PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 16 (30), 1973, pp. 43—47

ON GENERALIZED (i, j) — MODULAR QUASIGROUPS

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(Received June 4, 1973)

In one of his papers [1] S. Milić has discussed the (i, j)-modular law. Four n-ary quasigroups A, B, C, D satisfy the (i, j)-modular law if

$$(1) \qquad A(x_1^{i-1},\ B(y_1^n),\ x_{i+1}^n) = C(y_1^{j-1},\ D(x_1^{i-1},\ y_j,\ x_{i+1}^n),\ y_{j+1}^n),$$

for all x_r, y_s, where $r, s = 1, 2, \ldots, n$. Here we use the following notation (see G. Čupona [2]): $x_m{}^n$ means the sequence $x_m, x_{m+1}, \ldots, x_n$. If $m > n$, $x_m{}^n$ will be considered empty, and if $m = n$ then $x_m{}^n$ is the element x_m. Finally, ${}_x{}^n$ denotes sequence $x, x, \ldots, x$ where x is repeated n-times.

In a natural way we define an (i, j)-modular quasigroup A if it satisfies (1) for the case $A = B = C = D$.

Here we discuss the generalized case when A satisfies the equation

$$(2) \qquad A(x_{\sigma 1}^{\sigma(i-1)},\ A(y_{\tau 1}^{\tau n}),\ x_{\sigma(i+1)}^{\sigma n}) = A(y_1^{j-1},\ A(x_1^{i-1},\ y_j,\ x_{i+1}^n),\ y_{j+1}^n)$$

where A is an n-quasigroup defined on a nonempty set Q, i, j some fixed numbers, σ and τ permutations of the set $N = \{1, 2, \ldots, n\}$ with the property $\sigma i = \tau i = i$, $\sigma j = \tau j = j$. Such a quasigroup will be called generalized (i, j)-modular quasigroup. The full description of such quasigroup is given below.

We note that we have solved a particular case of a more general problem (see [4]):

Find all n-ary quasigroup operations A satisfying the equation

$$A(x_1^{i-1},\ A(x_i^{i+n-1}),\ x_{i+n}^{2n-1}) = A(y_1^{j-1},\ A(y_j^{j+n-1}),\ y_{j+n}^{2n-1}),$$

where $y_1{}^{2n-1}$ is a permutation of $x_1{}^{2n-1}$.

We shall prove the following

T h e o r e m. *All solutions of the functional equation*

$$(2) \qquad A(x_{\sigma 1}^{\sigma(i-1)},\ A(y_{\tau 1}^{\tau n}),\ x_{\sigma(i+1)}^{\sigma n}) = A(y_1^{j-1},\ A(x_1^{i-1},\ y_j,\ x_{i+1}^n),\ y_{j+1}^n),$$

where A is a n-quasigroup defined on a nonempty set Q, i, j some fixed numbers, σ and τ permutations of the set $N = \{1, 2, \ldots, n\}$ with the property $\sigma i = \tau i = i$, $\sigma j = \tau j = j$, are given by

$$A(x_1^n) = x_i \circ g \circ R(x_1^{i-1},\ x_{i+1}^{j-1},\ x_{j+1}^n) \circ x_j,$$

where $Q(\circ)$ is an aribitrary binary group, g an arbitrary element of Q and $Q(R)$ an arbitrary (n-2)-quasigroup satisfying the condition

$$R(x_{\sigma 1}^{\sigma(i-1)}, x_{\sigma(i+1)}^{\sigma(j-1)}, x_{\sigma(j+1)}^{\sigma n}) = R(x_{\tau 1}^{\tau(i-1)}, x_{\tau(i+1)}^{\tau(j-1)}, x_{\tau(j+1)}^{\tau n}) = R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^{n}).$$

(Here we suppose $i<j$, the proof is analogous if $i>j$)

