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SOME REMARKS ON THE STRUCTURE OF THE APPLICABILITY DOMAINS OF MATRIX OPERATORS AND METHODS

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Summary. In this Note we give some remarks on the structure of the applicability domains of matrix operators and convergence methods (Section 2) and various examples of matrices such that the applicability domains of the corresponding operators and methods are not BK-spaces[1] (Section 3).

1. Definitions and notations

1.1. Definition. Let $F=(f_{ij})$ be an infinite matrix of real (complex) scalars and let $t=(\tau_j)$ be a real(complex)-valued sequence. If the series

$$\sum_{j=0}^{\infty} f_{ij}\tau_j$$

converges for every $i=0, 1, 2, \ldots$, then we call the sequence

$$\left(\sum_{j=0}^{\infty} f_{ij}\tau_j\right)$$

a *transform* of the sequence $t=(\tau_j)$ by the *operator* (F) and denote it by $(F)(t)$. The set of all $t=(\tau_j)$ such that the transform $(F)(t)$ exists, we call the *applicability domain* of the operator (F) and denote it by $(F)^d$. The set of all $t=(\tau_j)$ for which $(F)(t)$ is a bounded sequence we call a *b-applicability domain* of the operator (F) and denote it by $(F)^b$.

1.2. Definition. If for $t=(\tau_j)\in(F)^d$ there exists

$$\lim_i \sum_{j=0}^{\infty} f_{ij}\tau_j,$$

it is called the *limit* of the sequence $t=(\tau_j)$ by the *method* (F') and we denote it by $(F')(t)$ and $(F')\text{-}\lim \tau_j$, respectively. The set of all $t=(\tau_j)$ such that $(F')\text{-}\lim \tau_j$ exists we call the *applicability domain*, or the *convergence domain*,

[1] i. e. Banach spaces in which the norm convergence implies the coordinate convergence

of the method (F') and denote it by $(F')^c$. The set of all $t=(\tau_j)\in(F')^c$ such that (F')-$\lim\tau_j=0$ we call the *o-applicability domain*, or the *o-convergence domain*, of the method (F') and denote it by $(F')^\circ$.

2. Applicability domains of matrix operators and methods as B_0K-spaces[1)]

Let $F=(f_{ij})$ be any matrix and let Y be a B_0K-space under the pseudonorms q_i $(i=0, 1, 2, \ldots)$. Denote by X the set of all $t=(\tau_j)\in(F)^d$ such that the transform $(F)(t)\in Y$. K. Zeller [1] proved three following results.

2.1. The linear space X is a B_0K-space with the pseudonorms

(1) $$p_k(t)=|\tau_k| \qquad (k=0, 1, 2, \ldots),$$

(2) $$p_k^1(t)=\sup_l\left|\sum_{j=0}^{l} f_{kj}\tau_j\right| \qquad (k=0, 1, 2, \ldots)$$

and

(3) $$\bar{p}_k^2(t)=q_k((F)(t)) \qquad (k=0, 1, 2, \ldots).$$

2.1.i. If the matrix $F=(f_{ij})$ is row-finite, then the linear space X is a B_0K-space under the pseudonorms (1) and (3).

2.1.ii. If $F=(f_{ij})$ is row-finite and of type U[2)], then X is a B_0K-space with the pseudonorms (3).

In this section of our paper we give the following completion of the above-mentioned results of Zeller:

2 1.iii. Proposition. *If the matrix $F=(f_{ij})$ is of the type U, then the linear space X is a B_0K-space under the pseudonorms* (2) *and* (3) (i. e. the pseudonorms (1) can be omitted).

(Now, obviously, 2.1.ii. is a consequence of 2.1.i. and 2.1.iii.)

