PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 16 (30), 1973 pp. 61—64

A GENERAL SPATIAL STRUCTURE ON CLASSES

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(Communicated January 20, 1973)

This note contains a recapitulation of so far involved spatial structures on classes and specification of a general form of these structures. In formation of spatial wholes on a level in $\mathcal{U}$ we started from a class of mathematical objects provided with a class of admissible rules. We have assumed that such a class has the structure of a fundamental semigroupoid. This structure is actually a basic structure on a class and can be involved on almost each class of objects on a level in $\mathcal{U}$. Because of that we have assumed it as our starting stage.

Our goal has been to organize such a basic world into a harmonic and fruitful spatial whole, a whole with precisely established internal relationships which will be capable of certain creative activities in itself. As we know, there are various ways to do it. We have regarded some of these ways. At first we regarded topological organization. This organization on a fundamental semigroupoid can be represented as a system

$$\langle p_{i+1}(t_{i+1});\ {}_c\mathfrak{F}_{i+1},\ {}_{cc}\mathfrak{F}_{i+1},\ \mathbf{O}_{i+1}\rangle,$$

where ${}_c\mathfrak{F}_{i+1}$ denotes *fc* and ${}_{cc}\mathfrak{F}_{i+1}$ *lcc* formation being defined on each subclass of $p_{i+1}(t_{i+1})$. $\mathbf{O}_{i+1}$ is an interior operator. This operator restricts the action of the formation ${}_{cc}\mathfrak{F}_{i+1}$ allowing so the inclusion of a class of objects for completing purposes. Hence we can write it as the system

$$\langle p_{i+1}(t_{i+1});\ {}_c\mathfrak{F}_{i+1},\ {}_{cc}\overset{*}{\mathfrak{F}}_{i+1}\rangle,$$

where ${}_{cc}\overset{*}{\mathfrak{F}}_{i+1}$ denotes the restricted *lcc* formation. According to [3] it is to be done on $< c_\beta$-subclasses of $p_{i+1}(t_{i+1})$.

Each of the above formations has a constructive character on the class of objects of $p_{i+1}(t_{i+1})$. So, the formation ${}_c\mathfrak{F}_{i+1}$ creates in $p_{i+1}(t_{i+1})$ from a subclass of $p_{i+1}(t_{i+1})$ an object called a sequent. This object is a constructive end of this subclass in $p_{i+1}(t_{i+1})$.

Furthermore we regarded pretopological organisations. Such an organization on a fundamental semigroupoid $p_{i+1}(t_{i+1})$ can be represented as the system

$$\langle p_{i+1}(t_{i+1});\ {}_c\mathfrak{F}_{i+1}\rangle.$$

List of used concepts: class, universe [1]; admissible rule, funhom = fundamental homomorphism, *fc* and *lcc* formation, filter [2]; topology, interior operator, continuous rule [3]; pretopology, intuitionistic topology [4].

This system is certainly only a weaker form of the above system. Hence we have that a pretopological organization is weaker than topological one.

Let us see now what does an intuitionistic topology on the regarded fundamental semigroupoid $p_{i+1}(t_{i+1})$. It gives a system like topological one only that the formations ${}_c\mathfrak{F}_{i+1}$ and ${}_{cc}\mathfrak{F}_{i+1}$ are now concerned not with all but with particularly chosen subclasses of $p_{i+1}(t_{i+1})$. This organization we represent as the following system:

$$\langle p_{i+1}(t_{i+1});\ {}_c\mathfrak{F}^{\bullet}_{i+1},\ {}_{cc}\mathfrak{F}^{\bullet}_{i+1}\rangle,$$

where the sing $\bullet$ means that the formations ${}_c\mathfrak{F}_{i+1}$ and ${}_{cc}\mathfrak{F}_{i+1}$ are concerned only with certain particularly chosen subclasses of the fundamental semigroupoid $p_{i+1}(t_{i+1})$ making constructively closed wholes from them.

