z}$ and let $x \in X$ and $y \in Y$ be such that ord$(x) = n$ and ord$(y) = m$. Then, the number ord(x, y), $(x, y) \in X \times Y$ can be considered as a "product" of numbers ord(x) and ord(y). Our first task is to show that this multiplication of numbers n and m is well-defined. That is, we must show that we obtain the same number regardless of the choice of spaces X and Y. We prove it first in a special case of numbers n and m.

2.1. *Let* X *and* Y *be arbitrary spaces from* $\mathcal{Z}$ *and* $X \times Y$ *their product space. Then, for each* $x \in X$ *and* $y \in Y$

(I) ord$(x) = n$ *and* ord$(y) = 0$ *imply* ord$(x, y) = n$,

(II) ord$(x) = n$ *and* ord$(y) = 1$ *imply* ord$(x, y) = 1$.

Proof. (I): The point y is isolated and $X \times \{y\}$ is a clopen subset of $X \times Y$. Since

$$p: X \times \{y\} \approx X, \quad p(x, y) = x$$

is a homeomorphism, ord$(x, y) =$ ord$(x) = n$.

(II): The point y has a clopen neighborhood V homeomorphic to the Cantor set C and the point (x, y) has the clopen neighborhood $X \times V$ being also homeomorphic to C. Hence, $\operatorname{ord}(x, y) = 1$.

In both cases, (I) and (II), the order of (x, y) does not depend upon the choice of X and Y.

If $x \in X$ and $y \in Y$, then evidently

$$\operatorname{ord}(x, y) = \operatorname{ord}(y, x),$$

where $(x, y) \in X \times Y$ and $(y, x) \in Y \times X$. This means that the "multiplication" of orders will be commutative.

2.2 *Let X and Y be arbitrary spaces from $\mathcal{Z}$ and $X \times Y$ their product space. If $x_0 \in X$, $\operatorname{ord}(x_0) = n$ and $y_0 \in Y$, $\operatorname{ord}(y_0) = m$, then $\operatorname{ord}(x_0, y_0)$ is a natural number uniquely determined by m and n.*

Proof. By 2.1, this statement is true for all n and $m = 0, 1$. Assume it is true for all n and $m < t$. Now we prove that the statement will be true for all n and for $m = t$. Then by induction, it will be true for all n and all m.

If $n = 0, 1$ and $m = t$, then $\operatorname{ord}(x, y) = \operatorname{ord}(y, x)$ and 2.1 imply our statement. Assume now it is true for all $n < k$ and $m \leqslant t$. Let $\operatorname{ord}(x_0) = k$, $\operatorname{ord}(y_0) = t$. Then, let U_{x_0} and V_{y_0} be two canonical neighborhoods of x_0 and y_0, respectively. We have for $x \in U_{x_0}$ and $y \in V_{y_0}$,

$$\rho_x = \operatorname{ord}(x) = 0, \ldots, k-2, k; \quad \rho_y = \operatorname{ord}(y) = 0, \ldots, t-2, t.$$

Let

$$s = \max\{\operatorname{ord}(x, y) \mid (x, y) \in U_{x_0} \times V_{y_0} \text{ and } (\rho_x, \rho_y) \neq (k, t)\},$$

where all numbers $\operatorname{ord}(x, y)$ are uniquely determined by ρ_x and ρ_y. Now two cases are possible:

(a) there is no $(x, y) \in X \times Y$ such that $\operatorname{ord}(x, y) = s - 1$.

(b) there is a pair $(x, y) \in X \times Y$ such that $\operatorname{ord}(x, y) = s - 1$.

By 1.2, in case (a): $\operatorname{ord}(x_0, y_0) \geqslant s$ and in case (b): $\operatorname{ord}(x_0, y_0) \geqslant s + 2$. For $x' \in U_{x_0}$, $\operatorname{ord}(x') = k$ and $y' \in V_{y_0}$, $\operatorname{ord}(y') = t$, we also have $\operatorname{ord}(x', y') \geqslant s$ or $\operatorname{ord}(x', y') \geqslant s + 2$ according to whether (a) or (b). Indeed, taking two neighborhoods $U_{x'} \subseteq U_{x_0}$ of x' and $V_{y'} \subseteq V_{y_0}$ of y' we get the same set of numbers

$$\{\operatorname{ord}(x, y) \mid (x, y) \in U_{x'} \times U_{y'} \text{ and } (\rho_x, \rho_y) \neq (k, t)\}.$$

Thus, in case (a), there is no $(x, y) \in U_{x_0} \times V_{y_0}$ such that $\operatorname{ord}(x, y) = s - 1$ and $\operatorname{ord}(x_0, y_0) = s$ and, in case (b), there is no $(x, y) \in U_{x_0} \times V_{y_0}$ such that $\operatorname{ord}(x, y) = s + 1$ and $\operatorname{ord}(x_0, y_0) = s + 2$. Both numbers are determined uniquely by the numbers ρ_x and ρ_y and independently of choice of X and Y.