Proof of 2.1.iii. Firstly, observe that

$$p_k^1(t)=0 \quad (k=0, 1, 2, \ldots), \qquad \bar{p}_k^2(t)=0 \quad (k=0, 1, 2, \ldots)$$

implies $t=0$. Further, let (t_m), $t_m=(\tau_j^m)\in X$ $(m=0, 1, 2, \ldots)$, be a Cauchy sequence concerning the pseudonorms (2) and (3), i. e. let

(a) $$p_k^1(t_m-t_n)\to 0 \quad (m\to\infty,\ n\to\infty;\ k=0, 1, 2, \ldots)$$

and

(b) $$\bar{p}_k^2(t_m-t_n)=q_k((F)(t_m)-(F)(t_n))\to 0 \ (m\to\infty,\ n\to\infty;\ k=0, 1, 2, \ldots).$$

Then (see, for inst., [3], III, Theorem 10) the property U implies

$$p_k(t_m-t_n)\to 0 \quad (m\to\infty,\ n\to\infty;\ k=0, 1, 2, \ldots).$$

[1)] i. e. B_0-spaces (see [2], 1) in which the norm convergence implies the coordinate convergence;

[2)] that is: a solution $t=(\tau_j)$ of the system $\sum_{j=0}^{\infty} f_{ij}\tau_j=\sigma_i$ $(i=0, 1, 2, \ldots)$ is unique, whenever it exists.

Hence, there exists a sequence $t=(\tau_j)$ such that

$$p_k(t_m-t)\to 0 \quad (m\to\infty;\ k=0, 1, 2, \dots).$$

It follows from (a) that this sequence $t=(\tau_j)$ satisfies

$$\text{(a')} \qquad \sup_l \left| \sum_{j=0}^{l} f_{kj}(\tau_j^m-\tau_j)\right| \to 0 \quad (m\to\infty;\ k=0, 1, 2, \dots).$$

This implies that the series

$$\sum_{j=0}^{\infty} f_{ij}\tau_j$$

converges for every $i=0, 1, 2, \dots,$ i.e. $t=(\tau_j)\in(F)^d$. Now (a') means the same as

$$p_k^1(t_m-t)\to 0 \quad (m\to\infty;\ k=0, 1, 2, \dots).$$

At the same time, there exists a sequence $s=(\sigma_i)\in Y$ such that

$$\text{(b')} \qquad q_k((F)(t_m)-s)\to 0 \quad (m\to\infty;\ k=0, 1, 2, \dots).$$

Then

$$\left|\sum_{j=0}^{\infty} f_{ij}\tau_j-\sigma_i\right| \le \left|\sum_{j=0}^{\infty} f_{ij}(\tau_j-\tau_j^m)\right| + \left|\sum_{j=0}^{\infty} f_{ij}\tau_j^m-\sigma_i\right| \le p_i^1(t_m-t)+$$

$$+\left|\sum_{j=0}^{\infty} f_{ij}\tau_j^m-\sigma_i\right| \quad (i=0, 1, 2, \dots)$$

and the property K of the space Y imply $s=(F)(t)\in Y$. Therefore (b') means the same as

$$\bar{p}_k^2(t_m-t)\to 0 \quad (m\to\infty;\ k=0, 1, 2, \dots).$$

This completes the proof of our proposition.

Taking for Y the space of all sequences, then the space of bounded, convergent or null-convergent sequences, we obtain as X the linear space $(F)^d$, $(F)^b$, $(F')^c$ and $(F')^\circ$, respectively. Therefore the propositions 2.1, 2.1.i, 2.1.ii and 2.1.iii have consequences concerning the linear spaces $(F)^d$, $(F)^b$, $(F')^c$ and $(F')^\circ$. Some of them, as 2.1.b, 2.1.i.b and 2.1.ii.b in the case of $(F')^c$, are explicitly cited in [1], Theorem 5.1. They are here to have a completeness.

