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EXISTENCE THEOREMS FOR THE SYSTEM $\begin{cases} x = H(x, y) \\ y = K(x, y) \end{cases}$ IN LOCALLY CONVEX SPACES

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In this paper we shall prove existence theorems for the system $x = H(x, y)$ $y = K(x, y)$ where $H: U \times V \to U$, $K: U \times V \to V$, U is a closed subset of locally convex space E and V is a closed convex subset of locally convex space F.

1. Introduction. In the introduction we have included a summary of the basic definitions and theorems to be used in the sequel [5].

Let Λ be an arbitrary set, E be a locally convex space, M be a subset of E and T be a mapping of $\Lambda \times M$ into E. The generalised sequence $\{T_\alpha\}$ is defined in the following way:

$$\begin{cases} T_o = \overline{co}\, T(\Lambda \times M) & \\ T_\alpha = \overline{co}\, T(\Lambda \times (M \cap T_{\alpha-1})) & \alpha - 1 \text{ exists} \\ T_\alpha = \bigcap_{\beta < \alpha} T_\beta & \alpha - 1 \text{ does not exist} \end{cases}$$

where α are ordinal numbers and $\overline{co}$ is the closed convex hull.

Lemma a) *Every set T_α is closed and convex*

b) $T(\Lambda \times (M \cap T_\alpha)) \subseteq T_{\alpha+1}$

c) *If $\eta < \alpha$ then $T_\alpha \subseteq T_\eta$*

d) $T(\Lambda \times (M \cap T_\alpha)) \subseteq T_\alpha$

e) *There exists the ordinal number δ such that $T_\alpha = T_\delta$ for every $\alpha > \delta$*

Definition 1 *$T_\delta = T^\infty(\Lambda \times M)$ is the limit domain of values of the mapping T on the set $\Lambda \times M$. The mapping T is a limiting compact mapping if the set $\overline{T(\Lambda \times (M \cap T^\infty(\Lambda \times M)))}$ is compact*

Definition 2 *Let $\mathfrak{M}$ be a subset of 2^E and $Q \in \mathfrak{M}$ implies $\overline{co}\, Q \in \mathfrak{M}$. Further, let $(A, \leqslant)$ be a partially ordered set. The measure of noncompactness ψ is a function $\psi: \mathfrak{M} \to A$ such that $\psi(\overline{co}\, Q) = \psi(Q)$.*

D e f i n i t i o n 3. *The mapping T is ψ-densifying if the implication $\{\psi[T(\Lambda\times Q)]\geqslant\psi(Q)\}\Rightarrow\{\overline{Q}$ is compact$\}$ holds.*

T h e o r e m A. *Suppose M is a closed subset of E, Λ is a compact topological space, T is a continuous ψ-densifying mapping and the measure ψ is monotone i.e. $Q_1\subseteq Q_2$ implies that $\psi(Q_1)\leqslant\psi(Q_2)$. Then T is a limiting compact mapping on $\Lambda\times M$.*

Examples of the measure of noncompactness.

1. *Kuratowski's measure of noncompactness.*

Suppose E is a uniform space, P is a family of uniform continuous pseudometrics on $E\times E$, $\mathfrak{M}$ the family of all sets in E which are bounded in respect to all $p\in P$, A a partially ordered set of functions $a:P\to[0,\infty)$ where $\leqslant$ is defined as follows $a_1\leqslant a_2\Leftrightarrow\forall(p\in P)[a_1(p)\leqslant a_2(p)]$. Kuratowski's measure of noncompactness is the function $[d(Q)](p)=\inf\{\varepsilon>0$, there exists a finite number of sets $S_1, S_2,\ldots, S_n$ such that $Q=\bigcup_{i=1}^{n}S_i$ and $d'(S_i)(p)\leqslant\varepsilon\ i=1,2,\ldots,n\}$ (d' is diameter of a set).

2. *Hausdorff's measure of noncompactness.*
$\eta_R:\mathfrak{M}\to A$ is the function $[\eta_R(Q)](p)=\inf\{\varepsilon>0$, Q has in R a finit ε-net in respect to the preudometric $p\}$, where R is a subset in E.