Proof. Following [4] we introduce the main parastrophes of the n-quasigroup A

$$^{\sigma^{-1}}A(x_1^n) = A(x_{\sigma 1}^{\sigma n}), \quad ^{\tau^{-1}}A(x_1^n) = A(x_{\tau 1}^{\tau n}).$$

The equation (2) becomes

$$(2') \qquad ^{\sigma^{-1}}A(x_1^{i-1}, {}^{\tau^{-1}}A(y_1^n), x_{i+1}^n) = A(y_1^{j-1}, A(x_1^{i-1}, y_j, x_{i+1}^n), y_{j+1}^n).$$

The equation (2′) satisfies the conditions of the Theorem 2.3 from [1], and according to that Theorem, we have

$$(3) \qquad \begin{cases} ^{\sigma^{-1}}A(x_1^n) = \alpha(\beta x_i \circ K(x_1^{i-1}, x_{i+1}^n)) \\ ^{\tau^{-1}}A(x_1^n) = \beta^{-1}(P(x_1^{j-1}, x_{j+1}^n) \circ \gamma x_j) \\ A(x_1^n) = \alpha(P(x_1^{j-1}, x_{j+1}^n) \circ x_j) \\ A(x_1^n) = \gamma x_i \circ K(x_1^{i-1}, x_{i+1}^n) \end{cases}$$

where α, β, γ, are permutations of the set Q, K and P $(n-1)$-quasigroups and $Q(\circ)$ a binary group.

We define the main parastrophes $^{\bar\sigma}K$, $^{\bar\tau}P$ of the $(n-1)$-quasigroups K, P in the following way:

$$^{\bar\sigma}K(x_1^{i-1}, x_{i+1}^n) = K(x_{\sigma^{-1}1}^{\sigma^{-1}(i-1)}, x_{\sigma^{-1}(i+1)}^{\sigma^{-1}n}),$$
$$^{\bar\tau}P(x_1^{j-1}, x_{j+1}^n) = P(x_{\tau^{-1}1}^{\tau^{-1}(j-1)}, x_{\tau^{-1}(j+1)}^{\tau^{-1}n}),$$

where $\bar\sigma$ is defined on $N\backslash\{i\}$, and respectively $\bar\tau$ is defined on $N\backslash\{j\}$.

Then the equations (3) can be written in the form

$$(4) \qquad A(x_1^n) = \alpha(\beta x_i \circ {}^{\bar\sigma}K(x_1^{i-1}, x_{i+1}^n)),$$

$$(5) \qquad A(x_1^n) = \beta^{-1}({}^{\bar\tau}P(x_1^{j-1}, x_{j+1}^n) \circ \gamma x_j),$$

$$(6) \qquad A(x_1^n) = \alpha(P(x_1^{j-1}, x_{j+1}^n) \circ x_j),$$

$$(7) \qquad A(x_1^n) = \gamma x_i \circ K(x_1^{i-1}, x_{i+1}^n).$$

From (4), (6) it follows

$$\alpha(\beta x_i \circ {}^{\bar\sigma}K(x_1^{i-1}, x_{i+1}^n)) = \alpha(P(x_1^{j-1}, x_{j+1}^n) \circ x_j),$$

and if we put $x_i = a$, such that $\beta a = e$, where e is the unit of the group $Q(\circ)$, we get

$$(8) \qquad {}^{\bar\sigma}K(x_1^{i-1}, x_{i+1}^n) = R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) \circ x_j,$$

where R is a $(n$-$2)$-quasigroup defined by

$$R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) = P(x_1^{i-1}, a, x_{i+1}^{j-1}, x_{j+1}^n).$$

Hence, (4) becomes

$$A(x_1^n) = \alpha(\beta x_i \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) \circ x_j). \tag{4'}$$

From (4′) and (6) we obtain

$$P(x_1^{j-1}, x_{j+1}^n) = \beta x_i \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n). \tag{9}$$

From (4′) and (7) it follows

$$\alpha(\beta x_i \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) \circ x_j) = \gamma x_i \circ K(x_1^{i-1}, x_{i+1}^n),$$

and for $x_i = b$, such that $\gamma b = e$, we have

$$K(x_1^{i-1}, x_{i+1}^n) = \alpha(c \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) \circ x_j), \tag{10}$$

where $\beta b = c$.

