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ON INFINITARY QUASIGROUPS

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The notion of infinitary operation in implicit form can be found in different fields of mathematics. Up to now that notion was not submitted to special investigations. There are some attempts to consider those operations. In particular can be mentioned the paper of Madevski Ž., Trpenovski B., Čupona G. [1].

By an infinitary operation A defined on a set Q we understand a mapping

$$a_1, a_2, \dots, a_n, \dots \rightarrow b,$$

where all a_i and b belong to the set Q. As in the n-ary case we shall write

$$A(a_1, a_2, \dots, a_n, \dots) = b. \tag{1}$$

The set Q together with the infinitary operation A we call an infinitary operative and denote by $Q(A)$. The infinitary operative $Q(A)$ such that the set Q contains only one element will be called trivial.

We shall use the following notations. The sequence $x_m, x_{m+1}, \dots, x_n$ will be denoted by x_m^n. If $m>n$, x_m^n will be considered empty, and if $m=n$ then x_m^n is the element x_m. By $\overset{n}{x}$ we denote the sequence $x, x, \dots. x$ where x is repeated n times. The symbol $\overset{o}{x}$ denotes the empty sequence.

The infinite sequence $x_m, x_{m+1}, \dots, x_n, \dots.$ will be denoted by x_m^∞ (m finite natural number). The infinite sequence $x, x, \dots, x, \dots$ we denote by $\overset{\infty}{x}$.

Therefore, the equation (1) can be written in the form $A(a_1^\infty) = b$.

The notion of quasigroup in infinitary case can be introduced in a natural way.

D e f i n i t i o n 1. A set Q together with an infinitary operation A we call an infinitary quasigroup (briefly ∞-quasigroup), if the equation

$$A(a_1^{i-1}, x, a_{i+1}^\infty) = b$$

has a unique solution x for all a_1^∞, $b \in Q$ and for every positive integer i

This concept can be found already in [1]. In that paper the authors prove that nontrivial infinitary groups, i.e. ∞-quasigroups satisfying associative laws, which will be defined later, do not exist.

In the present paper we prove more stronger result: there do not exist nontrivial (i, j)-associative infinitary quasigroups, where i, j are fixed natural numbers. It is proved that there exist infinitary quasigroups and loops of any finite order and also of infinite order. Further, the equation of general associativity for infinitary case is solved. Finally, some ordinary concepts of the theory of quasigroups are given for infinitary case, and some specific questions of that theory are considered.

§ 1. (i, j)-associative ∞-quasigroups

We shall start with some examples of ∞-quasigroups. First we give the definition of an infinitary loop.

Definition 2. The element e of the infinitary operative $Q(A)$ is called a unity if

$$A(\overset{i-1}{e}, x, \overset{\infty}{e}) = x,$$

for all $x \in Q$ and every $i = 1, 2, \ldots, n, \ldots$.

If an infinitary quasigroup $Q(A)$ contains at least one unity then $Q(A)$ is called an infinitary loop (∞-loop).

We shall prove the existence of infinitary quasigroups and loops.

Let D be the set of all real numbers and let D^∞ be the set of all infinite sequences of elements belonging to D.

On the set D^∞ we define a binary relation in the following way. The sequences $\alpha = a_1^\infty$ and $\beta = b_1^\infty$ are equivalent, $\alpha \sim \beta$, if and only if $\sum_{i=1}^{\infty} |a_i - b_i| < \infty$. It is obvious that $\alpha \sim \alpha$, and that from $\alpha \sim \beta$ it follows $\beta \sim \alpha$. We shall show that from $\alpha \sim \beta$, $\beta \sim \gamma$, where $\gamma = c_1^\infty$, it follows $\alpha \sim \gamma$. Indeed, we have

$$\sum_{i=1}^{\infty} |a_i - c_i| = \sum_{i=1}^{\infty} |(a_i - b_i) + (b_i - c_i)| \leqslant \sum_{i=1}^{\infty} |a_i - b_i| + \sum_{i=1}^{\infty} |b_i - c_i| < \infty$$

that is, $\alpha \sim \gamma$. So, "$\sim$" is an equivalence relation.

On the set D we define an infinitary operation which we denote by A. In every equivalence class K, defined by the relation "$\sim$", we choose one representative $\sigma = s_1^\infty$. Let 0 correspond to that representative, that is, $A(\sigma) = 0$. If $\alpha = a_1^\infty \in K$, that is $\alpha \sim \sigma$, then

$$\sum_{i=1}^{\infty} |a_i - s_i| < \infty. \tag{2}$$

The element $A(\alpha)$ we define as following:

$$A(\alpha) = \sum_{i=1}^{\infty} (a_i - s_i). \tag{3}$$

From (2) we see that the series on the right side of (3) is absolutely convergent.