Suppose S is a closed, convex set, U is an open set, $U\cap S\neq\varnothing$, $U_S=U\cap S$, $\overline{U}_S$ and $\dot{U}_S$ are the closure and boundary of U_S in the induced topology. Further, suppose that T is a completely continuous mapping of U_S into S and $Tx\neq x$ for every $x\in\dot{U}_S$. Under these conditions one can define the function $\gamma_o(I-T, U_S)$ which plays an important role in the fixed point theory.

Let T be a limiting compact mapping, $S=T^{\infty}(U_R)$ and R be a closed, convex subset of E. Then there exists $\gamma_o(I-T, U_S)$ and $\gamma(I-T, U_S)$ is by definition $\gamma_o(I-T, U_S)$. The function has two important properties:

1. If $I-T_1\sim I-T_2$ on $\overline{U}_R$ in respect to R then $\gamma(I-T_1, U_R)=\gamma(I-T_2, U_R)$ where $\sim$ is the relation of homotopy
2. If $\gamma(I-T, U_R)\neq 0$, then there exists at least one element $x\in U_R$ such that $Tx=x$

2. *Fixed point theorems.*

The following theorem is a generalization of theorem 1 in [2].

T h e o r e m 1 *Let E be a locally convex space sequentially complete p_i, $i\in\mathcal{J}$ be a saturated family of seminorms defining the topology of E, f be a mapping of $\mathcal{J}$ into $\mathcal{J}$, M be a closed subset in E and T be a mapping of M into M satisfying the following conditions:*

1. *For every $i\in\mathcal{J}$ there exists $q(i)\geqslant 0$ such that for every $x, y\in M$: $:p_i(Tx-Ty)\leqslant q(i)p_{f(i)}(x-y)$*

2. *There exists $x_o\in M$ such that for every $i\in\mathcal{J}$ the series* $\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-2}q[f^k(i)]\right)p_{f^{n-1}(i)}(Tx_0-x_0)=S(i)$ is convergent, $q[f^{-1}(i)]=1$, $q[f^0(i)]=q(i)$, $f^n(i)=f[f^{n-1}(i)]$.

Then there exists one and only one solution of the equation $x=Tx$ which satisfies the condition:

$\lim_{n\to\infty}\left(\prod_{k=0}^{n-2} q[f^k(i)]\right) p_{f^{n-1}(i)}(x-x_0)=0$ *for every $i\in\mathcal{J}$ and the inequality*

$$p_{f^k(i)}(x-x_0)\leq\frac{S(i)-S_k(i)}{\prod_{r=0}^{k-1} q[f^r(i)]} \tag{1}$$

for every $i\in\mathcal{J}$ and $k=0, 1, \ldots.$ where $S_k(i)$ is the partial sum of the series $S(i)$.

P r o o f: We shall construct the sequence $\{x_n\}\subset M$ in such a way that $x_n=Tx_{n-1}$ $n=1, 2, \ldots.$ Then we have:

$$p_i(x_2-x_1)\leq q(i)\,p_{f(i)}(x_1-x_0)$$
$$p_i(x_3-x_2)\leq q(i)\,q[f(i)]\,p_{f^2(i)}(x_1-x_0)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$p_i(x_{n+1}-x_n)\leq\left(\prod_{r=0}^{n-1} q[f^r(i)]\right)p_{f^{n+1}(i)}(x_1-x_0).$$

From this it follows that:

$$\sum_{r=1}^{n} p_i(d_r)\leq p_i(Tx_0-x_0)+q(i)\,p_{f(i)}(Tx_0-x_0)+\cdots\cdots+\left(\prod_{r=0}^{n-2} q[f^r(i)]\right)p_{f^{n-1}(i)}(Tx_0-x_0)$$

where $d_n=x_n-x_{n-1}$. Since the series $\sum_{n=1}^{\infty}\left(\prod_{r=0}^{n-2} q[f^r(i)]\right)p_{f^{n-1}(i)}(Tx_0-x_0)$ is convergent and $x_n=x_0+\sum_{r=1}^{n} d_r$, we conclude that there exists $x=\lim_{n\to\infty} T^n x_0$.

