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BANACH'S MAPPINGS AND SOME GENERALIZATIONS

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Introduction

Applications having all characteristic properties mentioned in the well-known Banach's fixed-point theorem [1] are usually called Banach's applications. Here we shall consider applications of the form $T: X^k \to X (k \in N)$. We shall say that $\xi \in X$ is a fixed point of T if it is such a point for the application $\mathcal{T}(x) = T(x, x \ldots, x)$

Definition. Let X be a metric space and ρ its metric. We say that the application $T: X^k \to X (k \in N)$ is a F_λ-contraction in X if for all $x, y \in X^k$ there exist nonnegative numbers $\alpha(x, y)$, $\beta(x, y)$, $\gamma(x, y)$, $\delta(x, y)$ and $q_i(x, y)$ $(i = 1, \ldots, k)$ such that

$$(1) \quad \operatorname{Sup}_{x, y \in X^k} k \left\{ \alpha(x, y) + \left(1 + \frac{1}{k}\right)(\beta(x, y) + \gamma(x, y)) + \delta(x, y) + \sum_{i=1}^{k} q_i(x, y) \right\} = \lambda k < 1$$

and

$$(F_\lambda) \quad \rho[Tx, Ty] \leqq \alpha(x, y)\rho[u_1, Tx] + \beta(x, y)\rho[v_1, Ty] + \gamma(x, y)\rho[u_1, Ty] + \delta(x, y)\rho[v_1, Tx] + \sum_{i=1}^{k} q_i(x, y)\rho[u_i, v_i]$$

where $x = (u_1, \ldots, u_k)$, $y = (v_1, \ldots, v_k)$.

Main results

Theorem 1: *Let the application $T: X^k \to X (k \in N)$ be a F_λ-contraction in the complete metric space X. Then;*

(I) *The application $\mathcal{T}(x) \overset{\text{def}}{=} T(x, \ldots, x)$ has a unique fixed point $\xi \in X$.*

(II) *ξ is the limit of the sequence*

$$(2) \quad x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1}), \ (n \in N),$$

indenpendently of the choice of $\bar{x}_1 \overset{\text{def}}{=} (x_1, x_2, \ldots, x_k) \in X^k$

(III) *The convergence rapidity of the sequence* $\{x_{n+k}\}$ *to the point* ξ *can be appreciated by*

$$\rho[x_{n+k}, \xi] \leqq \frac{\theta^n}{1-\theta} \max_{i=1,2,\ldots,k} \left\{ \frac{\rho(x_i, x_{i+1})}{\theta^i} \right\}, \ \theta \in (0, 1); \qquad n = 1, 2, \ldots.$$

Proof. We show firstly that (2) is a Cauchy sequence. Since $T: X^k \to X (k \in N)$ is a F_λ-contraction, we have by definition

$$\rho[x_{n+k}, x_{n+k+1}] = \rho[T(x_n, \ldots, x_{n+k-1}), T(x_{n+1}, \ldots, x_{n+k})] \leqq$$

$$\leqq \alpha(\bar{x}_n, \bar{x}_{n+1}) \rho[x_n, x_{n+k}] + \beta(\bar{x}_n, \bar{x}_{n+1}) \rho[x_{n+1}, x_{n+k+1}] +$$

$$+ \gamma(\bar{x}_n, \bar{x}_{n+1}) \rho[x_n, x_{n+k+1}] + \delta(\bar{x}_n, \bar{x}_{n+1}) \rho[x_{n+1}, x_{n+k}] +$$

$$+ \sum_{i=1}^{k} q_i(\bar{x}_n, \bar{x}_{n+1}) \rho[x_{n+i-1}, x_{n+i}]$$

where $\bar{x}_n \overset{\text{def}}{=} (x_n, \ldots, x_{n+k-1})$ and this inequality implies

(3) $$(1-\beta-\gamma) \rho[x_{n+k}, x_{n+k+1}] \leqq (q_1+\alpha+\gamma) \rho[x_n, x_{n+1}] +$$

$$+ (\max_{i=2,\ldots,k} q_i + \alpha + \beta + \gamma + \delta) \{\rho[x_{n+1}, x_{n+2}] + \cdots + \rho[x_{n+k-1}, x_{n+k}]\}$$

for $n \in N$.