We shall show that the equation

$$A(a_1^{k-1},\ x,\ a_{k+1}^{\infty})=b, \tag{4}$$

has a unique solution for all a_1^{k-1}, a_{k+1}^{∞}, b and every $k=1.2,\ldots,n,\ldots$ Let the sequence $\alpha'=a_1^{k-1}$, 0, a_{k+1}^{∞} belong to the class K with the representative $\sigma=s_1^{\infty}$. Then by the definition of A,

$$A(\alpha')=\sum_{\substack{i=1\\ i\neq k}}^{\infty}(a_i-s_i)-s_k=b'.$$

The sequence $\alpha''=a_1^{k-1}$, t, a_{k+1}^{∞}, where t is an arbitrary real number, obviously belongs to the class K. We have

$$A(\alpha'')=\sum_{\substack{i=1\\ i\neq k}}^{\infty}(a_i-s_i)+(t-s_k)=t+b'. \tag{5}$$

Comparing (4) and (5) we see that the solution of the equation (4) is number $b-b'$ and that solution is unique.

The analogous method we can apply if instead of D we take the set N of all integers. In this case, $\alpha\sim\beta$ means that the sequences $\alpha=a_1^{\infty}$ and $\beta=b_1^{\infty}$ differ only on finite number of places. Indeed, the condition $\sum_{i=1}^{\infty}|a_i-b_i|<\infty$ is fulfilled if and only if $a_i\neq b_i$ for finite number of indexes.

In the same way we can construct infinitary quasigroups on the set Q with n elements, where n is arbitrary natural number. We take that Q is an additive group of integers modulo n. In that case, $\alpha\sim\beta$ if and only if the sequences $\alpha=a_1^{\infty}$ and $\beta=b_1^{\infty}$ differ only on finite number of places and the sum $\sum_{i=1}^{\infty}(a_i-s_i)$, for two sequences $\alpha=a_1^{\infty}$ and $\sigma=s_1^{\infty}$ from the same equivalence class, is always well defined.

Now we show the existence of infinitary loops.

On the set D of all real numbers we define the equivalence relation "$\sim$" as in the preceding example. Let K_a be the equivalence class uniquely determined by an element $a\in Q$, $\overset{\infty}{a}\in K_a$. Then the sequence $\alpha=0,\ \overset{\infty}{a}$ also belongs to K_a. Taking that the representative of the class K_a is α, we define an infinitary quasigroup operation A on D as in the preceding example (then $A(\alpha)=0$).

The element a is a unity of the infinitary quasigroup $D(A)$. Indeed,

$$A(x,\ \overset{\infty}{a})=x,$$

$$A(\overset{k}{a},\ x,\ \overset{\infty}{a})=(a-0)+0+\cdots+0+(x-a)+0+\cdots=x.$$

If $a,b\in D$, $a\neq b$ then it is obvious that the sequences $\overset{\infty}{a}$ and $\overset{\infty}{b}$ are not equivalent. So, choosing in such a way representatives of all classes K_i we get that every element of D is a unity.

With different choice of representatives from the classes K_i we can get that the set of all unities is an arbitrary subset $M \subset D$.

Definition 3. An infinitary operative $Q(A)$ is called (i, j)-associative if it satisfies the indentity

$$A(x_1^{i-1}, A(x_i^{\infty}), y_1^{\infty}) = A(x_1^{j-1}, A(x_j^{\infty}), y_1^{\infty}), \tag{6}$$

for all $x_1^{\infty}, y_1^{\infty} \in Q$.

(Of course, we suppose $i \neq j$).

Definition 4. An infinitary operative $Q(A)$ is called an infinitary semigroup if it satisfies the indentity (6) for all i and j.

Definition 5. An infinitary quasigroup which is infinitary semigroup is called an infinitary group.

Examples of infinitary semigroups are given in [1]. Also it is proved there that nontrivial infinitary groups do not exist.

Now we shall consider (i, j)-associative infinitary quasigroups.

Lemma 1. *If an infinitary quasigroup $Q(A)$ is (i, j)-associative then it is $(j, 2j\text{-}i)$-associative.* (Here we suppose $i < j$).

Proof. Let a be arbitrary element from Q. Then, since $Q(A)$ is (i, j)-associative, we have

$$A(\overset{i-1}{a}, A(x_1^{j-1}, A(x_j^{\infty}), y_1^{\infty}), \overset{\infty}{a}) = A(\overset{i-1}{a}, x_1^{j-i}, A(x_{j-i+1}^{j-1}, A(x_j^{\infty}), y_1^{\infty}), \overset{\infty}{a}) =$$

$$= A(\overset{i-1}{a}, x_1^{j-i}, A(x_{j-i+1}^{2j-i-1}, A(x_{2j-i}^{\infty}), y_1^{\infty}), \overset{\infty}{a}) = A(\overset{i-1}{a}, A(x_1^{2j-i-1}, A(x_{2j-i}^{\infty}), y_1^{\infty}), \overset{\infty}{a}).$$

Since A is infinitary quasigroup then

$$A(x_1^{j-1}, A(x_j^{\infty}), y_1^{\infty}) = A(x_1^{2j-i-1}, A(x_{2j-i}), y_1^{\infty}),$$

that is, A is $(j, 2j-i)$-associative.