Further, we shall prove that:

$$\lim_{n\to\infty}\left(\prod_{k=0}^{n-2} q[f^k(i)]\right)p_{f^{n-1}(i)}(x-x_0)=0.$$

For every $k, n\in N$ we have:

$$p_{f^k(i)}(x_n-x_0)\leq p_{f^k(i)}(x_n-x_{n-1})+p_{f^k(i)}(x_{n-1}-x_{n-2})+\cdots+$$
$$+p_{f^k(i)}(x_1-x_0)\leq\left(\prod_{r=0}^{n-2} q[f^r(f^k(i))]\right)p_{f^{n+k-1}(i)}(Tx_0-x_0)+\cdots+$$
$$+q[f^k(i)]\,p_{f^{k+1}(i)}(Tx_0-x_0)+p_{f^k(i)}(Tx_0-x_0)=\prod_{r=0}^{n-2} q[f^{r+k}(i)]\times$$
$$\times p_{f^{n+k-1}(i)}(Tx_0-x_0)+\cdots\cdots+p_{f^k(i)}(Tx_0-x_0).$$

Since $S(i)-S_k(i)=\left(\prod_{r=0}^{k-1} q[f^r(i)]\right)p_{f^k(i)}(Tx_0-x_0)+$

$$+\left(\prod_{r=0}^{k} q[f^r(i)]\right)p_{f^{k+1}(i)}(Tx_0-x_0)+\cdots\cdots=\prod_{r=0}^{k-1} q[f^r(i)]\times$$

$$\times [p_{f^k(i)}(Tx_0 - x_0) + q[f^k(i)] p_{f^{k+1}(i)}(Tx_0 - x_0) + \cdots\cdots] =$$

$$= \left(\prod_{r=0}^{k-1} q[f^r(i)]\right) A_k(i)$$

we have $p_{f^k(i)}(x_n - x_0) \leqslant A_k(i)$. When $n \to \infty$ from this we obtain:

$$p_{f^k(i)}(x - x_0) \leqslant \frac{S(i) - S_k(i)}{\prod_{r=0}^{k-1} q[f^r(i)]}$$

and if $k \to \infty$ it follows $\lim_{n \to \infty} \left(\prod_{r=0}^{n-1} q[f^r(i)]\right) p_{f^n(i)}(x - x_0) = 0$.

Finally, we shall prove the uniqueness of the solution in M which also satisfies condition (1). Let on the contrary, x and y be two solutions of the equation $Tx = x$ then:

$$p_i(x - y) = p_i(Tx - Ty) \leqslant q(i)\, p_{f(i)}(x - y) \leqslant$$

$$\leqslant \left(\prod_{r=0}^{n} q[f^r(i)]\right) p_{f^{n+1}(i)}(x - y) \leqslant$$

$$\leqslant \left(\prod_{r=0}^{n} q[f^r(i)]\right) [p_{f^{n+1}(i)}(x - x_0) + p_{f^{n+1}(i)}(y - x_0)]$$

and if $n \to \infty$ we obtain $p_i(x - y) = 0$ for every $i \in \mathcal{J}$. Consequently $x = y$.

C o r o l l a r y 1 [2]. *We suppose*:

1. *For every* $i \in \mathcal{J}$ *there exists* $q(i) \geqslant 0$ *such that*:

   $p_i(Tx - Ty) \leqslant q(i)\, p_{f(i)}(x - y)$ *for every* $x, y \in M$

2. *For every* $i \in \mathcal{J}$ *there exists* $n(i) \in N$ *such that for every* $n \geqslant n(i)$ $q[f^n(i)] \leqslant q(i) < 1$

3. *There exists* $x_0 \in M$ *such that* $p_{f^n(i)}(x_0 - Tx_0) \leqslant m(i) < \infty$ *for every* $i \in \mathcal{J}$ *and* $n \geqslant 0$.