By 2.1, the applicability domain $(F)^d$ of a matrix operator (F) is a B_0K-space under the (homogeneous) pseudonorms (1), (2) and

$$\text{(3')} \qquad p_k^2(t)=\left|\sum_{j=0}^{\infty} f_{kj}\tau_j\right| \quad (k=0, 1, 2, \dots).$$

Since

$$\text{(c)} \qquad p_k^2(t)\le p_k^1(t) \quad (k=0, 1, 2, \dots),$$

the pseudonorms (3') can ce omitted (see [1], Theorem 3.3), i.e. we have

2.1.a. The applicability domain $(F)^d$ of an operator (F) is a B_0K-space under the pseudonorms (1) and (2).

BK-space. In the case of existence of $\lim_i f_{ij}$ $(j=0, 1, 2, \ldots, j_0)$ (or $\lim_i f_{ij} = 0$ $(j=0, 1, 2, \ldots, j_0)$), we have a matrix F such that the convergence domain $(F')^c$ (o-convergence domain $(F')^{\circ}$, respectively) of the method (F') is not a BK-space. Or, equivalently, there is not a B-space isomorphic to some of the considered B_0K-spaces (that is to the space of all sequences under the pseudonorms (1)) (see [2], 1.52).

In the theory of operators, as well as in the theory of summability methods, matrices of the type (*) do not have special importance. In the theory of summability more interesting, for example, are matrices F such that the methods (F') are o-regular[1] (especially, when the methods (F') are regular). Therefore we give examples of o-regular matrices F such that the linear spaces $(F)^d$, $(F)^b$, $(F')^c$ and $(F')^{\circ}$ are not spaces BK. In fact, we prove that to any o-regular (consequently, to any regular) matrix G we can correspond a whole class of matrices F such that each of the linear spaces $(F)^d$, $(F)^b$, $(F')^c$ and $(F')^{\circ}$ is not a space BK. At the same time, this shows that the matrices with such property are numerous in the set of all matrices.

Example. Consider a matrix $G=(g_{ij})$ such that the corresponding method (G') is o-regular. Then define the matrix $F=(f_{ij})$ as follows:

$$f_{ij} = \begin{cases} g_{im} & \text{for } j=2m, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, o-regularity of the method (G') implies o-regularity of the method (F').

By the assertion 2.1.a, the applicability domain $(F)^d$ of the operator (F) is a B_0K-space with the pseudonorms (1) and (2). Suppose that at the same time $(F)^d$ is a BK-space under a (homogeneous) norm $n(t)$. Then (see [1], Theorem 4.5) the norm $n(t)$ is equivalent to the family of pseudonorms (1) and (2), from which follows that the B_0K-space $(F)^d$ and the BK-space $(F)^d$ are isomorphic. Therefore (see [2], 1.52) there exist indices k_1 and k_2 such that the norm $n(t)$ is equivalent to the (homogeneous) pseudonorm (in fact, norm)

$$n_1(t) = \max[p_0(t), p_1(t), \ldots, p_{k_1}(t); \quad p_0^1(t), p_1^1(t), \ldots, p_{k_2}^1(t)].$$

Hence, $t=(\tau_j) \in (F)^d$ and $n_1(t)=0$, i.e.

$$p_k(t)=0 \quad (k=0, 1, \ldots, k_1) \quad \text{and} \quad p_k^1(t)=0 \quad (k=0, 1, \ldots, k_2),$$

imply $t=0$. By the definitions of the pseudonorms $p_k(t)$ and $p_k^1(t)$, this means that $t=(\tau_j) \in (F)^d$,

(d) $$\tau_k = 0 \quad (k=0, 1, \ldots, k_1)$$

and

(e) $$\sum_{j=0}^{l} f_{kj}\tau_j = 0 \quad (k=0, 1, \ldots, k_2;\ l=0, 1, 2, \ldots)$$

imply

(f) $$\tau_j = 0 \quad (j=0, 1, 2, \ldots).$$

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[1] i. e. regular for null-convergent sequences

