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ASYMPTOTIC PROPERTIES OF CONVOLUTION PRODUCTS OF SEQUENCES

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1. The convolution product of two sequences (a_n) and (p_n) is defined to be the sequence

$$c_n=\sum_{k=1}^{n} p_k\, a_{n-k}, \qquad n=1,\, 2,\, \ldots \tag{1.1}$$

If (a_n) is a sequence of positive numbers converging to a positive number and if the series $\sum_{k=1}^{\infty} p_k$ converges absolutely, then it is easy to see that

$$\lim_{n\to\infty} c_n/a_n=\sum_{k=1}^{\infty} p_k .$$

The following more general result can be found in [7]:

Let (p_n) be a sequence of real numbers such that the series $\sum_{k=1}^{\infty} p_k\, x^k$ has a positive radius of convergence R. If

$$\lim_{n\to\infty} a_{n+1}/a_n=\lambda^{-1} \tag{1.2}$$

exists, and if $0<\lambda<R$, then

$$\lim_{n\to\infty} c_n/a_n=\sum_{k=1}^{\infty} p_k\, \lambda^k .$$

The example $p_n=n^{-5/4}$, $a_n=e^{-\sqrt{n}}$, $n=1,\, 2,\, \ldots$ shows that the condition $0<\lambda<R$ in the above theorem is essential. We have in this case

$$R=1=\lambda^{-1}=\lim_{n\to\infty} a_{n+1}/a_n \quad \text{and} \quad c_n/a_n\to\infty\ (n\to\infty).$$

A result of this type is usually called a direct theorem. The major portion of this paper is concerned with the converse problem which can be stated as follows. Suppose that

$$\lim_{n\to\infty} c_n/a_n=C \qquad (0<C<\infty). \tag{1.3}$$

Is it true then that the sequence (a_{n+1}/a_n) is convergent? First results of this type were given by N. G. de Bruijn and P. Erdös in the early 1950's ([1], [2], [3]), but instead of considering a sequence (a_n) satisfying the asymptotic relation (1.3) they have assumed that

$$a_0 = 1, \quad a_n = \sum_{k=1}^{n} p_k a_{n-k}, \qquad n = 1, 2, \ldots \tag{1.4}$$

where (p_n) is a sequence of positive numbers. Independently, in 1962, A. M. Garsia [5] investigated essentially the same problem.

We shall study here necessary and sufficient conditions in order that the asymptotic relation (1.3) imply the existence of $\lim_{n\to\infty} a_{n+1}/a_n$. In Theorems I–4 we shall make the following assumptions:

(1) (p_n) is a sequence of nonnegative numbers with $p_1 > 0$ and $R \in (0, \infty)$ is the radius of convergence of series $\sum_{k=1}^{\infty} p_k x^k$;

(2) (a_n) is a sequence of positive numbers satisfying the relation

$$(C + \varepsilon_n) a_n = \sum_{k=0}^{n} p_k a_{n-k}, \qquad n = 0, 1, 2, \ldots \tag{1.5}$$

where $0 < C < \infty$ and $\varepsilon_n \to 0 \; (n \to \infty)$.

Our first result in this direction can be then stated as follows:

Theorem 1. *If* (1) *and* (2) *hold, then the necessary and sufficient condition for the convergence of the sequence* (a_{n+1}/a_n) *is that, for every fixed* A,

$$\lim_{n\to\infty} \left(\sum_{k=A}^{n+1} p_k \frac{a_{n+1-k}}{a_n} - \sigma \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} \right) = 0. \tag{1.6}$$

Here

$$\sigma = \begin{cases} R^{-1}, & \text{if } \sum_{k=1}^{\infty} p_k R^k \leqslant C \\ \gamma^{-1}, & \text{if } \sum_{k=1}^{\infty} p_k R^k > C \end{cases} \tag{1.7}$$

where $\gamma \in (0, R)$ *is such that* $\sum_{k=1}^{\infty} p_k \gamma^k = C$.

This result was proved by N. G. de Bruijn and P. Erdös in [2] the special case when (a_n) satisfies (1.4) instead of (1.5).

In some cases it can be shown that (1.6) can be replaced by a simpler condition.

Theorem 2. *If* (1) *and* (2) *hold and if* $\sum_{k=1}^{\infty} p_k R^k \geqslant C$, *then the necessary and sufficient condition for the convergence of* (a_{n+1}/a_n) *is that*

$$\limsup_{n\to\infty} \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} \to 0 \qquad (A \to \infty). \tag{1.8}$$

A special case of this theorem when (a_n) satisfies (1.4) instead of (1.5) was proved earlier by A M. Garsia, S. Orey and E. Rodemich [4].

The condition (1.8) is satisfied, in particular, if

$$\limsup_{n\to\infty} p_{n+1}/p_n \leqslant \sigma.$$

Using this fact we shall obtain the following extension of another result of A. M. Garsia, S. Orey and E. Rodemich [4]:

Theorem 3. *If* (1) *and* (2) *hold and if* (p_n) *is a sequence of positive numbers such that* $\sum_{k=1}^{\infty} p_k R^k \geqslant C$ *and*

$$\limsup_{n\to\infty} p_{n+1}/p_n \leqslant \sigma, \tag{1.9}$$

then

$$\lim_{n\to\infty} a_{n+1}/a_n = \sigma.$$

Finally, if one knows that $\sum_{k=1}^{\infty} p_k R^k > C$, it is possible to replace (1.8) by a still simpler necessary and sufficient condition:

Theorem 4. *If* (1) *and* (2) *hold and if* $\sum_{k=1}^{\infty} p_k R^k > C$, *then the sequence* (a_{n+1}/a_n) *converges if and only if*

$$\limsup_{n\to\infty} \sqrt[n]{a_n} > R^{-1}. \tag{1.10}$$

A continuous analog of this theorem for convolution products of measurable functions was proved recently by D. Drasin ([6], Th. 6).

2. In the following sections we shall give the proofs of Theorems 1 – 4. The method used here for the proofs of Theorems 1 – 3 is essentially an extension of the method which de Bruijn and Erdös have used in [2].

This paper is based on a portion of my Ph.D. dissertation written at The Ohio State University. I would like to take this opportunity to thank Professor Ranko Bojanic for his generous help during the preparation of my dissertation.

3.1. The necessity part of the proof of Theorems 1, 2 and 4 is based on the following Lemma.

Lemma 1. *If* (1) *and* (2) *hold and if* $\lim_{n\to\infty} a_{n+1}/a_n$ *exists, then*

$$\lim_{n\to\infty} a_{n+1}/a_n = \sigma$$

where σ *is the number defined in* (1.7).