Theorem 1. *There does not exist a nontrivial (i, j)-associative infinitary quasigroup.*

Proof. Let $Q(A)$ be (i, j)-associative infinitary quasigroup. By Lemma 1. $Q(A)$ is $(j, 2j-i)$-associative.

Let a, b be arbitrary elements of the set Q. If $A(\overset{\infty}{a}) = c$ then there exists an element $d \in Q$ such that $b = A(c, \overset{k-1}{a}, d, \overset{\infty}{a})$, where $k = j - i$.

Then,

$$A(\overset{i-1}{a}, b, \overset{\infty}{a}) = A(\overset{i-1}{a}, A(c, \overset{k-1}{a}, d, \overset{\infty}{a}), \overset{\infty}{a}) = A(\overset{i-1}{a}, c, \overset{k-1}{a}, A(d, \overset{\infty}{a}), \overset{\infty}{a}) =$$

$$= A(\overset{i-1}{a}, A(\overset{\infty}{a}), \overset{k-1}{a}, A(d, \overset{\infty}{a}), \overset{\infty}{a}) = A(\overset{j-1}{a}, A(\overset{\infty}{a}), \overset{k-1}{a}, A(d, \overset{\infty}{a}), \overset{\infty}{a}) =$$

$$= A(\overset{j-1}{a}\ c, \overset{k-1}{a}, A(d, \overset{\infty}{a}), \overset{\infty}{a}) = A(\overset{j-1}{a}, A(c, \overset{k-1}{a}, d, \overset{\infty}{a}), \overset{\infty}{a}) = A(\overset{j-1}{a}, b, \overset{\infty}{a}),$$

where we have used (i, j) and $(j, 2j-i)$-associativity of the infinitary quasigroup A.

Hence, for every $a, b \in Q$

$$A(\overset{i-1}{a}, b, \overset{\infty}{a}) = A(\overset{j-1}{a}, b, \overset{\infty}{a}).$$

Let x_1^∞, y be arbitrary elements from Q. Then there exists z such that $x_i = A(\overset{i-1}{y}, z, \overset{\infty}{y})$, and we have

$$A(x_1^\infty) = A(x_1^{i-1}, A(\overset{i-1}{y}, z, \overset{\infty}{y}), x_{i+1}^\infty) = A(x_1^{i-1}, A(\overset{j-1}{y}, z, \overset{\infty}{y}), x_{i+1}^\infty) =$$

$$= A(x_1^{i-1}, \overset{k}{y}, A(\overset{i-1}{y}, z, \overset{\infty}{y}), x_{i+1}^\infty) = A(x_1^{i-1}, \overset{k}{y}, x_i^\infty).$$

So, we have proved that for arbitrary elements x_1^∞, y

$$A(x_1^\infty) = A(x_1^{i-1}, \overset{k}{y}, x_i^\infty). \tag{7}$$

If in (7) we put $u = x_i = x_{i+k} = \cdots = x_{i+nk} = \cdots$ and $x_m = y$, for all $m \neq i + nk$, $m > i$, we get

$$A(x_1^{i-1}, u, \overset{k-1}{y}, u, \overset{k-1}{y}, u, \ldots) = A(x_1^{i-1}, y, \overset{k-1}{y}, u, \overset{k-1}{y}, u, \ldots)$$

that is, $u = y$.

That means that Q contains only one element and the theorem is proved.

Remark. It is proved in [1] that there does not exist a nontrivial infinitary group. In the note at the end of the mentioned paper, it is said that it can be easily seen that all the lemmas, which are proved there, are valid under the supposition that infinitary operation is only (1, 2)-associative. We remark that in the proof of Lemma 1 from [1] is used (2, 3)-associativity and not only (1, 2)-associativity.

§ 2. Generalized associativity on infinitary quasigroups

First we note that the definition of an infinitary quasigroup given above is for a countable set of the order type ω of variables: the mapping A which defines infinitary operation maps a countable set of order type ω $x_1, x_2, \ldots, x_n, \ldots$ into some element y from the same set Q. This set of variables has the order type ω, and we shall say that the operation A has also the type ω.

Here we introduce infinitary quasigroup of the type $\omega + k$.

The mapping A: $a_1, a_2, \ldots, a_n, \ldots, b_1, b_2, \ldots, b_k \to c$ defined on the set $Q^{\omega+k}$ (where by $Q^{\omega+k}$ we denote the set of all infinite sequences of order type $\omega + k$ of elements from Q) is called infinitary operation of the type $\omega + k$. We write in this case $A(a_1^\infty, b_1^k) = c$. The type of A we shall denote by $|A|$.

Definition 6. An infinitary operative $Q(A)$ of the type $\omega + k$ is called infinitary quasigroup of the type $\omega + k$ if the equations

$$A(a_1^{i-1}, x, a_{i+1}^\infty, b_1^k) = c, \qquad A(a_1^\infty, b_1^{i-1}, x, b_{i+1}^k) = c,$$

have an unique solution x for all a_p, b_q, $c \in Q$ and for all positive integers i in the first equation and for all $i = 1, 2, \ldots, k$ in the second equation.

