PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 16 (30), 1973 pp. 49—52

SPECTRUM OF THE TOTAL GRAPH OF A GRAPH

Dragoš M. Cvetković

(Received May 7, 1973)

The total graph $T(G)$ of a graph G is the graph whose set of vertices is the union of the set of vertices and of the set of edges of G, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident. The spectrum of a graph is the spectrum of its adjacency matrix.

In this paper we shall derive a relationship between the spectra of a regular graph and its total graph. Besides, some corollaries of this relationship are discussed. For the survey of results about total graphs see [1].

The line graph $L(G)$ of a graph G is the graph whose vertices are edges of G, where two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G have one vertex in common. Let A and B be adjacency matrices for G and $L(G)$ and let R be the vertex-edge incidence matrix of G. The well-known relations

$$RR^T = A + rI, \qquad R^T R = B + 2I,$$

where r is the degree of the regular graph G and I a unit matrix, will be used many times in the following.

It can easily be seen, that, by a suitable numbering of vertices, the adjacency matrix of $T(G)$ can be represented in the following form

$$\begin{Vmatrix} A & R \\ R^T & B \end{Vmatrix}.$$

Let $P_H(\lambda)$ be the characteristic polynomial of the adjacency matrix of the graph H.

If G has n vertices and m edges, we have

$$P_{T(G)} = \begin{vmatrix} \lambda I + rI - RR^T & -R \\ -R^T & \lambda I + 2I - R^T R \end{vmatrix}$$

$$= \begin{vmatrix} (\lambda + r) I - RR^T & -R \\ -(\lambda + r + 1) R^T + R^T RR^T & (\lambda + 2) I \end{vmatrix}$$

$$= \begin{vmatrix} (\lambda+r)I - RR^T + \frac{R}{\lambda+2}(-(\lambda+r+1)R^T + R^T RR^T) & 0 \\ -(\lambda+r+1)R^T + R^T RR^T & (\lambda+2)I \end{vmatrix}$$

$$= (\lambda+2)^m \det \left(\lambda I - A + \frac{1}{\lambda+2}(A+rI)(A-(\lambda+1)I)\right)$$

$$= (\lambda+2)^{m-n} \det (A^2 - (2\lambda - r + 3)A + (\lambda^2 - (r-2)\lambda - r)I)$$

$$= (\lambda+2)^{m-n} \prod_{i=1}^{n} (\lambda_i^2 - (2\lambda - r + 3)\lambda_i + \lambda^2 - (r-2)\lambda - r)$$

$$= (\lambda+2)^{m-n} \prod_{i=1}^{n} (\lambda^2 - (2\lambda_i + r - 2)\lambda + \lambda_i^2 + (r-3)\lambda_i - r),$$

λ_i $(i = 1, \ldots, n)$ being eigenvalues of A.

Thus, $T(G)$ has (for $r > 1$) m—n eigenvalues equal to -2 and the following $2n$ eingenvalues

$$\frac{1}{2}(2\lambda_i + r - 2 \pm \sqrt{4\lambda_i + r^2 + 4}), \quad i = 1, \ldots, n.$$

In further discussion we shall consider the connected graph only.

Note that $-r \leqslant \lambda_i \leqslant r$, $i = 1, \ldots, n$. Consider the functions

$$f_1(x) = \frac{1}{2}(2x + r - 2 + \sqrt{4x + r^2 + 4}), \quad f_2(x) = \frac{1}{2}(2x + r - 2 - \sqrt{4x + r^2 + 4}).$$

Both are increasing on the segment $[-r, r]$ for $r \neq 2$. For $r > 2$ the first maps this segment on the segment $[-2, 2r]$ and the second one on the segment $[-r, r-2]$. So, eigenvalues of $T(G)$ lie in the segment $[-r, 2r]$. (This holds for $r = 1$, too!). The greatest eigenvalue is naturally equal to $2r$. Eigenvalue $r-2$ appears always in the spectrum. The least eigenvalue is equal to $-r$ if and only if G is bipartite. Multiplicity of eigenvalue -2 in $T(G)$ is equal to $m - n + p_{-r} + p_{-1}$, where p_λ is the multiplicity of the eigenvalue λ in G and $r > 2$.