Proof of Lemma 1. Suppose that

$$\lim_{n\to\infty} a_{n+1}/a_n = \alpha. \tag{3.1.1}$$

We shall prove that $\alpha=\sigma$, where σ is defined by (1.7). By (1.5) we have

$$(C+\varepsilon_n)\, a_n \geqslant a_0 p_n$$

and so

(3.1.2) $$\limsup_{n\to\infty} \sqrt[n]{a_n} \geqslant R^{-1} = \limsup_{n\to\infty} \sqrt[n]{p_n}.$$

If (3.1.1) holds, it follows from (3.1.2) that we always have $\alpha \geqslant R^{-1}$.

Suppose first that $\sum_{k=1}^{\infty} p_k R^k \leqslant C$. We have to show that $\alpha=R^{-1}$. If we had $\alpha>R^{-1}$, we would have

$$C=\lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_k a_{n-k}}{a_n} = \sum_{k=1}^{\infty} p_k \alpha^{-k} < \sum_{k=1}^{\infty} p_k R^k,$$

which is impossible. Hence, we have

$$\alpha = R^{-1} = \sigma.$$

Next, suppose that $\sum_{k=1}^{\infty} p_k R^k > C$. From (1.5), we have

$$C+\varepsilon_n \geqslant \sum_{k=1}^{A} p_k \frac{a_{n-k}}{a_n}$$

for any $n>A$ and it follows immediately that

$$C \geqslant \sum_{k=1}^{A} p_k \alpha^{-k}.$$

Since A can be chosen arbitrarily large it follows that

$$C \geqslant \sum_{k=1}^{\infty} p_k \alpha^{-k}.$$

If $\alpha=R^{-1}$, then $C \geqslant \sum_{k=1}^{\infty} p_k R^k$ which is impossible. Hence we must have $\alpha>R^{-1}$ and

$$C=\lim_{n\to\infty} \sum_{k=1}^{n} p_k \frac{a_{n-k}}{a_n} = \sum_{k=1}^{\infty} p_k \alpha^{-k}.$$

Hence, by (1.7), we have $\alpha^{-1}=\gamma=\sigma^{-1}$, or $\alpha=\sigma$. This completes the proof of Lemma 1.

3.2. *Proof of Theorem* 1. We shall first prove the necessity part of Theorem 1. Let

(3.2.1) $$\varphi_n(A) = \sum_{k=A}^{n+1} p_k \frac{a_{n+1-k}}{a_n} - \sigma \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}.$$

By (1.5), we have

$$\varphi_n(A) = C\left(\frac{a_{n+1}}{a_n} - \sigma\right) - \sum_{k=1}^{A-1} p_k \frac{a_{n-k}}{a_n}\left(\frac{a_{n-k+1}}{a_{n-k}} - \sigma\right) + \varepsilon_{n+1}\frac{a_{n+1}}{a_n} - \sigma\varepsilon_n.$$

Now if (1) and (2) hold and the sequence (a_{n+1}/a_n) is convergent, we have, by Lemma 1,

$$\lim_{n\to\infty} a_{n+1}/a_n = \sigma$$

and the necessity of condition (1.6) follows.

In the proof of the sufficiency part of Theorem 1 we shall always assume that conditions (1) and (2), and (1.6) hold.

The first step in the proof consists in showing that

$$0<\lambda=\liminf_{n\to\infty}\frac{a_{n+1}}{a_n}\leqslant\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}=\Lambda<\infty. \tag{3.2.2}$$

These inequalities can be proved easily by the arguments used in the proof of Theorem 7 in [3].

For the remaining part of the proof we need the following lemma:

Lemma 2. *If $\sigma<\Lambda<\infty$ and if (n_i) is a sequence such that*

$$\lim_{i\to\infty}\frac{a_{n_i+1}}{a_{n_i}}=\Lambda, \tag{3.2.3}$$

then for each fixed positive integer j such that $p_j>0$, we have

$$\lim_{i\to\infty}\frac{a_{n_i+1-j}}{a_{n_i-j}}=\Lambda, \tag{3.2.4}$$

and for any positive integer A, we have

$$\limsup_{i\to\infty}\sum_{k=A}^{n_i}p_k\frac{a_{n_i-k}}{a_{n_i}}=0. \tag{3.2.5}$$

Likewise, if $0<\lambda<\sigma$ and if (m_i) is a sequence such that

$$\lim_{i\to\infty}\frac{a_{m_i+1}}{a_{m_i}}=\lambda, \tag{3.2.6}$$

then for each fixed positive integer j such that $p_j>0$, we have

$$\lim_{i\to\infty}\frac{a_{m_i+1-j}}{a_{m_i-j}}=\lambda. \tag{3.2.7}$$

Proof of Lemma 2. By (1.5) and (3.2.1), we have

$$\varphi_n(A)=\frac{a_{n+1}}{a_n}(C+\varepsilon_{n+1})-(\Lambda+\varepsilon)(C+\varepsilon_n)+ \tag{3.2.8}$$
$$+\sum_{k=1}^{A-1}p_k\left(\frac{(\Lambda+\varepsilon)\,a_{n-k}-a_{n+1-k}}{a_n}\right)+(\Lambda-\sigma+\varepsilon)\sum_{k=A}^{n}p_k\frac{a_{n-k}}{a_n}.$$

Let (n_i) be a sequence such that (3.2.3) holds. Since

$$\frac{a_{k+1}}{a_k}<\Lambda+\varepsilon\quad\text{for } k>N_\varepsilon, \tag{3.2.9}$$

we see that

$$(\Lambda+\varepsilon)\,a_{n-k}-a_{n+1-k}>0 \quad \text{for} \quad n\geqslant N_\varepsilon+A \quad \text{and} \quad 1\leqslant k<A.$$

From this inequality and (3.2.8) follows that

$$|\varphi_{n_i}(A)|>\frac{a_{n_i+1}}{a_{n_i}}(C+\varepsilon_{n_i+1})-(\Lambda+\varepsilon)(C+\varepsilon_{n_i})+(\Lambda-\sigma+\varepsilon)\sum_{k=A}^{n_i}p_k\frac{a_{n_i-k}}{a_{n_i}}.$$

Let $i\to\infty$. Since $\Lambda>\sigma$ we have, by (1.6) and (3.2.3)

$$\limsup_{i\to\infty}\sum_{k=A}^{n_i}p_k\frac{a_{n_i-k}}{a_{n_i}}<\varepsilon C/(\Lambda-\sigma),$$

and (3.2.5) follows, since ε can be chosen arbitrarily small.