We shall consider functional equation of generalized associativity on intinitary quasigroups:

$$A(x_1^{i-1},\ B(x_i^{\infty},\ y_1^r),\ y_{r+1}^{\infty}) = C(x_1^{j-1},\ D(x_j^{\infty},\ y_1^s),\ y_{s+1}^{\infty}), \tag{8}$$

where A, B, C, D are infinitary quasigroups of types $|A| = \omega$, $|B| = \omega + r$, $|C| = \omega$, $|D| = \omega + s$, defined on the same nonempty set Q, i, j some fixed natural numbers and r, s non-negative integers.

We shall give general solution of the equation (8). Without loss of generality we way assume that $i \leqslant j$.

Case I. $i=j$.

Subcase I_1. $r = s \geqslant 0$.

If we substitute in (8) variables x_1^{i-1}, y_{r+1}^{∞} by some fixed elements a_1^{i-1}, b_{r+1}^{∞} of the set Q we get

$$\alpha B(x_{i+1}^{\infty},\ y_1^r) = \beta D(x_{i+1}^{\infty},\ y_1^r),$$

where α, β are permutations of the set Q defined by $\alpha x = A(a_1^{i-1},\ x,\ b_{r+1}^{\infty})$, $\beta x = C(a_1^{i-1},\ x,\ b_{r+1}^{\infty})$, that is

$$D = \theta B, \qquad (\theta = \gamma^{-1} \alpha).$$

Putting in (8) θB instead of D we obtain

$$A(x_1^{i-1},\ B(x_i^{\infty},\ y_1^r),\ y_{r+1}^{\infty}) = C(x_1^{i-1},\ \theta B(x_i^{\infty},\ y_1^r),\ y_{r+1}^{\infty}),$$

or

$$A(x_1^{i-1},\ z,\ y_{r+1}^{\infty}) = C(x_1^{i-1},\ \theta z,\ y_{r+1}^{\infty}).$$

So, in this case all solutions of (8) are given by

$$D = \theta B, \quad A(x_1^{i-1},\ z,\ y_{r+1}^{\infty}) = C(x_1^{i-1},\ \theta z,\ y_{r+1}^{\infty}), \tag{9}$$

where θ is an arbitrary permutation of the set Q, B and C arbitrary infinitary quasigroups of types $|B| = \omega + r$, $|C| = \omega$ and D and A are determined by the equations (9).

Subcase I_2. $s > r \geqslant 0$. (If $r > s \geqslant 0$, the solution is analogous).

If in (8) we fix the variables x_1^{i-1}, y_{s+1}^{∞} by elements a_1^{i-1}, b_{s+1}^{∞} we get

$$A_1(B(x_i^{\infty},\ y_1^r),\ y_{r+1}^s) = \gamma D(x_i^{\infty},\ y_1^s),$$

where γ is the permutation of the set Q defined by

$$\gamma x = C(a_1^{i-1},\ x,\ b_{s+1}^{\infty}), \text{ and } A_1(z,\ y_{r+1}^s) = A(a_1^{i-1},\ z,\ y_{r+1}^s,\ b_{s+1}^{\infty}),$$

that is

$$D(x_i^{\infty},\ y_1^s) = K(B(x_i^{\infty},\ y_1^r),\ y_{r+1}^s), \tag{10}$$

where $K = \gamma^{-1} A_1$.

Putting D back in (8) we obtain

$$A(x_1^{i-1},\ B(x_i^\infty,\ y_1^r),\ y_{r+1}^\infty)=C(x_1^{i-1},\ K(B(x_i^\infty,\ y_1^r),\ y_{r+1}^s),\ y_{s+1}^\infty),$$

or

$$A(x_1^{i-1},\ z,\ y_{r+1}^\infty)=C(x_1^{i-1},\ K(z,\ y_{r+1}^s),\ y_{s+1}^\infty), \tag{11}$$

In this case all solutions of the equation (8) are given by the equations (10) and (11), where B and C are arbitrary infinitary quasigroups of types $|B|=\omega+r$, $|C|=\omega$, K arbitrary quasigroup of arity $s-r+1$ and A and D are determined by the equations (11) and (10).

Case II. $i<j$.

Subcase II$_1$. $r\geqslant s\geqslant 0$.

Substituting in (8) x_1^{i-1}, y_{r+1}^∞ by a_1^{i-1}, b_{r+1}^∞ we get

$$\alpha B(x_{i+1}^\infty,\ y_1^r)=C_1(x_i^{j-1},\ D(x_j^\infty,\ y_1^s),\ y_{s+1}^r),$$

where α is the permutation of Q defined by $\alpha x=A(a_1^{i-1},\ x,\ b_{r+1}^\infty)$ and $C_1(x_i^{j-1},\ z,\ y_{s+1}^r)=C(a_1^{i-1},\ x_i^{j-1},\ z,\ y_{s+1}^r,\ b_{r+1}^\infty)$, that is

$$B(x_{i+1}^\infty,\ y_1^r)=K(x_i^{j-1},\ D(x_j^\infty,\ y_1^s),\ y_{s+1}^r), \tag{12}$$

where $K=\alpha^{-1}C_1$.