Using (8), (10) and the definition of the parastrophe $\overline{\sigma}K$, we shall have

$$R(x_{\sigma 1}^{\sigma(i-1)}, x_{\sigma(i+1)}^{\sigma(j-1)}, x_{\sigma(j+1)}^{\sigma n}) \circ x_j = {}^{\overline{\sigma}}K(x_{\sigma 1}^{\sigma(i-1)}, x_{\sigma(i+1)}^{\sigma n}) = \tag{11}$$
$$= K(x_1^{i-1}, x_{i+1}^n) = \alpha(c \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) \circ x_j).$$

From (4) and (5) it follows

$$\alpha(\beta x_i \circ {}^{\overline{\sigma}}K(x_1^{i-1}, x_{i+1}^n)) = \beta^{-1}({}^{\overline{\tau}}P(x_1^{j-1}, x_{j+1}^n) \circ \gamma x_j),$$

that is,

$$\beta\alpha(\beta x_i \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) \circ x_j) = {}^{\overline{\tau}}P(x_1^{j-1}, x_{j+1}^n) \circ \gamma x_j,$$

and from there

$$\beta\alpha(\beta x_i \circ R(x_{\tau 1}^{\tau(i-1)}, x_{\tau(i+1)}^{\tau(j-1)}, x_{\tau(j+1)}^{\tau n}) \circ x_j) = {}^{\overline{\tau}}P(x_{\tau 1}^{\tau(j-1)}, x_{\tau(j+1)}^{\tau n}) \circ \gamma x_j =$$
$$= P(x_1^{j-1}, x_{j+1}^n) \circ \gamma x_j = \beta x_i \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) \circ \gamma x_j,$$

where we used (9) and the definition of the parastrophe $\overline{\tau}P$.

Hence,

$$\beta\alpha(x_i \circ R(x_{\tau 1}^{\tau(i-1)}, x_{\tau(i+1)}^{\tau(j-1)}, x_{\tau(j+1)}^{\tau n}) \circ x_j) = \tag{12}$$
$$= x_i \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) \circ \gamma x_j.$$

From (4′) and (7) with the help of (10), we obtain

$$\alpha(\beta x_i \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) \circ x_j) = \gamma x_i \circ \alpha(c \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^n) \circ x_j). \tag{13}$$

From (13), replacing all x_k ($k \neq i, j, k \in N$) with a_k, such that $R(a_1^{i-1}, a_{i+1}^{j-1}, a_{j-1}^n) = e$, it follows $\alpha(x_i \circ x_j) = \gamma\beta^{-1} x_i \circ \alpha L_c x_j$, that is, α is a quasiautomorphism*) of the binary group $Q(\circ)$. Hence, α can be represented as $\alpha = R_d \theta$, where θ is an automorphism of the group $Q(\circ)$ and R_d a translation of $Q(\circ)$.

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*) A permutation α of the group $Q(\circ)$ is called quasiautomorphism if $\alpha(x \circ y) = \beta x \circ \gamma y$ for some other permutations β and γ [3]. Every quasiautomorphism α can be represented as $\alpha = L_p \theta$ or $\alpha = R_q \theta'$, where θ and θ' are automorphisms, $L_p x = p \circ x$ and $R_q x = x \circ q$, $p, q \in Q$.