Now we can give the following definition: *Let X and Y be two spaces from $\mathcal{Z}$. Let $x \in X$, $y \in Y$ and $\operatorname{ord}(x) = n$, $\operatorname{ord}(y) = m$. Then the product of orders n and m is the number $\operatorname{ord}(x, y) = n \times m$.*

By 2.2, $n \times m$ is well-defined. As we have already noticed it, this multiplication is commutative.

Since $(X\times Y)\times Z$ is homeomorphic to $X\times(Y\times Z)$ under the homeomorphism carrying $((x, y), z)$ onto $(x, (y, z))$, we also have

$$\operatorname{ord}(x, y)\times\operatorname{ord}(z)=\operatorname{ord}(x)\times\operatorname{ord}(y, z),$$

what means that the multiplication of accumulation orders is associative. Hence, $(N, \times)$ *is a semigroup.* Moreover this semigroup has 0 as the neutral element and $(N, \times)$ *is a monoid.*

Proving 2.1, we already know that $n\times 0=n$ and $n\times 1=1$. Now we prove

2.3. (III) $n\times 2=\begin{cases} n+2, & n=3k \\ n, & n\neq 3k\end{cases}$

(IV) $n\times 3=\begin{cases} n+3, & n=2k \\ n, & n=2k+1\end{cases}$ (V) $n\times 4=\begin{cases} n+4, & n=3k \\ n, & n=3k+1 \\ n+2, & n=3k+2\end{cases}$

(VI) $n\times 6=\begin{cases} 1, & n=1 \\ n+6, & n\neq 1.\end{cases}$

(VII) (a) $n\times 5=(n\times 2)\times 3$ *and* (b) $n\times 7=(n\times 3)\times 4$.

Proof. In proving (III), (IV), (V) and (VI), we use induction on n. All these formulae are true for $n=0, 1$. We will suppose they are true for all $n<t$ and a fixed m, $(m=2, 3, 4, 6)$. Then, we choose an $x_0\in X$ such that $\operatorname{ord}(x_0)=t$ and a $y_0\in Y$ such that $\operatorname{ord}(y_0)=m$. The points x_0 and y_0 have the canonical neighborhoods U_{x_0} for x_0 and V_{y_0} for y_0. Then $t\times m=\operatorname{ord}(x_0, y_0)$. Since $t\times m$ is uniquely determined by $\rho_x\times\rho_y$, $(\rho_x, \rho_y)=(t, m)$ where $\rho_x=\operatorname{ord}(x)$, $x\in U_{x_0}$ and $\rho_y=\operatorname{ord}(y)$, $y\in V_{y_0}$, we will indicate all possible values of ρ_x and ρ_y in each of the particular cases of this proof. The maximal value s of $\rho_x\times\rho_y$ and the cases when $s-1$ or $s+1$ do not exist will be purely a matter of verification based upon the induction hypothesis or upon some of formulae in 2.3 which we will have already proved. Most of such details we will leave out.

(III): Proof of (III) splits into three cases: (a) $t=3k$, (b) $t=3k+1$ and (c) $t=3k+2$.

(a) $\rho_x=0, \ldots, 3k-2, 3k$ $\rho_y=0, 2.$

Then, $(3k-3)\times 2=3k-1$ and $s=(3k)\times 0=3k$. Maximality of s is proved verifying that $\rho_x\times\rho_y\neq 3k+1$. Hence, $(3k)\times 2=3k+2$.

(b) $\rho_x=0, \ldots, 3k-1, 3k+1$ $\rho_y=0, 2.$

Then, $(3k+1)\times 0=3k+1=s$ and $\rho_x\times\rho_y\neq 3k$. Hence, $(3k+1)\times 2=3k+1$.

(c) $\rho_x=0, \ldots, 3k, 3k+2$ $\rho_y=0, 2.$

Then, $(3k+2)\times 0=3k+2$ and $\rho_x\times\rho_y\neq 3k+1$. Indeed, $(3k)\times 2=3k+2$ and for $\rho_x<3k$, $\rho_x\times 2\leq 3k-1$.