Putting B back in (8) we have

$$A(x_1^{i-1},\ K(x_i^{j-1},\ D(x_j^\infty,\ y_1^s),\ y_{s+1}^r),\ y_{r+1}^\infty)=C(x_1^{j-1},\ D(x_j^\infty,\ y_1^s),\ y_{s+1}^\infty),$$

or

$$C(x_1^{j-1},\ z,\ y_{s+1}^\infty)=A(x_1^{i-1},\ K(x_i^{j-1},\ z,\ y_{s+1}^r),\ y_{r+1}^\infty). \tag{13}$$

All solutions of (8) are given by (12) and (13) where A and D are arbitrary infinitary quasigroups of types $|A|=\omega$, $|D|=\omega+s$, K arbitrary quasigroup of arity $j-i+r-s+1$ and B and C are determined by the equations (12) and (13).

Subcase II$_2$. $s>r\geqslant 0$.

In this case we obtain general solution by an analogous method as in [2].

If in (8) we substitute the variables x_1^{i-1}, y_{s+1}^∞ by elements a_1^{i-1}, b_{s+1}^∞ from Q, we get

$$A_1(B(x_i^{j-1},\ x_j^\infty,\ y_1^r),\ y_{r+1}^s)=C_1(x_i^{j-1},\ D(x_j^\infty,\ y_1^r,\ y_{r+1}^s)), \tag{14}$$

where $A_1(z,\ y_{r+1}^s)=A(a_1^{i-1},\ z,\ y_{r+1}^s,\ b_{s+1}^\infty)$, $C_1(x_i^{j-1},\ z)=C(a_1^{i-1},\ x_i^{j-1},\ z,\ b_{s+1}^\infty)$.

If we define

$$A_1(z,\ y_{r+1}^s)=\bar{A}_1(z,(y_{r+1}^s)),$$

$$B(x_i^{j-1},\ x_j^\infty,\ y_1^r)=\bar{B}((x_i^{j-1}),\ (x_j^\infty,\ y_1^r)),$$

$$C_1(x_i^{j-1},\ z)=\bar{C}_1((x_i^{j-1}),\ z),$$

$$D(x_j^\infty,\ y_1^r,\ y_{r+1}^s)=\bar{D}((x_j^\infty,\ y_1^r),(y_{r+1}^s)),$$

and denote $(x_i^{j-1})=X$, $(x_j^{\infty},\ y_1^r)=Y$, $(y_{r+1}^s)=Z$, the equation (14) becomes

$$\overline{A}_1(\overline{B}(X,\ Y),Z)=\overline{C}_1(X,\overline{D}(Y,Z)),$$

where $\overline{A}_1$, $\overline{B}$, $\overline{C}_1$, $\overline{D}$ are GD-groupoids (see [2]), defined on

$$\overline{A}_1: Q\times Q^{s-r}\to Q,\qquad \overline{C}_1: Q^{j-i}\times Q\to Q,$$

$$\overline{B}: Q^{j-i}\times Q^{\omega+r}\to Q,\qquad \overline{D}: Q^{\omega+r}\times Q^{s-r}\to Q.$$

These GD-groupoids satisfy the conditions of Theorem 2 from [2], and by this theorem

$$\overline{B}(X,\ Y)=\alpha^{-1}(\gamma X\circ\delta Y),\qquad \overline{D}(Y,Z)=\varphi^{-1}(\delta Y\circ\beta Z)$$

that is,

$$B(x_i^{\infty},\ y_1^r)=\alpha^{-1}(\gamma(x_i^{j-1})\circ\delta(x_j^{\infty},\ y_1^r)),\tag{15}$$

$$D(x_j^{\infty},\ y_1^s)=\varphi^{-1}(\delta(x_j^{\infty},\ y_1^r)\circ\beta(y_{r+1}^s)),\tag{16}$$

where α, φ are permutations of the set Q, $Q(\circ)$ binary group, γ and β quasigroups of arities $|\gamma|=j-i$, $|\beta|=s-r$ and δ an infinitary quasigroup of type $\omega+r$.

Using (15) and (16), we put back B and D in (8) and obtain

$$\begin{aligned}A(x_1^{i-1},\ \alpha^{-1}(\gamma(x_i^{j-1})\circ\delta(x_j^{\infty},\ y_1^r)),y_{r+1}^{\infty})=\\=C(x_1^{j-1},\ \varphi^{-1}(\delta(x_j^{\infty},\ y_1^r)\circ\beta(y_{r+1}^s)),y_{s+1}^{\infty}),\end{aligned}\tag{17}$$

and if in (17) we substitute the variables x_i^{j-1} by elements c_i^{j-1} such that $\gamma(c_i^{j-1})=e$, where e is the unity of the group $Q(\circ)$, we shall have

$$A(x_1^{i-1},\alpha^{-1}\delta(x_j^{\infty},\ y_1^r),\ y_{r+1}^{\infty})=K(x_1^{i-1},\ \ \delta(x_j^{\infty},\ y_1^r)\circ\beta(y_{r+1}^s),\ y_{s+1}^{\infty}),$$

where

$$K(x_1^{i-1},\ y,\ y_{s+1}^{\infty})=C(x_1^{i-1},\ c_i^{j-1},\ \varphi^{-1}y,\ y_{s+1}^{\infty}).$$