Then there exists one and only one solution of the equation $x = Tx$ *which also satisfies the condition*:

4. $p_{f^n(i)}(x - x_0) \leqslant p(i, x) < \infty$. $n \geqslant 0$.

P r o o f: Since

$$\sum_{n=0}^{\infty} \left(\prod_{k=0}^{n-1} q[f^k(i)]\right) p_{f^n(i)}(Tx_0 - x_0) \leqslant \sum_{n=0}^{\infty} \left(\prod_{k=0}^{n-1} q[f^k(i)]\right) m(i)$$

we shall apply D'Alambert's criterion on the series $\sum_{n=0}^{\infty} a_n(i)$ where $a_n(i) = \prod_{k=0}^{n-1} q[f^k(i)]$. Then we obtain:

$$\frac{a_{n+1}(i)}{a_n(i)} = \frac{\prod_{k=0}^{n} q[f^n(i)]}{\prod_{k=0}^{n-1} q[f^k(i)]} = q[f^n(i)] \leqslant q(i) < 1$$

and the proof is complete.

Theorem 2. *Let G be a closed and convex subset of the topological, Hausdorff locally convex, complete space E and S, T two mappings of G into E satisfying the following conditions:*

1. *For every $x, y \in G$, $Tx + Sy \in G$*

2. a) *For every $i \in \mathcal{J}$ there exists $q(i) \geqslant 0$ such that $p_i(Tx - Ty) \leqslant q(i) p_{f(i)}(x-y)$ for every $x, y \in G$*

b) *For every $i \in \mathcal{J}$ and $n \in N$ there exist $a_n(i) \geqslant 0$ and $g(i) \in \mathcal{J}$ such that for every $x \in E$, $n \geqslant N$ the inequality $p_{f^n(i)}(x) \leqslant a_n(i) p_{g(i)}(x)$ holds*

c) *The series:*

$$\sum_{n=1}^{\infty} \left(\prod_{k=0}^{n-2} q[f^k(i)] \right) a_{n-1}(i)$$

is convergent

3. *The mapping S is continuous and SF is relatively compact set.*
Then there exists at least one point $x_0 \in G$ such that $Sx_0 + Tx_0 = x_0$.

Proof: Let us consider the mapping $y \to Ty + Sx$ where x is a fixed element of G. Since the mapping T satisfies the condition 2. the mapping $y \to Ty + Sx$ satisfies all the conditions of the theorem 1. and so there exists $Rx \in G$ such that $Rx = TRx + Sx$. The uniqueness of the fixed point of the mapping $y \to Ty + Sx$ follows by conditions 2. b) and 2. c). Using the inequality:

$$p_i(Rx - Rx_0) \leqslant p_{g(i)}(Sx - Sx_0) \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-2} q[f^k(i)] \right) a_{n-1}(i)$$

we can prove that the mapping R is continuous and the set RG is compact. From this fact and Tihonov's fixed point theorem we conclude [2] that there exists $z \in G$ such that $Rz = z$ i.e. $z = Tz + Sz$

Theorem 3. *Let E be a locally convex space sequential complete, M be a subset closed in E, Λ be a topological space and Φ be a mapping of $M \times \Lambda$ into M. Further, suppose that $\Phi_x : \lambda \to \Phi(x, \lambda)$ is continuous in $\lambda \in \Lambda$ for each $x \in M$ and that $\Phi_\lambda : x \to \Phi(x, \lambda)$ satisfies the following conditions*

1. *For every $i \in \mathcal{J}$ and $\lambda \in \Lambda$ there exist $f_\lambda : \mathcal{J} \to \mathcal{J}$ and $q_\lambda(i) \geqslant 0$ such that:*

$$p_i(\Phi_\lambda x - \Phi_\lambda y) \leqslant q_\lambda(i) p_{f_\lambda(i)}(x-y) \text{ for every } x, y \in M$$