Next, suppose that $p_j>0$ and that $1\leqslant j\leqslant A-1$. Since $\sigma<\Lambda$ and $(\Lambda+\varepsilon)\,a_{n-k}-a_{n+1-k}\geqslant 0$, we have, again by (3.2.8)

$$\varphi_{n_i}(A)\geqslant\left(\frac{a_{n_i+1}}{a_{n_i}}-(\Lambda+\varepsilon)\right)C+p_j\left(\frac{(\Lambda+\varepsilon)\,a_{n_i-j}-a_{n_i+1-j}}{a_{n_i}}\right)\geqslant$$

$$\geqslant\left(\frac{a_{n_i+1}}{a_{n_i}}-(\Lambda+\varepsilon)\right)C+p_j\frac{a_{n_i-j}}{a_{n_i}}\left(\Lambda-\frac{a_{n_i+1-j}}{a_{n_i-j}}\right).$$

From this inequality and (3.2.9) follows that

$$-\varepsilon\leqslant\Lambda-\frac{a_{n_i+1-j}}{a_{n_i-j}}\leqslant\frac{1}{p_j}\,\frac{a_{n_i}}{a_{n_i-j}}\left(\left(\Lambda+\varepsilon-\frac{a_{n_i+1}}{a_{n_i}}\right)C+\varphi_{n_i}(A)\right).$$

Since $a_{n_i}/a_{n_i-j}\leqslant(\Lambda+\varepsilon)^j$, by (3.2.9) and $\lim_{i\to\infty}\varphi_{n_i}(A)=0$, it follows that

$$\lim_{i\to\infty}\frac{a_{n_i+1-j}}{a_{n_i-1}}=\Lambda.$$

for each j such that $p_j>0$. This proves (3.2.4). The proof of the remaining statements is similar.

To prove the sufficiency of condition (1.6) it is clearly sufficient to show that $\Lambda\leqslant\sigma$ and $\sigma\leqslant\lambda$. By (3.2.2) there exists a sequence (n_i) such that (3.2.3) holds with $0<\Lambda<\infty$. Suppose that $\Lambda>\sigma$. Then, by Lemma 2, we have also for all k such that $p_k>0$

$$\frac{a_{n_i-k}}{a_{n_i}}\to\Lambda^{-k}\qquad(i\to\infty).$$

Using (1.5) we find that

$$\left|C-\sum_{k=1}^{A-1}p_k\frac{a_{n_i-k}}{a_{n_i}}\right|\leqslant\sum_{k=A}^{n_i}p_k\frac{a_{n_i-k}}{a_{n_i}}+|\varepsilon_{n_i}|.$$

Hence

$$\left|C-\sum_{k=1}^{A-1}p_k\Lambda^{-k}\right|<\limsup_{i\to\infty}\sum_{k=A}^{n_i}p_k\frac{a_{n_i-k}}{a_{n_i}}.$$

Since A can be chosen arbitrarily large, it follows, by (3.2.5), that

$$C=\sum_{k=1}^{\infty} p_k \Lambda^{-k}.$$

We can now easily show that our hypothesis $\Lambda>\sigma$ leads to a contradiction.

Suppose that first $C \geqslant \sum_{k=1}^{\infty} p_k R^k$. By (1.7) we have then $\sigma=R^{-1}$. Since $\Lambda>\sigma=R^{-1}$, we have

$$C=\sum_{k=1}^{\infty} p_k \Lambda^{-k}<\sum_{k=1}^{\infty} p_k R^k<C$$

which is impossible. Hence $\Lambda \leqslant \sigma$.

Next, suppose that $C<\sum_{k=1}^{\infty} p_k R^k$. By (1.7) we have $\sigma=\gamma^{-1}$, where $\sum_{k=1}^{\infty} p_k \gamma^k=C$. Since $\Lambda>\sigma=\gamma^{-1}$, we have

$$C=\sum_{k=1}^{\infty} p_k \Lambda^{-k}<\sum_{k=1}^{\infty} p_k \gamma^k=C$$

which is again impossible. Hence $\Lambda \leqslant \sigma$.

To complete the proof of the theorem we have to show that $\sigma \leqslant \lambda$. Suppose that $\sigma>\lambda$. We have, by (1.5) and (3.2.1),

$$C\left(\frac{a_{n+1}}{a_n}-(\lambda-\varepsilon)\right)=\varepsilon_n-\varepsilon_{n+1} \frac{a_{n+1}}{a_n}+\sum_{k=1}^{A-1} p_k\left(\frac{a_{n+1-k}-(\lambda-\varepsilon) a_{n-k}}{a_n}\right)+$$
$$+\varphi_n(A)+(\sigma-\lambda+\varepsilon) \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}.$$

Let (m_i) be a sequence so that (3.2.6) holds and let A be such that $p_A>0$. If $\varepsilon>0$ and $k>N_{\varepsilon}$; then $a_{k+1}-(\lambda-\varepsilon) a_k \geqslant 0$. Hence, for $m_i>A+N_{\varepsilon}$, we have

$$C\left(\frac{a_{m_i+1}}{a_{m_i}}-(\lambda-\varepsilon)\right) \geqslant \varepsilon_{m_i}-\varepsilon_{m_i+1} \frac{a_{m_i+1}}{a_{m_i}}+\varphi_{m_i}(A)+(\sigma-\lambda+\varepsilon) \sum_{k=A}^{m_i} p_k \frac{a_{m_i-k}}{a_{m_i}}.$$

Let $i \rightarrow \infty$. Since $\sigma>\lambda$, we have, by Lemma 2,

$$\varepsilon C \geqslant(\sigma-\lambda) p_A \lambda^{-A},$$

a contradiction, since ε can be chosen arbitrarily small. Thus we must have $\sigma \leqslant \lambda$, and Theorem 1 is proved.