Hence,

$$A(x_1^{i-1},\alpha^{-1}x,y_{r+1}^{\infty})=K(x_1^{i-1},x\circ\beta(y_{r+1}^s),\ y_{s+1}^{\infty}).\tag{18}$$

If in (17) we substitute x_j^{∞}, y_1^r by elements c_j^{∞}, d_1^r such that $\delta(c_j^{\infty},\ d_1^r)=e$, we have

$$C(x_1^{j-1},\ \varphi^{-1}\beta(y_{r+1}^s),\ y_{s+1}^{\infty})=A(x_1^{i-1},\ \alpha^{-1}\gamma(x_i^{j-1}),\ y_{r+1}^{\infty}),\tag{19}$$

and by (18) and (19) it follows

$$C(x_1^{j-1},\ \varphi^{-1}\beta(y_{r+1}^s),\ y_{s+1}^{\infty})=K(x_1^{i-1},\ \gamma(x_i^{j-1})\circ\beta(y_{r+1}^s),\ y_{s+1}^{\infty}),$$

i.e.

$$C(x_1^{j-1},\ y,\ y_{s+1}^{\infty})=K(x_1^{i-1},\ \gamma(x_i^{j-1})\circ\varphi y,\ y_{s+1}^{\infty}).$$

So, in this case all solutions of (8) are given by

$$(20)\quad \begin{cases} A(x_1^{i-1},\ x,\ y_{r+1}^{\infty}) = K(x_1^{i-1},\ \alpha x \circ \beta(y_{r+1}^{s}),\ y_{s+1}^{\infty}), \\ B(x_i^{\infty},\ y_1^{r}) = \alpha^{-1}(\gamma(x_i^{j-1}) \circ \delta(x_j^{\infty},\ y_1^{r})), \\ C(x_1^{j-1},\ y,\ y_{s+1}^{\infty}) = K(x_1^{i-1},\ \gamma(x_i^{j-1}) \circ \varphi y,\ y_{s+1}^{\infty}), \\ D(x_j^{\infty},\ y_1^{s}) = \varphi^{-1}(\delta(x_j^{\infty},\ x_1^{r}) \circ \beta(y_{r+1}^{s})), \end{cases}$$

where α, φ are permutations of the set Q, $Q(\circ)$ binary group, β and γ quasigroups of arities $|\beta| = s-r$, $|\gamma| = j-i$, and δ, K infinitary quasigroups of types $|\delta| = \omega + r$, $|K| = \omega$.

All the preceding results can be summarized in

Theorem 2. *All solutions of the equation*

$$(8)\qquad A(x_1^{i-1},\ B(x_i^{\infty},\ y_1^{r}),\ y_{r+1}^{\infty}) = C(x_1^{j-1},\ D(x_j^{\infty},\ y_1^{s}),\ y_{s+1}^{\infty}),$$

are given by the following relations:

I_1 $(i=j,\quad r=s\geqslant 0)$:

$$D=\theta B,\quad A(x_1^{i-1},\ z,\ y_{r+1}^{\infty}) = C(x_1^{i-1},\ \theta z,\ y_{r+1}^{\infty}),$$

I_2 $(i=j,\quad s>r\geqslant 0)$:

$$D(x_i^{\infty},\ y_1^{s}) = K(B(x_i^{\infty},\ y_1^{r}),\ y_{r+1}^{s}),$$

$$A(x_1^{i-1},\ z,\ y_{r+1}^{\infty}) = C(x_1^{i-1},\ K(z,\ y_{r+1}^{s}),\ y_{s+1}^{\infty}),$$

$|K| = s-r+1,$

II_1 $(i<j,\quad r\geqslant s\geqslant 0)$:

$$B(x_{i+1}^{\infty},\ y_1^{r}) = K(x_i^{j-1},\ D(x_j^{\infty},\ y_1^{s}),\ y_{s+1}^{r}),$$

$$C(x_1^{j-1},\ z,\ y_{s+1}^{\infty}) = A(x_1^{i-1},\ K(x_i^{j-1},\ z,\ y_{s+1}^{r}),\ y_{r+1}^{\infty}),$$

$|K| = j-i+r-s+1,$

II_2 $(i<j,\quad s>r\geqslant 0)$:

$$A(x_1^{i-1},\ x,\ y_{r+1}^{\infty}) = K(x_1^{i-1},\ \alpha x \circ \beta(y_{r+1}^{s}),\ y_{s+1}^{\infty}),$$

$$B(x_i^{\infty},\ y_1^{r}) = \alpha^{-1}(\gamma(x_i^{j-1}) \circ \delta(x_j^{\infty},\ y_1^{r})),$$

$$C(x_1^{j-1},\ y,\ y_{s+1}^{\infty}) = K(x_1^{i-1},\ \gamma(x_i^{j-1}) \circ \varphi y,\ y_{s+1}^{\infty}),$$

$$D(x_j^{\infty},\ y_1^{s}) = \varphi^{-1}\delta(x_j^{\infty},\ y_1^{r}) \circ \beta(y_{r+1}^{s})),$$

$|K| = \omega,\quad |\delta| = \omega + r,\quad |\beta| = s-r,\quad |\gamma| = j-i.$

All operations on the right sides of these equalities are arbitrary, $(\circ)$-arbitrary binary group.