(IV): Proof of (IV) splits into two cases: (a) $t=2k$ and (b) $t=2k+1$.

$$\text{(a)}\quad \rho_x=0, \dots, 2k-2, 2k \qquad \rho_y=0, 1, 3.$$

Then, $(2k-2)\times 3=2k+1=s$, $(2k)\times 0=2k$ and $\rho_x\times\rho_y\neq 2k+2$. Hence, $(2k)\times 3=2k+3$.

$$\text{(b)}\quad \rho_x=0, \dots, 2k-1, 2k+1 \qquad \rho_y=0, 1, 3.$$

Then, $(2k-2)\times 3=2k+1=s$ and $\rho_x\times\rho_y\neq 2k$. Hence, $(2k+1)\times 3=2k+1$.

(V): Proof of (V) splits into three cases: (a) $t=3k$, (b) $t=3k+1$ and (c) $t=3k+2$.

$$\text{(a)}\quad \rho_x=0, \dots, 3k-2, 3k \qquad \rho_y=0, 1, 2, 4.$$

Then, $(3k-3)\times 4=3k+1$, $(3k)\times 2=3k+2=s$ and $\rho_x\times\rho_y\neq 3k+3$. Hence, $(3k)\times 4=3k+4$.

$$\text{(b)}\quad \rho_x=0, \dots, 3k-1, 3k+1 \qquad \rho_y=0, 1, 2, 4.$$

Then, $(3k-3)\times 4=3k+1=s$ and $\rho_x\times\rho_y\neq 3k$. Indeed, $(3k+1)\times 2=3k+1$ and for $\rho_x\leqslant 3k-1$, $\rho_x\times 2\leqslant 3k-1$. Further, $(3k-1)\times 4=3k+1$, $(3k-2)\times 4=3k-2$ and for $\rho_x\leqslant 3k-4$, $\rho_x\times 4\leqslant 3k-2$.

$$\text{(c)}\quad \rho_x=0, \dots, 3k, 3k+2 \qquad \rho_y=0, 1, 2, 4.$$

Then, $(3k)\times 4=3k+4=s$ and $\rho_x\times\rho_y\neq 3k+3$. Hence, $(3k+2)\times 4=3k+4$.

(VII) (a): Since the multiplication of orders is associative, we have

$$n\times 5=n\times(2\times 3)=(n\times 2)\times 3.$$

(VI): This formula is true for $n=0, 1, 2, 3, 4, 5$, since we have already proved it. Suppose $t>5$.

$$\rho_x=0, \dots, t-2, t \qquad \rho_y=0, 1, 2, 3, 4, 6.$$

Then, $(t-3)\times 6=t+3$, $(t-2)\times 6=t+4=s$ and $\rho_x\times\rho_y\neq t+5$. Hence, $t\times 6=t+6$.

(VII) (b): We have $n\times 7=n\times(3\times 4)=(n\times 3)\times 4$.

If $m\neq 1$ is a natural number, then m can be written in a unique way as $6k+r$, where $r=0, 2, 3, 4, 5, 7$ and $k=0, 1, \dots$ When we write $m=6k+r$, this representation is to be understood.

2.4. *If* $n=6k_1+r_1$ *and* $m=6k_2+r_2$, *then*

$$n\times m=6(k_1+k_2)+r_1\times r_2.$$

Proof. By 2.3 (VI) $6k+r=(6k)\times r$ for $r\neq 1$. Thus,

$$(6k_1+r_1)\times(6k_2+r_2)=(6k_1+r_1)\times[(6k_2)\times r_2]$$
$$=[(6k_1+r_1)\times 6k_2]\times r_2=[6(k_1+k_2)+r_1]\times r_2$$
$$=[6(k_1+k_2)]\times(r_1\times r_2)=6(k_1+k_2)+r_1\times r_2.$$

By 2.4, the multiplication of any two orders m and n is reducible to the multiplication of their remainders. Here is the multiplication table for remainders

	0	2	3	4	5	7
0	0	2	3	4	5	7
2	2	2	5	4	5	7
3	3	5	3	7	5	7
4	4	4	7	4	7	7
5	5	5	5	7	5	7
7	7	7	7	7	7	7

2.5. *The monoid* $(N, \times)$ *has six and only six prime numbers*: 0, 1, 2, 3, 4, 6.