3.3. *Proof of Theorem 2.* We shall first prove the necessity of condition (1.8). If $\lim_{n \rightarrow \infty} a_{n+1} / a_n$ exists, then by Lemma 1, it is equal to σ where σ is defined by (1.7). We shall first consider the case $\sum_{k=1}^{\infty} p_k R^k=C$. Then $\sigma=R^{-1}$. By (1.5), we have

$$C+\varepsilon_n=\sum_{k=1}^{A-1} p_k \frac{a_{n-k}}{a_n}+\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}.$$

Since

$$\lim_{n\to\infty} \frac{a_{n-k}}{a_n} = R^k, \qquad k=1, 2, \dots$$

it follows that

$$\limsup_{n\to\infty} \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} = C - \sum_{k=1}^{A-1} p_k R^k.$$

Hence, we have

$$\lim_{A\to\infty} \left(\limsup_{n\to\infty} \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}\right) = 0.$$

Next, suppose that $\sum_{k=1}^{\infty} p_k R^k > C$. Then by Lemma 1, we have $\sigma = \gamma^{-1}$ and $\sum_{k=1}^{\infty} p_k \gamma^k = C$. Hence

$$\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} = C + \varepsilon_n - \sum_{k=1}^{A-1} p_k \frac{a_{n-k}}{a_n}.$$

Since

$$\lim_{n\to\infty} \frac{a_{n-k}}{a_n} = \sigma^{-k} = \gamma^k, \qquad k=1, 2, \dots$$

we find that

$$\limsup_{n\to\infty} \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n} = C - \sum_{k=1}^{A-1} p_k \gamma^k.$$

Let $A\to\infty$, and (1.8) follows.

The proof of the sufficiency of condition (1.8) is based on the following lemmas:

Lemma 3. *If the condition* (1.8) *holds, we have*

$$0 < \lambda = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leqslant \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \Lambda < \infty.$$

Proof of Lemma 3. By (1.8), for any $0 < \varepsilon < \frac{1}{2} C$, we can choose first A_ε and $N_\varepsilon > A_\varepsilon$ such that for $n > N_\varepsilon$ we have

$$\sum_{k=A_\varepsilon}^{n} p_k \frac{a_{n-k}}{a_n} \leqslant \frac{1}{2}(C+\varepsilon).$$

If $n > A_\varepsilon + N_\varepsilon$ we have

$$(C+\varepsilon_{n+1}) \frac{a_{n+1}}{a_n} = \sum_{k=1}^{A_\varepsilon - 1} p_k \frac{a_{n+1-k}}{a_n} + \frac{a_{n+1}}{a_n} \sum_{k=A_\varepsilon}^{n+1} p_k \frac{a_{n+1-k}}{a_{n+1}} \leqslant$$

$$\leqslant \frac{1}{2}(C+\varepsilon) \frac{a_{n+1}}{a_n} + \sum_{k=1}^{A_\varepsilon} p_k \frac{a_{n+1-k}}{a_n}$$

or

$$\left(\frac{1}{2}(C-\varepsilon) + \varepsilon_{n+1}\right) \frac{a_{n+1}}{a_n} \leqslant \sum_{k=1}^{A_\varepsilon} p_k \frac{a_{n+1-k}}{a_n}.$$

Since

$$\limsup_{n\to\infty} \frac{a_{n-1}}{a_n} \leqslant C/p_1$$

we have

$$\limsup_{n\to\infty} \frac{a_{n-k}}{a_n} \leqslant (C/p_1)^k,$$

and so

$$\frac{1}{2}\left(C-\varepsilon\right)\Lambda \leqslant \sum_{k=1}^{A_\varepsilon} p_k (C/p_1)^{k-1} < \infty.$$

Lemma 4. *Suppose that condition* (1.8) *holds. If the sequence* (n_i) *is such that*

(3.3.1) $$\lim_{i\to\infty} \frac{a_{n_i+1}}{a_{n_i}} = \Lambda < \infty,$$

then, for fixed j *such that* $p_j > 0$,

(3.3.2) $$\lim_{i\to\infty} \frac{a_{n_i+1-j}}{a_{n_i-j}} = \Lambda.$$

Likewise, if

(3.3.3) $$\lim_{i\to\infty} \frac{a_{m_i+1}}{a_{m_i}} = \lambda > 0$$

then

(3.3.4) $$\lim_{i\to\infty} \frac{a_{m_i+1-j}}{a_{m_i-j}} = \lambda,$$

for each fixed j *such that* $p_j > 0$.

Proof of Lemma 4. If $\varepsilon > 0$ and $k > N_\varepsilon$ we have

(3.3.5) $$a_{k+1}/a_k < \Lambda + \varepsilon.$$

Using (1.5), for $n_i > A + N_\varepsilon$, we find that

$$C\left((\Lambda+\varepsilon)\, a_{n_i} - a_{n_i+1}\right) = \sum_{k=1}^{A-1} p_k\left((\Lambda+\varepsilon)\, a_{n_i-k} - a_{n_i+1-k}\right) +$$
$$+ (\Lambda+\varepsilon)\, a_{n_i}(W_{n_i}(A) - \varepsilon_{n_i}) - a_{n_i+1}(W_{n_i+1}(A) - \varepsilon_{n_i+1})$$

where

(3.3.6) $$W_n(A) = \sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}.$$

By (3.3.5), we have, for $j < A$ such that $p_j > 0$,

$$(\Lambda+\varepsilon)\, a_{n_i-k} - a_{n_i+1-k} \geqslant 0 \quad \text{for} \quad n_i > A + N_\varepsilon \quad \text{and} \quad 1 \leqslant k \leqslant A-1.$$

Hence

$$C\left(\Lambda+\varepsilon - \frac{a_{n_i+1}}{a_{n_i}}\right) \geqslant p_j \frac{a_{n_i-j}}{a_{n_i}}\left(\Lambda+\varepsilon - \frac{a_{n_i+1-j}}{a_{n_i-j}}\right) - \frac{a_{n_i+1}}{a_{n_i}}\left(W_{n_i+1}(A) - \varepsilon_{n_i+1}\right) - (\Lambda+\varepsilon)\,\varepsilon_{n_i}$$

$$\geqslant p_j(\Lambda+\varepsilon)^{-j}\left(\Lambda - \frac{a_{n_i+1-j}}{a_{n_i-j}}\right) - \frac{a_{n_i+1}}{a_{n_i}}\left(W_{n_i+1}(A) - \varepsilon_{n_i+1}\right) - (\Lambda+\varepsilon)\,\varepsilon_{n_i}.$$

Hence

$$-\varepsilon\leqslant\Lambda-\frac{a_{n_i+1-j}}{a_{n_i-j}}\leqslant\frac{(\Lambda+\varepsilon)^j}{p_j}\left(C\left(\Lambda+\varepsilon-\frac{a_{n_i+1}}{a_{n_i}}\right)+\frac{a_{n_i+1}}{a_{n_i}}(W_{n_i+1}(A)-\varepsilon_{n_i+1})+(\Lambda+\varepsilon)\varepsilon_{n_i}\right).$$

From (3.3.1) it follows that

$$-\varepsilon\leqslant\liminf_{i\to\infty}\left(\Lambda-\frac{a_{r_i+1-j}}{a_{n_i-j}}\right)\leqslant\limsup_{i\to\infty}\left(\Lambda-\frac{a_{n_i+1-j}}{a_{n_i-j}}\right)\leqslant$$

$$\leqslant\frac{(\Lambda+\varepsilon)^j}{p_j}(\varepsilon C+\Lambda\limsup_{i\to\infty}W_{n_i+1}(A))$$

and so $\lim\limits_{i\to\infty}\dfrac{a_{n_i+1-j}}{a_{n_i-j}}=\Lambda,$

since ε can be chosen arbitrarily small and A can be chosen arbitrarily large. This proves (3.3.2). The proof of (3.3.4) can be obtained by the same argument.