§ 3. Some remarks on ∞-quasigroups

1° We have introduced ∞-quasigroups of two types: ω and $\omega + k$. This fact suggests a further generalization of ∞-guasigroups. Here we use some notations and results from [3]. We shall use the notations X_α, Y_β, Z_γ and so on, for linear ordered sequences of the order types α, β, γ, respectively. Let Q be an arbitrary set. The set of all sequences of the order type α of elements from Q we denote by Q^{α}. The set Q together with the mapping $A: Q^{\alpha} \to Q$ is called operative of the type α, and we write $A(X_{\alpha}) = y$, where X_{α} is the set of variables of the type α, and the operative will be denoted as usual by $Q(A)$. We shall say that the operation (mapping) A has the type α which will be denoted by $|A|$, i.e. $|A| = \alpha$.

Now let us introduce the notion of quasigroup of the type α. Let X_{α} be a linear ordered sequence of variables of the type α, and let x be an arbitrary variable from X. Let X_{α_1} be the set of all variables from X_{α} which are less than x and X_{α_2} the set of all elements from X_{α} which are greater than x. Both X_{α_1} and X_{α_2} are linear ordered and they have the order types α_1 and α_2 respectively. Hence α can be represented as $\alpha_1 + 1 + \alpha_2$. The operative $Q(A)$ of the type α is a quasigroup if the equation

$$A(C_{\alpha_1}, x, C_{\alpha_2}) = b, \tag{21}$$

where C_{α_1} and C_{α_2} are arbitrary linear ordered sequences from Q^{α_1} and Q^{α_2} respectively, b is an arbitrary element from Q, has a unique solution.

The definitions of the ∞-quasigroups of the type ω and $\omega + k$ are particular cases of the definition given above.

2° The type of a ∞-quasigroup generalizes the notion of the arity of finitary operation. The question which arises for infinitary case is what operations should ce considered equal. It is natural to consider two operations A and B, defined on the same set Q, equal if they have the same type α and $A(C_{\alpha}) = B(C_{\alpha})$ for all sequences C_{α} from Q^{α}.

This definition introduces a new kind of parastrophs which will be considered bellow (see 5°).

3° The notion of isotopy may be introduced in the same way as we did for n-ary quasigroups. It can be done for quasigroups of any type, but we restrict ourselves to quasigroups of the type ω. Two quasigroups B and A of the type ω defined on the same set Q are called isotopic if there exists a sequence $T = \alpha_0^{\infty}$ of permutations of Q such that $B(x_1^{\infty}) = \alpha_0^{-1} A(\{\alpha_i x_i\}_{i=1}^{\infty})$. As usually we introduce the notion of principal isotopy, isomorphism, autotopy, see [4]. The LP-isotopy can be also easily introduced. Let $\bar{a} = a_1^{\infty}$ be some sequence from Q^{ω}. The mapping $L_i(\bar{a}): x \to A(a_1^{i-1}, x, a_{i+1}^{\infty})$ is called the i-translation with respect to $\bar{a}$. Of course it is a permutation of Q. The isotopy $T = \alpha_0^{\infty}$ where $\alpha_0 = 1$, $\alpha_i = L_i^{-1}(\bar{a})$ is called LP-isotopy. As for finitary case we can prove that every LP-isotope of a quasigroup (of the type ω) is a loop of the type ω with the unity $e = A(a_1^{\infty})$. The usual theoremes for isotopy are also true for infinitary case.

4° Let $Q(A)$ be a given quasigroup of the type α. The equation (21) defines an inverse quasigroup operation. The notion of an inverse quasigroup operation is more clear for the quasigroup operations of the type ω. We consider the equation $A(a_1^{i-1}, x, a_{i+1}^{\infty}) = b$. This equation defines a new operation ${}^{(i)}A$ as following: $x = {}^{(i)}A(a_1^{i-1}, b, a_{i+1}^{\infty})$. The inverse operation is a particular case of a parastroph, see [4], [5].

Here we shall define the parastroph of an infinitary quasigroup in an other way which is equivalent to usual one for finitary case.

