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AN APPLICATION OF BERGMAN-WHITTAKER OPERATOR

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1. Integral operators p in the sense of Bergman [1] map a set of analytic functions f of one or several complex variables into a set of solutions ψ of various partial differential equations: $\psi = p\{f\}$. Operators p are constructed in such a way that (as many as possible) properties of f are transplanted into solutions ψ. Some properties, e. g. theorems on zeros or, more generally, on value distribution of f are however lost in the application of p. Nevertheless, these can be used in proving some properties of solutions ψ^{-1} inverse to ψ (in the sense of the composition defined below).

In particular let $\vec{x} = (x, y, z)$, $u = x + (iy + z)\dfrac{\zeta}{2} + (iy - z)\dfrac{\zeta^{-1}}{2}$; then the Bergman-Whittaker operator

$$B_3\{f(u, \zeta)\} = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} f(u, \zeta)\frac{d\zeta}{\zeta}$$

generates harmonic functions $h(\vec{x}) = B_3\{f(u, \zeta)\}$, [1, p. 43]. An algebra of $h(\vec{x})$ is obtained by Bergman by introducing the composition $h_1(\vec{x}) * h_2(\vec{x}) = B_3\{f_1 f_2\}$, [2].

It is shown in the present paper that using some properties of the coefficients in the expansion of a harmonic function $h(\vec{x})$ in a series of spherical harmonics, one can obtain informations about the singularities of a harmonic function inverse to $h(\vec{x})$.

2. Let $H(\vec{x})$ be an axially symmetric harmonic function regular inside the unit sphere S around the origin, and let $f(u)$ be an analytic function regular in the unit circle $|u| < 1$. Then, $H(\vec{x})$ can be represented in S by means of the Bergman-Whittaker operator

$$H(\vec{x}) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} f(u)\frac{d\zeta}{\zeta} \tag{1}$$

(where the associate function f depends on u only) (Cf. [4, p. 172 et seq.]).

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Moreover; the expansion of $H(\vec{x})=\bar{H}(r, \theta)$ in spherical harmonics has the form

$$(2) \qquad H(\vec{x})=\sum_{\nu=0}^{\infty} a_\nu r^\nu P_\nu(\cos\theta),$$

where $P_\nu(\cos\theta)$ are Legendre polynomials [4, p. 47].

Further let $G_\alpha(\vec{x})$ be the harmonic function inverse to $H(\vec{x})-\alpha$:

$$(3) \qquad G_\alpha(\vec{x})=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\{f(u)-\alpha\}^{-1}\frac{d\zeta}{\zeta},$$

where α is any complex number.

We prove the following

Theorem. *If the expansion of $H(\vec{x})$ in spherical harmonics has Hadamard's gaps, i. e. if*

$$(4) \qquad H(\vec{x})=\sum_{k=0}^{\infty} a_k r^{n_k} P_{n_k}(\cos\theta), \quad r<1,$$

where

$$(5) \qquad n_{k+1}/n_k \geqslant q>1, \quad k\geqslant 1;$$

then either of the conditions

$$(6a) \qquad \sum |a_k|=\infty, \text{ with (5) holding for } q>q_0 \ (\approx 100),$$

$$(6b) \qquad \limsup_{k\to\infty}|a_k|>0, \text{ with (5) holding for some } q>1,$$

leads to the following conclusion:

There exist infinitely many singularities of the harmonic function $G_\alpha(\vec{x})$; these are branch lines situated on the sphere $x^2+y^2+z^2=|A_\nu(\alpha)|^2$, where $A_\nu(\alpha)$, $\nu=1, 2, \ldots$ are zeros of the function $f(u)-\alpha$.

Proof of the theorem. There follows from (3) that each solution of the equation $f(u)-\alpha=0$, being a pole of the associate function $\{f(u)-\alpha\}^{-1}$ is a singularity of $G_\alpha(\vec{x})$. Consequently, one has to establish the existence of infinitely many solutions $A_\nu(\alpha)$ of the above equation in the interior of the unit circle $|u|<1$.

To that end notice that the relations (1), (4), (5) and

$$r^\nu P_\nu(\cos\theta)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1} u^\nu\frac{d\zeta}{\zeta}$$

([4. p, 165]) imply that the Taylor series of $f(u)$ has the form

$$(7) \qquad f(u)=\sum_{k=0}^{\infty} a_k u^{n_k},$$

converges for $|u|<1$, and n_k satisfy (5).

Now, the existence of infinitely many $A_\nu(\alpha)$ for $|u|<1$ and for any complex α, is granted by the condition (6a), according to a theorem of *M* and *G*. Weiss [5], or, by the condition (6b), according to a theorem of W. H. J. Fuchs [6].

To complete the proof, one has to show that the singularities are of the type described in the Theorem. This, however, is achieved (even for some more general cases) by the following procedure introduced by Bergman in [3]:

Applying to $\{f(u)-\alpha\}^{-1}$ the theorem of Mittag-Leffler and then integrating termwise, one obtains

$$G_\alpha(\vec{x}) = \sum_{\nu=1}^{\infty} \{S_\nu(\vec{x}) - P_\nu(\vec{x})\} + E(\vec{x}), \tag{8}$$

where

$$S_\nu(\vec{x}) = \sum_{\mu=1}^{m_\nu} A_{\nu\mu} \frac{\partial^\mu}{\partial x^\mu} \{(x - A_\nu(\alpha))^2 + y^2 + z^2\}^{1/2}, \tag{9}$$

$P_\nu(\vec{x})$ is a harmonic polynomial and $E(\vec{x})$ is an entire harmonic function; the series (8) converges uniformly in any closed bounded domain of $x^2+y^2+z^2<\infty$ which does not contain singularities of $S_\nu(\vec{x})$. It is readily seen from (9) that, if $I_m A_\nu(\alpha) \neq 0$, the curves

$$x = R_e A_\nu(\alpha), \quad y^2+z^2 = \{I_m A_\nu(\alpha)\}^2$$

are branch lines (shrinking to points if $I_m A_\nu(\alpha)=0$) of $G_\alpha(\vec{x})$ satisfying

$$x^2+y^2+z^2 = |A_\nu(\alpha)|^2.$$

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