Now we can prove the sufficiency of condition (1.8). Let (n_i) be a sequence so that (3.3.1) holds. From (1.5) it follows that for $n_i>A$

$$\left|C-\sum_{k=1}^{A-1}p_k\frac{a_{n_i-k}}{a_{n_i}}\right|\leqslant\sum_{k=A}^{n_i}p_k\frac{a_{n_i-k}}{a_{n_i}}+|\varepsilon_{n_i}|.$$

From Lemma 4, we find that

$$\left|C-\sum_{k=1}^{A-1}p_k\Lambda^{-k}\right|\leqslant\limsup_{i\to\infty}\sum_{k=A}^{n_i}p_k\frac{a_{n_i-k}}{a_{n_i}}.$$

Finally, using condition (1.8), we find that

$$C=\sum_{k=1}^{\infty}p_k\Lambda^{-k}.$$

Similarly, by (3.3.4), we have

$$C=\sum_{k=1}^{\infty}p_k\lambda^{-k}.$$

Since $\sum\limits_{k=1}^{\infty}p_kR^k\geqslant C,$ we have $\lambda=\xi^{-1}=\Lambda,$ where ξ is the unique number in $(0, R]$ such that $\sum\limits_{k=1}^{\infty}p_k\xi^k=C.$ Hence Theorem 2 is proved.

4. *Proof of Theorem 3.* As usual, we shall prove first that Λ is finite. By (1.9) we can find M such that

$$p_{k+1}/p_k\leqslant M,\qquad k=1,2,\ldots$$

By (1.5), we have

$$(C+\varepsilon_{n+1})\frac{a_{n+1}}{a_n}=p_1+\sum_{k=1}^{n}\frac{p_{k+1}}{p_k}p_k\frac{a_{n-k}}{a_n}\leqslant p_1+M(C+\varepsilon_n).$$

Hence,

$$\Lambda=\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}\leqslant\frac{p_1}{C}+M<\infty.$$

The following lemma is analog of Lemma 2.

Lemma 5. *If* $\sigma<\Lambda<\infty$ *and if* (n_i) *is a sequence such that*

$$\lim_{i\to\infty}\frac{a_{n_i+1}}{a_{n_i}}=\Lambda, \tag{4.1}$$

then for each fixed positive integer j *such that* $p_j>0$, *we have*

$$\lim_{i\to\infty}\frac{a_{n_i+1-j}}{a_{n_i-j}}=\Lambda \tag{4.2}$$

and

$$\limsup_{i\to\infty}\sum_{k=A}^{n_i}p_k\frac{a_{n_i-k}}{a_{n_i}}\to 0 \qquad (A\to\infty). \tag{4.3}$$

Proof of lemma 5. If $\varepsilon>0$ and $k>N_\varepsilon$ then

$$(\Lambda+\varepsilon)\,a_k-a_{k+1}\geqslant 0 \quad\text{and}\quad (\sigma+\varepsilon)p_k-p_{k+1}\geqslant 0.$$

Using (1.5) again, if $n>A+N_\varepsilon$, we have

$$(\Lambda+\varepsilon)\,(C+\varepsilon_n)\,a_n-(C+\varepsilon_{n+1})\,a_{n+1}=\sum_{k=1}^{A-1}p_k\,((\Lambda+\varepsilon)\,a_{n-k}-a_{n+1-k})+$$
$$+(\Lambda-\sigma)\sum_{k=A}^{n}p_k a_{n-k}+\sum_{k=A}^{n}((\sigma+\varepsilon)p_k-p_{k+1})\,a_{n-k}-p_A a_{n-A+1}.$$

The first 3 terms of the right hand side of this equation are nonnegative. Hence, we have

$$C\,((\Lambda+\varepsilon)\,a_n-a_{n+1})+(\Lambda+\varepsilon)\,\varepsilon_n a_n-\varepsilon_{n+1}a_{n+1}\geqslant \tag{4.4}$$
$$\geqslant(\Lambda-\sigma)\sum_{k=A}^{n}p_k a_{n-k}+\sum_{k=1}^{A-1}p_k\,((\Lambda+\varepsilon)\,a_{n-k}-a_{n+1-k})-p_A a_{n+1-A}.$$

Let $K\leqslant A\leqslant 2K$. Then

$$p_A a_{n+1-A}\geqslant(\Lambda-\sigma)\sum_{k=2K}^{n}p_k a_{n-k}+\sum_{k=1}^{K-1}p_k\,((\Lambda+\varepsilon)\,a_{n-k}-a_{n+1-k})-$$
$$-C\,((\Lambda+\varepsilon)\,a_n-a_{n+1})-(\Lambda+\varepsilon)\,\varepsilon_n a_n+\varepsilon_{n+1}a_{n+1}.$$

Hence,

$$\min_{K\leqslant A\leqslant 2K}p_A a_{n+1-A}\geqslant(\Lambda-\sigma)\sum_{k=2K}^{n}p_k a_{n-k}+\sum_{k=1}^{K-1}p_k\,((\Lambda+\varepsilon)\,a_{n-k}-a_{n+1-k})- \tag{4.5}$$
$$-C\,((\Lambda+\varepsilon)\,a_n-a_{n+1})-(\Lambda+\varepsilon)\,\varepsilon_n a_n+\varepsilon_{n+1}a_{n+1}.$$