2. *For every* $i \in \mathcal{J}$ *and* $n \in N$ *there exist* $a_n(i) \geqslant 0$, $Q_n(i) \geqslant 0$ *and* $g(i) \in \mathcal{J}$ *such that*:

a) $p_{f^n_\lambda(i)}(x) \leqslant a_n(i)\, p_{g(i)}(x)$ *for every* $\lambda \in \Lambda$ *and* $x \in M$

b) $q[f^n_\lambda(i)] \leqslant Q_n(i)$ *for every* $\lambda \in \Lambda$

c) *The series*:

$$\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-2} Q_k(i)\right) a_{n-1}(i) \text{ is convergent.}$$

Then the solution $x(\lambda)$ *of the equation* $x(\lambda)=\Phi[x(\lambda),\lambda]$ *is continuous in* $\lambda \in \Lambda$.

Proof: Condition a) implies that there exists a unique element $x(\lambda) \in M$ such that $x(\lambda)=\Phi[x(\lambda),\lambda]$, $\lambda \in \Lambda$ and $x(\lambda)$ can be obtained as the limit of the sequence $\{x_{n,\lambda}\}$, $x_{n,\lambda}=\Phi[x_{n-1,\lambda},\lambda]$. Furthermore, due to the condition c) of the theorem, it does not matter which is the first element $x_{0,\lambda} \in M$. If we apply theorem 1 taking for T the mapping Φ_λ we obtain from (1), when $k=0$, the inequality:

$$p_i(x-x_0)=p_i(x(\lambda)-x(\lambda_0)) \leqslant p_{g(i)}(\Phi_\lambda[x(\lambda_0)]-\Phi_{\lambda_0}[x(\lambda_0)] \times$$

$$\times \sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-2} q[f^k_\lambda(i)]\right) a_{n-1}(i) \leqslant p_{g(i)}(\Phi_\lambda[x(\lambda_0)]-\Phi_{\lambda_0}[x(\lambda_0)]) \times$$

$$\times \sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-2} Q_k(i)\right) a_{n-1}(i).$$

The mapping $\lambda \to \Phi(x,\lambda)$ is continuous so there exists a neighbourhood $V(\lambda_0) \subset \Lambda$ such that:

$$p_{g(i)}(\Phi[x(\lambda_0),\lambda]-\Phi[x(\lambda_0),\lambda_0]) \leqslant \varepsilon\left\{\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-2} Q_k(i)\right) a_{n-1}(i)\right\}^{-1}$$

and therefore $p_i(x(\lambda)-x(\lambda_0)) \leqslant \varepsilon$ for every $\lambda \in V(\lambda_0)$.

3. *Existence theorems for the system* $\begin{cases} x=H(x,y) \\ y=K(x,y) \end{cases}$

Theorem 4 *Suppose E is a locally convex space sequentially complete, F is a locally convex space, U is a closed subset of E, V is a convex, closed subset of F, H is a mapping of* $U \times V$ *into U and K is a mapping of* $U \times V$ *into V. Further, suppose that the following conditions are satisfied*:

1. *The mapping* $y \to H(x,y)$ *is continuous in* $y \in V$
2. *For every* $i \in \mathcal{J}$ *there exist* $q(i) \geqslant 0$ *and* $f: \mathcal{J} \to \mathcal{J}$ *such that*: $p_i(H(x_1,y)-$ $-H(x_2,y)) \leqslant q(i)\, p_{f(i)}(x_1-x_2)$ *for every* $x_1, x_2 \in U$, $y \in V$
3. *For every* $i \in \mathcal{J}$ *and* $n \in N$ *there exist* $a_n(i) \geqslant 0$ *and* $g(i) \in \mathcal{J}$ *such that*: $p_{f^n(i)}(x) \leqslant a_n(i)\, p_{g(i)}(x)$ *for every* $x \in E$, $n \in N$ *and the series*

$$\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-2} q[f^k(i)]\right) a_{n-1}(i) \text{ is convergent.}$$

4. *One of the following conditions is satisfied:*

4.1 *F is a semireflexive space, V is a bounded subset of F, K is a continuous, limiting compact mapping*

4.2 *On the set F is defined the measure of noncompactness ψ and K is a continuous, ψ-densifying mapping. Also the mapping ψ is monotone and has one of the following properties*

a) $\forall (x_0 \in V, Q \subseteq V, Q \neq \varnothing) [\psi(\{x_0\} \cup Q) = \psi(Q)]$

b) $(x_0 \in V, Q_1 \subseteq V, Q_2 \subseteq V) [\psi(x_0 + Q_1) = \psi(Q_1)$ *and* $\psi(Q_1 \cup Q_2) =$

$= max\, \{\psi(Q_1), \psi(Q_2)\}]$.