Let A be a ω-quasigroup (i.e. a quasigroup of the type ω) defined on the set Q. The ω-quasigroup B is finitely symmetric to A if $B(x_1^\infty) = A(x_{\sigma 1}^{\sigma n}, x_{n+1}^\infty)$ where $x_{\sigma 1}^{\sigma n}$ is a permutation of x_1^n. The quasigroup ${}^{(i)}B$ (of course of the type ω), where σ is an arbitrary permutation on finite number of places of the variables of A and i is an arbitrary positive integer, is called a parastroph of A and the mapping $A \to {}^{(i)}B$ is called parastrophy of A. It is easily seen that the usual properties of parastrophies (and connected notion of isostrophy) are true for the parastrophy of ω-quasigroups.

5° We may also consider the permutations on infinite places of variables. For example, let $Q(A)$ be a ω-quasigroup, $A(x_1^\infty) = y$. We may obtain the following operation A' as following:

$$A'(x_1, x_3, x_5, \ldots, x_2, x_4, x_6, \ldots) = y. \tag{22}$$

The operation A' has the type $\omega + \omega$. If we take in account our definition for the equality of two ∞-quasigroups we cannot say that A and A' defined in (22) are equal. Hence we need a new more adequate definition of the equality of two ∞-operations. On the other hand we may say that A' is a parastroph of A. In fact we have two kinds of parastrophy: one which does not change the type of the quasigroup and the second which does. This situation will arise when considering some kinds of functional equations on ∞-quasigroups.

6° Considering the functional equation (8) we note that in both sides of it we have a superposition of operations. The left side can be written in the following way:

$$(A \overset{i}{+} B)(x_1^{i-1}, x_i^\infty, y_1^r, y_{r+1}^\infty)$$

or

$$(A \overset{i}{+} B)(x_1^\infty, y_1^\infty).$$

Analogously the right side has the form

$$(C \overset{j}{+} D)(x_1^\infty, y_1^\infty),$$

i.e. the equation (8) can be written as

$$(A \overset{i}{+} B)(x_1^\infty, y_1^\infty) = (C \overset{j}{+} D)(x_1^\infty, y_1^\infty,)$$

or briefly, see the definition of the equality of two ∞-quasigroups,

$$A \overset{i}{+} B = C \overset{j}{+} D.$$

One of the authors has considered in [4], [5], the insertion algebra for n-ary quasigroups. The elements of this algebra Σ are the finitary operations defined on some set Q and the operations of Σ are the superpositions $(\overset{i}{+})$, $i = 1, 2, \ldots$.

We can extend the notion of insertion algebra for infinitary case. But the axioms of this algebra should be changed. For example the first axiom of the insertion algebra, considered in [5], page 11, is the following:

$$|A \overset{i}{+} B| = |A| + |B| - 1 \tag{23}$$

where, as we know, $|A|$ means the arity of A. If we restrict ourselves to the infinitary operations whose types are order types of well ordered sets, then (23) should be replaced by the following four axioms. Let $|A|=\alpha$, $|B|=\beta$, α, β are the types (for infinitary case) or arities (for finitary case) of A and B respectively. We have:

1) $|A\overset{i}{+}B|=|B|+|A|$, if $\alpha, \beta\geqslant\omega$,

2) $|A\overset{i}{+}B|=|B|+|A|-i$ if $\alpha<\omega$, $\beta\geqslant\omega$,

3) $|A\overset{i}{+}B|=|A|$, if $\alpha\geqslant\omega$,, $\beta<\omega$,

4) $|A\overset{i}{+}B|=|A|+|B|-1$, if $\alpha<\omega$, $\beta<\omega$.

Let us prove the relation 1). By the definition of $(\overset{i}{+})$ we have

$$A(x_1^{i-1},\ B(X_\beta),\ Y_\alpha)=(A\overset{i}{+}B)(x_1^{i-1},\ X_\beta,\ Y_\alpha), \tag{24}$$

where X_β and Y_α are well ordered sequences of the types β and α respectively. From (24) it follows:

$$|A\overset{i}{+}B|=(i-1)+\beta+\alpha=\beta+\alpha,$$
$$|A|=(i-1)+1+\alpha=\alpha,$$
$$|B|=\beta.$$

Hence, $|A\overset{i}{+}B|=\beta+\alpha=|B|+|A|$. We note here that we cannot write $|A\overset{i}{+}B|=$ $=|A|+|B|$ because the infinite types α and β do not commute.

The proof of 2)—4) is similar.

We note also that the superposition on infinite place can also be considered.

At the end we formulate some problems on ∞-quasigroups:

1. Solve the functional equation $A\overset{i}{+}B=C\overset{j}{+}D$ when the types are different from $\omega+k$.

2. Discuss the both notions of parastrophy given in 5°.

3. Define the (i, j)-associativity for the ∞-quasigroups of the types $\neq\omega$ and find its structure (of course if they exist).

4. Find other examples of ∞-quasigroups. Note that the example from §1 is not effective one.

5. Does there exist ∞-quasigroups satisfying some known identities? We note that the example of ∞-loop given in §1, which has the property that all elements are unity elements, satisfies the identities $A(\overset{k}{x}, y, \overset{\infty}{x})=y$, for all $x, y\in Q$, $k=1, 2, \ldots$.

6. Construct the theory of insertion algebras for infinitary case.

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