But, if $n>2K+N_\varepsilon$, we have

$$\min_{K\leqslant A\leqslant 2K}p_A a_{n+1-A}\leqslant\frac{1}{K}\sum_{k=K+1}^{2K}p_k a_{n+1-k}\leqslant \tag{4.6}$$
$$\leqslant\frac{1}{K}\sum_{k=K}^{2K-1}\frac{p_{k+1}}{p_k}p_k a_{n-k}\leqslant$$
$$\leqslant\frac{\sigma+\varepsilon}{K}\sum_{k=K}^{2K}p_k a_{n-k}\leqslant\frac{\sigma+\varepsilon}{K}(C+\varepsilon_n)\,a_n.$$

Combining (4.5) and (4.6), we find that (4.4) becomes

$$(4.7)\qquad \frac{\sigma+\varepsilon}{K}(C+\varepsilon_n)\,a_n+C((\Lambda+\varepsilon)\,a_n-a_{n+1})+(\Lambda+\varepsilon)\,\varepsilon_n a_n-\varepsilon_{n+1}a_{n+1}\geqslant$$
$$\geqslant\sum_{k=1}^{K-1}p_k((\Lambda+\varepsilon)\,a_{n-k}-a_{n+1-k})+(\Lambda-\sigma)\sum_{k=2K}^{n}p_k a_{n-k}.$$

Dividing both sides of (4.7) by a_n, if $j<K$, we have first

$$(4.8)\qquad \frac{\sigma+\varepsilon}{K}(C+\varepsilon_n)+C\left(\Lambda+\varepsilon-\frac{a_{n+1}}{a_n}\right)+(\Lambda+\varepsilon)\,\varepsilon_n-\varepsilon_{n+1}\frac{a_{n+1}}{a_n}\geqslant$$
$$\geqslant p_j(\Lambda+\varepsilon)^{-j}\left(\Lambda-\frac{a_{n+1-j}}{a_{n-j}}\right).$$

Let (n_i) be a sequence that (4.1) holds. Replacing n by n_i in (4.8), we find that

$$-\varepsilon<\Lambda-\frac{a_{n_i+1-j}}{a_{n_i-j}}\leqslant\frac{(\Lambda+\varepsilon)^j}{p_j}\left(\frac{\sigma+\varepsilon}{K}(C+\varepsilon_{n_i})+C\left(\Lambda+\varepsilon-\frac{a_{n_i+1}}{a_{n_i}}\right)+\right.$$
$$\left.+(\Lambda+\varepsilon)\,\varepsilon_{n_i}-\varepsilon_{n_i+1}\frac{a_{n_i+1}}{a_{n_i}}\right)$$

and (4.2) follows by first letting $i\to\infty$, then $K\to\infty$ and finally $\varepsilon\to 0$.

On the other hand by (4.7), we have

$$\frac{\sigma+\varepsilon}{K}(C+\varepsilon_n)\,a_n+C((\Lambda+\varepsilon)\,a_n-a_{n+1})+(\Lambda+\varepsilon)\,\varepsilon_n a_n-\varepsilon_{n+1}a_{n+1}\geqslant$$
$$\geqslant(\Lambda-\sigma)\sum_{k=2K}^{n}p_k a_{n-k}.$$

If (n_i) is as in (4.1), we have

$$\left(\frac{\sigma+\varepsilon}{K}+\varepsilon\right)C\geqslant(\Lambda-\sigma)\limsup_{i\to\infty}\sum_{k=2K}^{n_i}p_k\frac{a_{n_i-k}}{a_{n_i}}.$$

As $\varepsilon\to 0$, (4.3) follows.

To complete the proof of Theorem 3 we shall first show that $\Lambda\leqslant\sigma$. Suppose $\sigma<\Lambda$. Let (n_i) be a sequence so that (4.1) holds. By (1.5) we have

$$(C+\varepsilon_{n_i})\geqslant\sum_{k=1}^{A}p_k\frac{a_{n_i-k}}{a_{n_i}}$$

for any positive integer A. Using (4.2) and first letting $i\to\infty$, then $A\to\infty$, we find that

$$C\geqslant\sum_{k=1}^{\infty}p_k\Lambda^{-k}.$$

On the other hand from

$$C+\varepsilon_{n_i}=\sum_{k=1}^{A-1}p_k\frac{a_{n_i-k}}{a_{n_i}}+\sum_{k=A}^{n_i}p_k\frac{a_{n_i-k}}{a_{n_i}}$$

and the asymptotic relations (4.2) and (4.3) we find that

$$C\leqslant \sum_{k=1}^{\infty} p_k \Lambda^{-k}.$$

Hence

$$C=\sum_{k=1}^{\infty} p_k \Lambda^{-k}.$$

Now we shall show that the hypothesis $\Lambda>\sigma$ leads to a contradiction. Suppose first that $C=\sum_{k=1}^{\infty} p_k R^k$. We have then, by (1.7), $\sigma=R^{-1}$. Hence

$$C=\sum_{k=1}^{\infty} p_k \Lambda^{-k}<\sum_{k=1}^{\infty} p_k \sigma^{-k}=C$$

which is impossible. Next suppose that $C<\sum_{k=1}^{\infty} p_k R^k$. Then by (1.7) $\sigma=\gamma^{-1}$ and $\sum_{k=1}^{\infty} p_k \gamma^k=C$. Hence

$$C=\sum_{k=1}^{\infty} p_k \Lambda^{-k}<\sum_{k=1}^{\infty} p_k \sigma^{-k}=C,$$

which is again impossible. Hence $\Lambda\leqslant\sigma$.

Now it remains only to show that

$$\lim_{A\to\infty}\left(\limsup_{n\to\infty}\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}\right)=0.$$

Then from Theorem 2 it will follow that $\lim\limits_{n\to\infty} a_{n+1}/a_n$ exists.

If $\varepsilon>0$, and if $n\geqslant k\geqslant N_\varepsilon$, then

$$a_k/a_n\geqslant(\sigma+\varepsilon)^{-n+k}.$$

Let $n>A+N_\varepsilon$ we have, by (1.5),

$$\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}\leqslant C+\varepsilon_n-\sum_{k=1}^{A-1} p_k(\sigma+\varepsilon)^{-k}.$$

Hence

$$\limsup_{n\to\infty}\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_k}\leqslant C-\sum_{k=1}^{A-1} p_k(\sigma+\varepsilon)^{-k}.$$

If $\varepsilon\to 0$, we find that

$$\limsup_{n\to\infty}\sum_{k=A}^{n} p_k \frac{a_{n-k}}{a_n}\leqslant C-\sum_{k=1}^{A-1} p_k\sigma^{-k}. \tag{4.9}$$

But, if $C\leqslant\sum_{k=1}^{\infty} p_k R^k$, and σ is defined by (1.7), it is easy to see that

$$C=\sum_{k=1}^{\infty} p_k \sigma^{-k}.$$

Hence condition (1.8) follows immediately from inequality (4.9).