We can now generalize the above considerations. We see that all regarded organizations are to be represented as systems with two types of operations, those which form on a subclass of the regarded fundamantal semigroupoid its end and those which form its coend. We called them in [4] inductive and coinductive operations, respectively. If we denote the class of operations of the first type by $\mathbf{Ind}_{i+1}$ and of the second by $\mathbf{Coind}_{i+1}$, where $i+1$ denotes the class level of their elements then we would have that a general spatial organization could be represented as the following system:

$$\langle p_{i+1}(t_{i+1});\ \mathbf{Ind}_{i+1},\ \mathbf{Coind}_{i+1}\rangle,$$

where $p_{i+1}(t_{i+1})$ means a fundamental semigroupoid, i.e. a system $\langle p_{i+1}(t_{i+1});$ $\mathcal{D}_0, \mathcal{D}_1, \mathcal{C}\rangle$ consisting of a class $p_{i+1}(t_{i+1})$ and operations $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{C}$, see [2]. We have identified notations for the fundamental semigroupoid and its underlying class.

Domains of actions of operations entering into $\mathbf{Ind}_{i+1}$ and $\mathbf{Coind}_{i+1}$ are various and determined according to certain criterions. We discussed them in [4]. According to that discussion we can say generally that choices of domains of these operations depend on some fundamental semigroupoids and to be relative to objects of certain fundamental semigroupoids. Moreover we saw that we could utilize one type of operations for the purpose of choices and the second in constructive senses. If we summarize all this then we can say that a general spatial organization on the $(i+1)$th level in $\mathcal{U}$ can be represented as the system

$$\text{(GSS)} \qquad (\langle p_{i+1}(t_{i+1});\ \mathbf{Ind}_{i+1},\ \mathbf{Coind}_{i+1}\rangle,\ \mathbf{Ch}_{i+1}).$$

where $p_{i+1}(t_{i+1})$ is a fundamental semigroupoid, $\mathbf{Ind}_{i+1}$ and $\mathbf{Coind}_{i+1}$ are classes of inductive and coinductive operations, respectively and $\mathbf{Ch}_{i+1}$ is a class of choices—choice funhoms upon which actions of operations of the above classes depend. As we have already said, elements of the class $\mathbf{Ch}_{i+1}$ may depend upon certain fundamental semigroupoids and funhoms from them to the semigroupoid $p_{i+1}(t_{i+1})$ and may be relative to objects of some fundamental semigroupoids those which allow taking part in such an organization—formation. We have further that one type of the operations from $\mathbf{Ind}_{i+1}$ of $\mathbf{Coind}_{i+1}$ may be utilized for the choice purposes and the second for constructions. To mention that all this considerations are regarded in an $(i+2)$-class containing these semigroupoids and funhoms. We allow further the existence of connecting rules between objects of $\mathbf{Ch}_{i+1}$. Moreover there may exist certain structures on the $\mathbf{Ch}_{i+1}$ equipped with the class of rules. The existence of rules and structures on $\mathbf{Ch}_{i+1}$ will imply the existence of the same things on $\mathbf{Ind}_{i+1}$ and $\mathbf{Coind}_{i+1}$. We mention finally that choice funhoms may carry certain structures.

We may assume now that operations in the system (GSS) have more general forms. We regarded so far that elements of $\mathbf{Ind}_{i+1}$ and $\mathbf{Coind}_{i+1}$ are those ones forming from particularly chosen subclasses of p_{i+1} (t_{i+1}) objects called constructive ends and coends, respectively. We may assume, however, that the role of these objects take over certain subclasses of p_{i+1} (t_{i+1}). It means then that chosen subclasses are not constructively closed by objects but by certain subclasses of p_{i+1} (t_{i+1}). Instances of such operations are cylinder and cocylinder operations. To comprise all these cases we may assume generally that the classes $\mathbf{Ind}_{i+1}$ and $\mathbf{Coind}_{i+1}$ contain in themselves all creations being performable in the regarded fnndamental semigroupoid. In that case the system (GSS) will be the represent of a general mathematical activity on a class being equipped with the fundamental structure. Of course, dependences of activities are as in the above case, i.e. in the case of $\mathbf{Ch}_{i+1}$. We shall deal with this system and various creations within it more detailly in the next paper.