3. An application.

With every space $X \in \mathcal{Z}$, we can associate an increasing sequence of natural numbers being the accumulation orders of its points. Such a sequence is denoted by $s(X)$ and called the *accumulation spectrum* of X. For instance, if $X = X_0$ then $s(X) = 0$, $s(C) = (\varnothing, 1)$, $s(Z_2) = (0, \varnothing, 2)$, $s(Z_1 + Z_2) = (0, 1, 2)$. Generally, if $s(X)$ is finite, it has one of these two forms

$$(0, \dots, n-2, \varnothing, n) \quad \text{or} \quad (0, \dots, n-1, n).$$

In the former case we use the empty set to denote the fact that $X_{n-1} = \varnothing$.

Call a space X *full* whenever $\bar{X}_n \neq \varnothing$ implies $\bar{X}_n \approx C$, $n = 1, 2. \dots$ In [3], we have constructed an infinite sequence of full spaces $C_1, C_2, \dots, C_n, \dots$ being such that $(\bar{C}_n)_t \approx C$, $t = 1, 2, \dots, n-2, n$ and $(C_n)_{n-1} = \varnothing$. Note that $C_1 = C$ and that we will describe here the construction of C_2 and C_3.

Now we use a statement proved in [3]. Namely,

3.1. *Let X and Y be two spaces from $\mathcal{Z}$ such that their accumulation spectra are finite and equal and let* $\operatorname{card}(X_0) = \operatorname{card}(Y_0)$. *Then, X and Y are homeomorphic.*

Our next task here is to find infinitely many pairs of non-homeomorphic spaces from $\mathcal{Z}$ having their squares homeomorphic.

We need the following

3.2. *The product $X \times Y$ of two full spaces X and Y is a full space.*

Proof. Let X and Y be full and let $(x, y) \in (X \times Y)_n$, where $n > 0$. Then, either $x \in X_k$ for some $k > 0$ or $y \in Y_h$ for some $h > 0$. In the former case $(x, y) \in X_k \times \{y\} \subseteq (X \times Y)_n$ and in the latter $(x, y) \in \{x\} \times Y_h \subseteq (X \times Y)_n$. Since in either case X_k and Y_h have no isolated point, the point (x, y) is not isolated in $(X \times Y)_n$. Thus, $\overline{(X \times Y)}_n \approx C$.

3.3. *There exist infinitely many pairs of non-homeomorphic spaces in* $\mathcal{Z}$ *having their squares homeomorphic. Such pairs are*:

(a) $X = C_{6k+2} + C_{6k+3}, \quad Y = C_{6k+5}$

and

(b) $X = C_{6k+3} + C_{6k+4}, \quad Y = C_{6k+7},$

where $k = 0, 1, 2, \ldots$

Proof. Since the accumulation spectra of the spaces from each pair are:

(a) $s(X) = (0, \ldots, 6k+2, 6k+3), \quad s(Y) = (0, \ldots, 6k+5)$

(b) $s(X) = (0, \ldots, 6k+3, 6k+4), \quad s(Y) = (0, \ldots, 6k+7),$

X is not homeomorphic to Y. By 3.2, X^2 and Y^2 are full and, by 2.4 and by the multiplication table, it is easy to verify that $s(X^2) = s(Y^2)$. Then, 3.1 implies $X^2 \approx Y^2$.

Now suppose that $C_1 = C$ is realized in the usual way as a subset of the interval $[0, 1]$. Let C_2 be the Cantor discontinuum C plus the centers of all deleted intervals. The space C_3 will be constructed as a subset of the Euclidean plane as follows. Let

$$A_0 = C_2 \times \{0\}, \quad A_n = C_1 \times \left\{\frac{1}{n}\right\}$$

and let

$$C_3 = \cup \{A_n \mid n = 0, 1, \ldots\}.$$

Let $X = C_2 + C_3$ and $Y = (C_2 + C_3) \times (C_2 + C_3)$. Then, X and Y are not homeomorphic but their squares are.

REFERENCES

[1] R. H. Fox, *On a problem of S. Ulam concerning Cartesian products*, Fund. Math. 34 (1947), p. 278—287.

[2] K. Kuratowski, *Topology*, Moscow, vol. I (1966) and vol II (1969).

[3] M. M. Marjanović, *Exponentially complete spaces III*, Publ. Inst. Math. t. 14 (28), (1972), p. 97—109.

Added in proof. A simpler construction of the sequence of full spaces is as follows. The spaces C_2 is obtained by interpolation of a point (a copy of C_0) in each of deleted intervals of C. The space C_n is obtained from C_{n-1}, by interpolation of a copy of C_{n-2} in each of the deleted intervals. The copy of C_{n-2} is interpolated so that it does not intersect any of previously interpolated copies.