5. *Proof of Theorem 4.* (Necessity) If $\lim_{n\to\infty} a_{n+1}/a_n$ exists, by Lemma 1 it is equal to σ, where σ is defined by (1.7). Since $\sum_{k=1}^{\infty} p_k R^k > C$, we have $\sigma=\gamma^{-1}$, where $\sum_{k=1}^{\infty} p_k \gamma^k = C$. Hence

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \sigma = \gamma^{-1} \geqslant R^{-1}.$$

If we had $\sigma = R^{-1}$, it would follow that

$$C = \sum_{k=1}^{\infty} p_k \gamma^k = \sum_{k=1}^{\infty} p_k \sigma^{-k} = \sum_{k=1}^{\infty} p_k R^k > C$$

which is impossible. Hence $\sigma > R^{-1}$ and condition (1.10) is necessary.

In order to prove the sufficiency of condition (1.10) we define a sequence $(A(\rho))$ by the following relation

$$A_n(\rho) = \sum_{k=1}^{n} \rho^k a_k, \qquad n=1, 2, \dots$$

Let δ be the radius of convergence of the series $\sum_{k=1}^{\infty} a_k x^k$. Then, by (1.10), $0 \leqslant \delta < R$ and we have, for every $\rho > \delta$,

$$A_n(\rho) \to \infty \qquad (n\to\infty).$$

Using this result, it is easy to see that the following lemma is true.

Lemma 6 *For every* $\rho \in (\delta, R)$ *we have*

$$(5.1) \qquad (C + \eta_n(\rho)) A_n(\rho) = \sum_{k=1}^{n} p_k \rho^k A_{n-k}(\rho)$$

where $\eta_n(\rho) \to 0$ $(n\to\infty)$.

To prove that the limit of the sequence (a_{n+1}/a_n) exists, we shall prove first that there exists $c_r \in (0, 1)$, for some $r \in (\delta, R)$, such that

$$(5.2) \qquad \lim_{n\to\infty} \frac{r^n a_n}{A_n(r)} = c_r.$$

We have then

$$\frac{a_n}{a_{n+1}} = r \frac{r^n a_n}{A_n(r)} \frac{A_n(r)}{A_{n+1}(r)} \frac{A_{n+1}(r)}{r^{n+1} a_{n+1}}.$$

Since

$$\frac{A_n(r)}{A_{n+1}(r)} = 1 - \frac{r^{n+1} a_{n+1}}{A_{n+1}(r)}.$$

We have, by (5.2),

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = r c_r (1-c_r) \frac{1}{c_r} = r(1-c_r).$$

Thus, it remains only to prove that (5.2) holds. We have, for every $\rho \in (\delta, R)$,

$$0 \leqslant \alpha_\rho = \liminf_{n\to\infty} \frac{\rho^n a_n}{A_n(\rho)} \leqslant \limsup_{n\to\infty} \frac{\rho^n a_n}{A_n(\rho)} = \beta_\rho \leqslant 1.$$

To establish (5.2) we shall need two lemmas.

Lemma 7. *There exists $r \in (\delta, R)$ such that*

$$0 < \beta_r < 1. \tag{5.3}$$

Proof of Lemma 7. We first prove that $\beta_r < 1$ for every $r \in (\delta, R)$. Suppose that $\beta_r = 1$ for some $r \in (\delta, R)$. Then there exists a sequence (m_i) such that

$$\lim_{i \to \infty} \frac{r^{m_i} a_{m_i}}{A_{m_i}(r)} = 1,$$

or

$$\lim_{i \to \infty} \frac{A_{m_i - 1}(r)}{A_{m_i}(r)} = 0.$$

Hence, given $\varepsilon > 0$ there exists I_ε such that

$$\frac{A_{m_i - 1}(r)}{A_{m_i}(r)} < \varepsilon \quad \text{for all} \quad i > I_\varepsilon. \tag{5.4}$$

Since the sequence $(A_n(r))$ is monotone increasing, we have, by (5.1),

$$C + \eta_{m_i}(r) < \frac{A_{m_i - 1}(r)}{A_{m_i}(r)} \sum_{k=1}^{m_i} p_k r^k$$

and it follows from (5.4) that $C \leqslant \varepsilon \sum_{k=1}^{\infty} p_k r^k$.

Since $r < R$ and ε can be chosen arbitrarily small, we get $C \leqslant 0$, a contradiction.

Next, we prove that $\beta_r > 0$ for some $r \in (\delta, R)$. Suppose that this were not true. Then for any $r \in (\delta, R)$, we would have $\beta_r = 0$. This would imply that

$$\lim_{n \to \infty} \frac{r^n a_n}{A_n(r)} = 0 \quad \text{for every} \quad r \in (\delta, R).$$

Choose $\varepsilon \in (0, 1)$ and a number N, which depends on both ε and r, so that

$$\frac{r^n a_n}{A_n(r)} < \varepsilon \quad \text{for all} \quad n > N. \tag{5.5}$$

This means that

$$\frac{A_{n-1}(r)}{A_n(r)} > 1 - \varepsilon \quad \text{for all} \quad n > N. \tag{5.6}$$

Using (5.1) and (5.6), we find that

$$(C + \eta_n(r)) A_n(r) \geqslant \sum_{k=1}^{n-N} p_k r^k A_{n-k}(r) \geqslant A_n(r) \sum_{k=1}^{n-N} p_r r^k (1 - \varepsilon)^k.$$

Let $n \to \infty$. We then have

$$C \geqslant \sum_{k=1}^{\infty} p_k [r(1 - \varepsilon)]^k.$$

Hence we must have

$$C \geqslant \sum_{k=1}^{\infty} p_k r^k \quad \text{for every} \quad r \in (\delta, R)$$

and it would follow that

$$C \geqslant \sum_{k=1}^{\infty} p_k R^k.$$

But this is impossible in view of the condition

$$C < \sum_{k=1}^{\infty} p_k R^k.$$

This completes the proof of Lemma 7.