In the case $r = 2$ the function $f_2(x)$ has a minimum for $x = -\frac{7}{4}$. Since $f_2\left(-\frac{7}{4}\right) = -\frac{9}{4}$ the least eigenvalue of $T(G)$ is greater than $-\frac{9}{4}$. Equality can never hold, since an eigenvalue of a graph cannot be a rational non-integer number. But, since eigenvalues of a connected regular graph G of degree 2 with n vertices are $2\cos\frac{2\pi}{n} i$ $(i = 1, \ldots n)$, there exist graphs G for which the least eigenvalue of $T(G)$ is arbitrarily close to the lower bound $-\frac{9}{4}$.

The case $r=1$ is quite simple. G has eigenvalues $1,-1$, and $T(G)$ has eigenvalues $2,-1,-1$.

In [1] the following result is mentioned (for the proof see [2]):

The total graph of K_n (K_n denotes the complete graph with n vertices) is isomorphic to line-graph of K_{n+1}.

We can prove this statement by the use of graph spectra.

The case $n=1$ is clear. It is well-konwn that the spectrum of K_n consists of the number $n-1$ and $n-1$ numbers -1 for $n>1$. Using the above described relation between the spectra of G and $T(G)$, we get that the spectrum $T(K_n)$ contains the numbers $2n-2$, $n-3$ and -2 with multiplicities 1, n and $\frac{1}{2}(n^2-n-2)$ respectively. According to [3], [4], a graph with such a spectrum is isomorphic to $L(K_{n+1})$ except for $n=7$, in which case only three exceptional graphs exist. This case can be treated separately and it can be seen that $T(K_7)$ is not isomorphic to the mentioned exceptional graphs. Hence, $T(K_7)=L(K_8)$, which completes the proof.

Moreover, using graph spectra we can solve the „graph equation".

$$T(G)=L(H), \tag{1}$$

G being a regular graph, i.e. find all pairs of (connected) graphs (G, H) satisfying (1).

We can restrict ourselves to connected graphs. A sequence of solutions of (1) is (K_n, K_{n+1}), $n=1, 2, \ldots$. Naturally, from the solution (K_2, K_3) we get the solution $(K_2, K_{1,3})$, where $K_{m,n}$ denotes a bicomplete graph with parameters m, n. We shall show, that the graph equation (1) has, except the mentioned, no other solutions.

Let G be a regular graph of degree r ($r \geq 2$), which is not a complete graph. Consider the case $r>2$. In that case G has an eigenvalue λ smaller than -1. Since $f_2(-1)=-2$, the eigenvalue $f_2(\lambda)$ of $T(G)$ is smaller than -2. However, it is known that the least eigenvalue of $L(H)$ cannot be smaller than -2 (see, for example, [5]). Therefore if $r>2$, then (1) has no solutions but the mentioned ones.

In the case $r=0, 1$ we can simply solve the equation and we see that there are no new solutions.

The only interesting case is $r=2$. We have now $f_2(x)=x-\sqrt{x+2}$. Since $f_2(-2)=f_2(-1)=-2$, $T(G)$ has no eigenvalue smaller than -2 if and only if G has no eigenvalue greater than -2 and smaller than -1.

Since the eigenvalues of a connected regular graph of degree 2 with n vertices are given by $2\cos\frac{2\pi}{n}i$ ($i=0, 1, \ldots, n-1$) (see, for example, [6]), we see that the above will be true only for $n=3, 4, 6$. For $n=3$, G is a complete graph, and in the remaning two cases there do not exist solutions, which can be seen by direct consideration of all possible cases.

REFERENCES

[1] M. Behzad, G. Chartrand, *An introduction to total graphs, Theory of graphs* (Ed. P. Rosenstiehl), New York-Paris 1967, 31—33.

[2] M. Behzad, G. Chartrand, E. A. Nordhaus, *Triangles in line graphs and total graphs*, Indian J. Math., 10 (1968), no. 2, 109—120.

[3] A. J. Hoffman, *On the uniqueness of the triangular association scheme*, Ann. Math. Statist 31 (1960), 492—497.

[4] A. J. Hoffman, *On the exceptional case in a characterisation of the arc of a complete graph.* IBM J. Res. Develop. 4 (1960), 487—496.

[5] A. J. Hoffman, *Some recent results on spectral properties of graphs. Beiträge zur Graphentheorie*, Leipzig 1968, 75—80.

[6] L. Collatz, U. Sinogowitz, *Spektren endlicher Grafen.* Abh. Math. Sem, Univ. Hamburg 21 (1957) 63—77.