So, the equation (13) becomes

$$\theta\beta x_i \circ \theta R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^{n}) \circ \theta x_j \circ d = \gamma x_i \circ \theta c \circ \theta R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^{n}) \circ \theta x_j \circ d,$$

that is

(14) $$\beta x = \theta^{-1}\gamma x \circ c.$$

From (11) we have

(15) $$R(x_{\sigma 1}^{\sigma(i-1)}, x_{\sigma(i+1)}^{\sigma(j-1)}, x_{\sigma(j+1)}^{\sigma n}) \circ x_j = \theta c \circ \theta R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^{n}) \circ \theta x_j \circ d.$$

Putting

$$x_1^{i-1} = b_1^{i-1}, \; x_{i+1}^{j-1} = b_{i+1}^{j-1}, \; x_j = e, \; x_{j+1}^{n} = b_{j+1}^{n},$$

in (15) we get

$$d = (\theta c \circ \theta R(b_1^{i-1}, b_{i+1}^{j-1}, b_{j+1}^{n}))^{-1} \circ R(b_{\sigma 1}^{\sigma(i-1)}, b_{\sigma(i+1)}^{\sigma(j-1)}, b_{\sigma(j+1)}^{\sigma n}).$$

If we fix in (15),

$$x_1^{i-1} = b_1^{i-1}, \; x_{i+1}^{j-1} = b_{i+1}^{j-1}, \; x_{j+1}^{n} = b_{j+1}^{n},$$

we obtain that θ is an inner automorphism

(16) $$\theta x = d \circ x \circ d^{-1},$$

and

(17) $$\alpha x = d \circ x.$$

Hence, we get

(18) $$R(x_{\sigma 1}^{\sigma(i-1)}, x_{\sigma(i+1)}^{\sigma(j-1)}, x_{\sigma(j+1)}^{\sigma n}) = d \circ c \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^{n}).$$

Putting in (18) $x_r = f$ for all $r \neq i, j, r \in N$ we have $d \circ c = e$, that is,

(19) $$R(x_{\sigma 1}^{\sigma(i-1)}, x_{\sigma(i+1)}^{\sigma(j-1)}, x_{\sigma(j+1)}^{\sigma n}) = R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^{n}).$$

Fom (16) and (14) we have

(20) $$\beta x = d^{-1} \circ \gamma x \circ d \circ c = d^{-1} \circ \gamma x.$$

Now (12) can be written in the form

$$d^{-1} \circ \gamma (d \circ x_i \circ R(x_{\tau 1}^{\tau(i-1)}, x_{\tau(i+1)}^{\tau(j-1)}, x_{\tau(j+1)}^{\tau n}) \circ x_j = x_i \circ R(x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^{n}) \circ \gamma x_j,$$

and if we fix again $x_r = f$ for all $r \neq i, j, r \in N$, and denote $d \circ x_i \circ R(\overset{n-2}{f}) = y$, we obtain

$$\gamma (y \circ x_j) = y \circ \gamma x_j$$

and from the last equation for $x_j = e$, we get

(21) $$\gamma y = y \circ g,$$

where $g = \gamma e$.

Hence, (20) becomes

(20′) $$\beta x = d^{-1} \circ x \circ g.$$

Now the equation (12) can be written in the form

$$x_i \circ R\,(x_{\tau 1}^{\tau(i-1)},\ x_{\tau(i+1)}^{\tau(j-1)},\ x_{\tau(j+1)}^{\tau n}) \circ x_j \circ g = x_i \circ R\,(x_1^{i-1},\ x_{i+1}^{j-1},\ x_{j+1}^{n}) \circ x_j \circ g.$$

that is,

$$R\,(x_{\tau 1}^{\tau(i-1)},\ x_{\tau(i+1)}^{\tau(j-1)},\ x_{\tau(j+1)}^{\tau n}) = R\,(x_1^{i-1},\ x_{i+1}^{j-1},\ x_{j+1}^{n}). \qquad (22)$$

Hence, from (17), (20′) and (4′) quasigroup A can be represented as

$$A\,(x_1^n) = x_i \circ g \circ R\,(x_1^{i-1},\ x_{i+1}^{j-1},\ x_{j+1}^{n}) \circ x_j. \qquad (23)$$

So, we have proved first part of the Theorem.

By a straight forward computation, it is easy to verify the converse part.

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