From now on we shall fix the number $r \in (\delta, R)$ which is determined by Lemma 7. The result of Lemma 7 enables us to prove the following Lemma:

Lemma 8. *If (m_i) is a sequence of natural numbers so that*

$$\lim_{i\to\infty} \frac{r^{m_i} a_{m_i}}{A_{m_i}(r)} = \beta_r, \tag{5.7}$$

then, for each j such that $p_j > 0$, we have

$$\lim_{i\to\infty} \frac{r^{m_i - j} a_{m_i - j}}{A_{m_i - j}(r)} = \beta_r. \tag{5.8}$$

Likewise, if (n_i) is a sequence of natural numbers so that

$$\lim_{i\to\infty} \frac{r^{n_i} a_{n_i}}{A_{n_i}(r)} = \alpha_r, \tag{5.9}$$

then, for each j such that $p_j > 0$, we have

$$\lim_{i\to\infty} \frac{r^{n_i - j} a_{n_i - j}}{A_{n_i - j}(r)} = \alpha_r. \tag{5.10}$$

Proof of Lemma 8. We shall prove (5.8) only, the proof of (5.10) can be established by the same argument.

Choose $\varepsilon > 0$ such that $1 - \beta_r - \varepsilon > 0$. Next, choose N_ε such that $k > N_\varepsilon$, we have

$$\frac{r^k a_k}{A_k(r)} < \beta_r + \varepsilon, \tag{5.11}$$

or

$$\frac{A_{k-1}(r)}{A_k(r)} > 1 - \beta_r - \varepsilon. \tag{5.12}$$

Let A be a positive integer and let $m_i > A + N_\varepsilon$. Multiplying (5.1) by $\beta_r + \varepsilon$ and (1.5) by r^{m_i} and subtracting, we obtain

$$C((\beta_r + \varepsilon) A_{m_i}(r) - r^{m_i} a_{m_i}) + (\beta_r + \varepsilon) \eta_{m_i}(r) A_{m_i}(r) - \varepsilon_{m_i} r^{m_i} a_{m_i} = \tag{5.13}$$

$$= \left[\sum_{k=1}^{A-1} + \sum_{k=A}^{m_i}\right] p_k r^k ((\beta_r + \varepsilon) A_{m_i - k}(r) - r^{m_i - k} a_{m_i - k}).$$

Let

$$\psi_n(A)=\sum_{k=A}^{n} p_k r^k\left((\beta_r+\varepsilon)\frac{A_{n-k}(r)}{A_n(r)}-\frac{r^{n-k}a_{n-k}}{A_n(r)}\right).$$

Since $r^{n-k}a_{n-k}\leq A_{n-k}(r)$ and $(A_n(r))$ is an increasing sequence, we have

$$|\psi_n(A)|\leq\sum_{k=A}^{n} p_k r^k(\beta_r+\varepsilon+1)\leq 2\sum_{k=A}^{n} p_k r^k.$$

Since $r\in(\delta, R)$, we have

$$\limsup_{n\to\infty}|\psi_n(A)|\to 0 \qquad (A\to\infty). \tag{5.14}$$

Dividing (5.13) by $A_{m_i}(r)$ and using (5.12), for $j\leq A$ with $p_j>0$, we have

$$C\left((\beta_r+\varepsilon)-\frac{r^{m_i}a_{m_i}}{A_{m_i}(r)}\right)+(\beta_r+\varepsilon)\eta_{m_i}(r)-\frac{\varepsilon_{m_i}r^{m_i}a_{m_i}}{A_{m_i}(r)}-\psi_{m_i}(A)\geq \tag{5.15}$$

$$\geq p_j r^j\frac{A_{m_i-j}(r)}{A_{m_i}(r)}\left((\beta_r+\varepsilon)-\frac{r^{m_i-j}a_{m_i-j}}{A_{m_i-j}(r)}\right)\geq$$

$$\geq p_j r^j(1-\beta_r-\varepsilon)^j\left(\beta_r-\frac{r^{m_i-j}a_{m_i-j}}{A_{m_i-j}(r)}\right).$$

In order to simplify the notation we shall denote the left hand side of (5.15) by $T_{m_i}(A)$. By (5.14) and (5.7), we have

$$\limsup_{i\to\infty}T_{m_i}(A)\to\varepsilon C \qquad (A\to\infty).$$

Hence, by (5.15), we have

$$-\varepsilon\leq\beta_r-\frac{r^{m_i-j}a_{m_i-j}}{A_{m_i-j}(r)}\leq\frac{T_{m_i}(A)}{p_j r^j(1-\beta_r-\varepsilon)^j}.$$

By letting $A\to\infty$ and $\varepsilon\to 0$ we find that

$$\lim_{i\to\infty}\frac{r^{m_i-j}a_{m_i-j}}{A_{m_i-j}(r)}=\beta_r.$$

Using this result, we can prove that $\beta_r=\alpha_r$. By Lemma 8 there is a sequence (m_i) such that (5.7) holds. Consequently, we have, for all j such that $p_j>0$,

$$\lim_{i\to\infty}\frac{A_{m_i-1-j}(r)}{A_{m_i-j}(r)}=1-\beta_r. \tag{5.16}$$

By (5.1), for $m_i>A$, we have

$$(C+\eta_{m_i}(r))=\left[\sum_{k=1}^{A-1}+\sum_{k=A}^{m_i}\right]p_k r^k\frac{A_{m_i-k}(r)}{A_{m_i}(r)}.$$

By (5.16), since $(A_n(r))$ is an increasing sequence, we find that

$$C\leq\sum_{k=1}^{A-1}p_k r^k(1-\beta_r)^k+\sum_{k=A}^{\infty}p_k r^k.$$

Since A can be chosen arbitrarily large and since $r \in (\delta, R)$, we have

$$C \leqslant \sum_{k=1}^{\infty} p_k r^k (1-\beta_r)^k. \tag{5.17}$$

On the other hand using (5.1), we have

$$C + \eta_n(r) \geqslant \sum_{k=1}^{A} p_k r^k \frac{A_{n-k}(r)}{A_n(r)}.$$

Replacing n by m_i and using (5.16), we find that

$$C \geqslant \sum_{k=1}^{A} p_k r^k (1-\beta_r)^k.$$

Hence, as $A \to \infty$, we have

$$C \geqslant \sum_{k=1}^{\infty} p_k r^k (1-\beta_r)^k. \tag{5.18}$$

The equality

$$C = \sum_{k=1}^{\infty} p_k r^k (1-\beta)_r^k \tag{5.19}$$

follows by combining (5.17) and (5.18).

Similarly, we have

$$C = \sum_{k=1}^{\infty} p_k r^k (1-\alpha_r)^k. \tag{5.20}$$

From (5.19) and (5.20) follows that $\beta_r = \alpha_r$ and the theorem is proved.

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