Then there exists at least one element $z \in U \times V$ such that

$$z = (H(z), K(z)).$$

Proof: At first, we shall show that there exists a mapping $R: V \to V$ such that $R(y) = H(R(y), y)$ for every $y \in V$. If we apply Theorem 3 taking for Λ the topological space V (in the induced topology), for M the subset U and for $\Phi(x, \lambda)$ the mapping H we see that in this case $q_\lambda(i) = q(i)$ and $f_\lambda(i) = f(i)$. Since the mapping H satisfies the conditions 1., 2. and 3, it follows that all the conditions of Theorem 3 are satisfied and $x(\lambda) = Ry$. Let us now assume that the condition 4.1 holds. We define the mapping T of V into V by setting $Ty = K(R(y), y)$. It is evident that T is a continuous mapping of V into V. We shall prove that T is a limiting compact mapping showing that the set $T^\infty(V)$ is contained in the set $K^\infty(U \times V)$.

Let $\{T_\alpha^1\}$ and $\{T_\alpha\}$ be generalized sequences of sets which correspond to the mappings T and K respectively, namely:

$$\begin{cases} T_0^1 = \overline{co}\, T(V) & \\ T_\alpha^1 = \overline{co}\, T(T_{\alpha-1}^1) & \alpha - 1 \text{ exists} \\ T_\alpha^1 = \bigcap\limits_{\beta<\alpha} T_\beta^1 & \alpha - 1 \text{ does not exist} \end{cases}$$

$$\begin{cases} T_0 = \overline{co}\, K(U \times V) & \\ T_\alpha = \overline{co}\, K(U \times T_{\alpha-1}) & \alpha - 1 \text{ exists} \\ T_\alpha = \bigcap\limits_{\beta<\alpha} T_\beta & \alpha - 1 \text{ does not exist} \end{cases}$$

Using the transfinite induction it can be shown that $T_\alpha^1 \subseteq T_\alpha$ for every α. For $\alpha = 0$ we have $T_0^1 = \overline{co}\, T(V) = \overline{co}\, \{K(R(y), y) \mid y \in V\} \subseteq \overline{co}\, K(U \times V) = T_0$. Suppose that $T_\alpha^1 \subseteq T_\alpha$ for every $\alpha < \alpha_0$. We distinguish two cases:

1. $\alpha_0 - 1$ exists

In this case $T_{\alpha_0}^1 = \overline{co}\, T(T_{\alpha_0-1}^1) = \overline{co}\, \{K(R(y), y) \mid y \in T_{\alpha_0-1}^1\} \subseteq$

$\subseteq \overline{co}\, \{K(R(y), y) \mid y \in T_{\alpha_0-1}\} \subseteq \overline{co}\, K(U \times T_{\alpha_0-1}) = T_{\alpha_0}$

2. $\alpha_0 - 1$ does not exist

Then we have $T^1_{\alpha_0} = \bigcap_{\beta<\alpha_0} T^1_\beta \subseteq \bigcap_{\beta<\alpha_0} T_\beta = T_{\alpha_0}$

The set $T^\infty(V)$ is by definition T^1_δ where δ is that ordinal number for which $T^1_\delta = T^1_\alpha$ for every $\alpha > \delta$. Suppose $K^\infty(U \times V) = T_{\delta'}, \delta' \geqslant \delta$. It follows that $T^1_\delta = T^1_{\delta'} \subseteq T_{\delta'}$ i.e. $T^\infty(V) \subseteq K^\infty(U \times V)$. If we have $\delta \geqslant \delta'$ it follows by Lemma that $T^1_\delta \subseteq T^1_{\delta'} \subseteq T_{\delta'}$ so $T^\infty(V) \subseteq K^\infty(U \times V)$. Since $\overline{T(T^\infty(V))} = \overline{\{K(R(y), y) \mid y \in T^\infty(V)\}} \subseteq \overline{K(U \times K^\infty(U \times V))}$ and the set $\overline{K(U \times K^\infty(U \times V))}$ is compact, we conclude that the set $\overline{T(T^\infty(V))}$ is also compact i.e. T is a limiting compact operator.