We mention now some examples of the system (GSS) given in its more general form. We have already regarded the systems representing topological and intuitionistic organization. We have further various logical systems. These systems define only special kinds of this system. There are also many other examples as for instance Lawvere's toposes, sheaves, Grothendieck topologies, standard constructions, motions, and so on. We illustrate further some examples of activity on p_{i+1} (t_{i+1}). For instance by combining choices and certain operations of (GSS) we can get in p_{i+1} (t_{i+1}) some objects that we shall call class-objects. They are for instance simplicial ones. To form them, it is enough to start from two basic cells $\mathcal{H}_{i+1}: I_{i+1} \to F_{i+1}$ and $\mathcal{H}'_{i+1}: F^2_{i+1} \to F_{i+1}$, obtained by means of the identity funhom I_{i+1} and the funhom F_{i+1} of p_{i+1} (t_{i+1}) to itself, and then iteratively form more complex cells from these ones. Likewise we can start from some other cells or simplices. In such a way we shall obtain various kinds of many-valued funhoms from p_{i+1} (t_{i+1}) to itself. Instances of such funhoms are the funhoms $\mathfrak{F}_{i+1}$ and $\mathfrak{H}_{i+1}$ given in [2]. To mention that all these many-valued funhoms we may utilize as choice funhoms in certain further constructions. In that way a part of activity on p_{i+1} (t_{i+1}) we displace into the class $\mathbf{Ch}_{i+1}$. We may take this stipulation in general. Thus a part of activities on a fundamental semigroupoid is to be utilized in choices, i. e. it may be contained in $\mathbf{Ch}_{i+1}$.

We believe that if not all then much of mathematics can be developed within the system (GSS). This system is certainly a wide mathematical system. However, it is inconvenient for a general study because there may exist many different combinations and possibilities within it. On account of that it is better to specify some particular systems and deal with them. We do not know which such system is the best one. This is of course a matter of conviction. However, in the next papers we shall attempt to involve some systematization into the system (GSS) and so to answer, in some measure, this question.

Now we can regard a class of the structures (GSS) defined on various underlying fundamental semigroupoids. As we know, this class will be capable of studying if it is provided with a class of admissible rules. These rules are obviously funhoms which preserve some intrinsic properties of the regarded structures. It is obvious which properties characterize the structure (GSS). They are certainly operations specified according to certain choices. So, in the case of topology we can assume that chosen subclasses are filters, see [3]. As we know the essence of this structure consists in completing certain filters by inserting objects of a (discrete) class. We say then that a filter converges if it possesses d-limit in the fundamental semigroupoid. Continuous rules being admissible for classes of these structures preserve this property. In the case of intuitionistic topology we choose subclasses with respect to certain objects and claim that they allow formations of closed wholes from themselves. Admissible rules for a class of these structures have to preserve this property. A class of the structures (GSS) together with the class of admissible rules is a fundamental semigroupoid. Having defined this fundamental semigroupoid we can also impose upon it a structure of the type (GSS). Hence we can say that the general structure (GSS) exists on each level in $\mathcal{U}$.

Let us regard now a class of general spatial structures $(\mathrm{GSS})_i$, $i \in \mathcal{J}$ of different levels in $\mathcal{U}$. The question arises; in which way we can connect the structures of various levels. According to the axiom $A\,7$ of [1] we have that any system (GSS) on a level i is an element of a class on the next $(i+1)$th level. It means that there is a rule

$$U_{i(i+1)}: \mathcal{U}_i \to \mathcal{U}_{i+1}$$

for which there exists a rule $\mathcal{E}_{i(i+1)}: \mathcal{U}_i \to \mathcal{U}_{i+1}$ such that $\mathcal{E}_{i(i+1)}(t_{i-1}) \in U_{i(i+1)}(t_{i-1})$, for every $t_{i-1} \in \mathcal{U}_i$. Certainly, each class $U_{i(i+1)}(t_{i-1})$ being an element of $\mathcal{U}_{i+1}$ if provided with a class of admissible rules allows some structure (GSS). Moreover, $\mathcal{U}_{i+1}$ itself allows some general spatial structure in which $\mathcal{U}_i$ will play the role of the object 1_i, i.e. of the constructive end. Otherwise, the structure which it already possesses on the level i will be a choice structure of the (GSS) on the level $i+1$. For instance in the case of topological organization its elements are included as d-limit objects of filters of $\mathcal{U}_{i+1}$. The new structure of $\mathcal{U}_i$ we shall call a general spatial hyperstructure and denote by (GSHS). Topological hyperstructures are instances of the system (GSHS).

We shall not study further here the systems (GSS) and (GSHS). However, we shall return to them later to lighten them a little more. Then we shall also discuss the question of transferring of properties allong the levels. Before we shall finish with formalization of so far investigations.

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