We shall show that in the case of the considered matrix $F=(f_{ij})$ this implication is not true. Indeed, define the sequence $t=(\tau_j)$ as follows:

$$(**)\qquad \tau_j=\begin{cases} 0 & \text{for } j=0, 1, \ldots, k_1, \\ 0 & \text{for } j=k_1+1, k_1+2, k_1+3, \ldots \text{ and } j=2m, \\ \dfrac{1}{m+1} & \text{for } j=k_1+1, k_1+2, k_1+3, \ldots \text{ and } j=2m+1. \end{cases}$$

Since the method (F') is o-regular and $t=(\tau_j)$ is null-sequence, we have $t==(\tau_j)\in(F')^\circ$. This implies $t=(\tau_j)\in(F)^d$ (in the case of any matrix F the inclusion $(F')^\circ\subseteq(F')^c\subseteq(F)^b\subseteq(F)^d$ is valid).

Obviously, for the just defined matrix $F=(f_{ij})$ and the sequence $t=(\tau_j)$ the conditions (d) and (e) are satisfied. At the same time, the relation (f) is not true. The obtained contradiction shows that in the considered case the linear space $(F)^d$ cannot be a BK-space. The same matrix is an example of a matrix F such that each of the linear spaces $(F)^b$, $(F')^c$ and $(F')^\circ$ cannot be a space BK, too.

Namely, assuming the contrary, by 2.1.b, 1.52[2] and Theorem 4.5 of Zeller [1], in the new case we get to the indices k_1 and k_2 such that $t==(\tau_j)\in(F)^b$, $(F')^c$ or $(F')^\circ$, the conditions (d), (e) and

$$\text{(g)}\qquad \sum_{j=0}^{\infty} f_{ij}\tau_j=0 \quad (i=0, 1, 2, \ldots)$$

imply (f). As we said, the sequence $t=(\tau_j)$ defined by (**) lies in each of the sets $(F)^b$, $(F')^c$ and $(F')^\circ$. The same sequence $t=(\tau_j)$ satisfies (d), (e) and (g), but with the condition (f) this is not the case. This contradiction completes the proof that for the considered matrix F the linear spaces $(F)^d$, $(F)^b$, $(F')^c$ and $(F')^\circ$ cannot be BK-spaces.

Observe that instead of the considered sequence $t=(\tau_j)$ we can deal with the sequence $t=(\tau_j)$ defined in the following way:

$$\tau_j=\begin{cases} 0 & \text{for } j=0, 1, \ldots, k_1, \\ 0 & \text{for } j=k_1+1, k_1+2, k_1+3, \ldots \text{ and } j=2m, \\ a_m & \text{for } j=k_1+1, k_1+2, k_1+3, \ldots \text{ and } j=2m+1, \end{cases}$$

where (a_m) is any null-sequence such that $a_m\neq 0$ for infinitely many indices m.

Also it is clear that in a similar way we can construct many examples of matrices F such that $(F)^d$, $(F)^b$, $(F')^c$ and $(F')^\circ$ cannot be BK-spaces (or, as we said, such that there is not a B-space isomorphic to some of the B_0K-spaces $(F)^d$, $(F)^b$, $(F')^c$ and $(F')^\circ$).

REFERENCES

[1] Zeller, K.: *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. 53(1951), pp. 463—487.

[2] Mazur, S. et Orlicz, W.: *Sur les espaces métriques linéaires* (I), Stud. Math. 10(1948), pp. 184—208.

[3] Banach, S.: *Théorie des opérations linéaires*, Monografje Matematyczne I, New York, 1955.

[4] Zeller, K., Beekmann, W.: *Theorie der Limitierungsverfahren*, Ergebnisse der Maihematik und ihrer Grenzgebiete, Band 15, Berlin · Heidelberg · New York, 1970.

[5] Mazur, S. and Orlicz, W.: *Sur les méthodes linéaires de sommation*, C. R. Acad. Sci. 196(1933), pp. 32—34.

[6] Mazur, S. and Orlicz, W.: *On linear methods of summnbility*, Stud. Math. 14(1954), pp. 129—160.