Further, in [5], it was shown that if the space F is semireflexive and V is a bounded subset of T then $T^1_\alpha \neq \varnothing$ for every α, so $T^1_\delta = T^\infty(V) \neq \varnothing$. By Theorem 3.4.1 [5] if $U = F$ $\gamma(I-T, V) = \gamma(I-T, U_V) = 1$ and there exists at least one element $y_0 \in V$ such that $y_0 = Ty_0$. If we take for z element $(R(y_0), y_0)$ it follows that $z = (H(z), K(z))$ i.e. $R(y_0) = H(R(y_0)\ y_0 = K(R(y_0)\ y_0)$.

Suppose now that the condition 4.2 is satisfied. To show that T is ψ-densifying mapping, it is necessary to show that $\psi(T(Q)) \geqslant \psi(Q)$ implies compactness of the set $\bar{Q}$. Since $T(Q) = \{K(R(y), y) \mid y \in Q\} \subseteq K(U \times Q)$ and the mapping ψ is monoton we have $\psi(T(Q)) \leqslant \psi(K(U \times Q))$ and $\psi(K(U \times V)) \geqslant \psi(Q)$. From the fact that K is a ψ-densifying mapping, it follows that the set $\bar{Q}$ is compact i.e. T is a ψ-densifying mapping. Then there exists a compact set K such that $T(K) \subseteq K$ [5]. In fact this set $\overline{K_1}$, $K_1 = \{T^n x_0, n = 0, 1, \ldots\ldots\}$. This implies that $T^\infty(V) \neq \varnothing$ [5] and we obtain $z = (R(y_0), y_0)$ $y_0 = Tx_0$. This completes the proof.

The following theorem is a generalization of Theorem 4.

Theorem 5. *Suppose that the conditions* 1. *and* 4. *of Theorem* 4. *are satisfied. Suppose further that for every* $i \in \mathcal{J}$ *and* $k \in N$ *there exist* $q_k(i) \geqslant 0$ *and* $f: \mathcal{J} \to \mathcal{J}$ *such that*:

$$p_i(H^k_y(x_1) - H^k_y(x_2)) \leqslant q_k(i)\, p_{f(i)}(x_1 - x_2)$$

for every $x_1, x_2 \in U$, $y \in V$ *and the series* $\sum_k q_k(i)$ *it convergent, where* $H_y(x)$: $: x \to H(x, y)$.

Then there exists at least one element $z \in U \times V$ *such that*

$$z = (H(z),\ K(z)).$$

Proof: If we apply Theorem 2 in [3] we see that there exists the mapping $R: V \to V$ such that $Ry = H(R(y), y)$. As in the proof of Theorem 4 one can show that the mapping $Ty = K(R(y), y)$ has a fixed point $y_0 \in V$ and $z = (R(y_0), y_0)$.

Remark: If $H(x, y) = Ax + By$, A is a linear mapping which satisfies the condition:

(2) $$p_i(A^k x - A^k y) \leqslant q_k(i)\, p_{f(i)}(x - y)$$

for every $k=1, 2, \ldots$ and $\sum_{k=1}^{\infty} q_k(i)$, then we have $H_y^k(x)=A^k(x)+$
$+\sum_{r=0}^{k-1} A^r B(y)$ and $H_y^k(x_1)-H_y^k(x_2)=A^k(x_1)-A^k(x_2)$.

In [2] is given an example of a mapping (in the field of Mikusiński's operators) which satisfies the condition (2).

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