Hence,

$$\rho[x_{n+k}, x_{n+k+1}] \leqq \frac{1}{1-\beta-\gamma} \Big\{ (q_1+\alpha+\gamma) \rho(x_n, x_{n+1}) +$$

$$(\max_{i=2,\ldots,k} q_i + \alpha + \beta + \gamma + \delta) \sum_{i=2}^{k} \rho(x_{n+i-1}, x_{n+i}) \Big\} \leqq$$

$$\leqq \frac{\alpha+\beta+\gamma+\delta+\sum_{i=1}^{k} q_i}{1-(\beta+\gamma)} \sum_{i=1}^{k} \rho[x_{n+i-1}, x_{n+i}] \left(\sum_{i=1}^{k} \rho[x_{n+i-1}, x_{n+i}] \overset{\text{def}}{=} \sum \right) \leqq$$

$$\leqq \frac{\lambda - \frac{1}{k}(\beta+\gamma)}{1-(\beta+\gamma)} \sum = \frac{\lambda - \frac{1}{k} + \frac{1}{k}[1-(\beta+\gamma)]}{1-(\beta+\gamma)} \sum = \left[\frac{\lambda - \frac{1}{k}}{1-(\beta+\gamma)} + \frac{1}{k} \right] \sum \leqq$$

$$\leqq \left(\frac{\lambda - \frac{1}{k}}{1-\lambda} + \frac{1}{k} \right) \sum = \left(\lambda + \frac{\lambda - \frac{1}{k}}{1-\lambda} + \frac{1}{k} - \lambda \right) \sum =$$

$$= \left[\lambda + \frac{\lambda k - 1 + 1 - \lambda - \lambda k + \lambda^2 k}{k(1-\lambda)} \right] \sum = \left[\lambda + \frac{\lambda(\lambda k^{-1})}{k(1-\lambda)} \right] \sum \leqq \lambda \sum$$

i.e.

(4) $$\rho[x_{n+k}, x_{n+k+1}] \leqq \lambda \{\rho[x_n, x_{n+1}] + \cdots + \rho[x_{n+k-1}, x_{n+k}]\}$$

Now we use the following lemma, proved in [9]

Lemma. 1°. *Let the application* $f: R^k \to R$ ($k \in N$ and fixed) *be homogeneous and increasing with respect to every its real argument, and let* (x_n) *be a sequence of nonnegative numbers satisfying the condition*

$$x_{n+k} \leqq f(a_1 x_n, \ldots, a_k x_{n+k-1}) \quad (n=1, 2, \ldots,),$$

where $a_1, \ldots, a_k$ *are nonnegative constants. Then exists a positive number* θ *such that*

$$(5) \qquad x_n \leqq \mathscr{L}\, \theta^n \, (n=1, 2, \ldots), \quad \mathscr{L} = \max_{i=1, \ldots, k} \left\{ \frac{x_i}{\theta^i} \right\}$$

2°. *Especially, if* f *is continuous and*

$$f(a_1, \ldots, a_k) < 1$$

then we have (5) *with* $\theta \in (0, 1)$.

A remark. This lemma was proved in [9], and it is based upon a lemma of S. B. Prešić [2].

Applying this lemma with $f = \lambda \sum_{i=1}^{k} x_i$ to the sequence $(\rho[x_n, x_{n+1}])$, we obtain, according to (5)

$$\rho[x_n, x_{n+1}] \leqq \theta^n \max_{i=1, 2, \ldots, k} \left\{ \frac{\rho[x_i, x_{i+1}]}{\theta^i} \right\} \quad (n \in N;\ \theta \in (0, 1))$$

Hence, for $n, s \in N$

$$(6) \qquad \rho[x_n, x_{n+s}] \leqq \sum_{j=1}^{s} \rho[x_{n+j-1}, x_{n+j}] \leqq \max_{i=1, 2, \ldots, k} \left\{ \frac{\rho[x_i, x_{i+1}]}{\theta^i} \right\} \sum_{j=1}^{k} \theta^{n+j-1} \leqq$$
$$\leqq \frac{\theta^n}{1-\theta} \max_{i=1, \ldots, k} \left\{ \frac{\rho[x_i, x_{i+1}]}{\theta^i} \right\},$$

which implies that (x_n) is a Cauchy's sequence. Hence, the metric space X being complete, there exists

$$\xi = \lim_{n \to \infty} x_n.$$

Let us prove that ξ is a fixed point of T (in the sense precised above). We get, according to our hypotheses on T,

$$\rho[x_{n+k}, T(\xi, \ldots, \xi)] = \rho[T(x_n, \ldots, x_{n+k-1}), T(\xi, \ldots, \xi)] \leqq$$
$$\leqq \alpha(\overline{x_n}, \overline{\xi})\, \rho[x_n, x_{n+k}] + \beta(\overline{x_n}, \overline{\xi})\, \rho[\xi, T(\xi, \ldots, \xi)] +$$
$$+ \gamma(\overline{x_n}, \overline{\xi})\, \rho[x_n, T(\xi, \ldots, \xi)] + \delta(\overline{x_n}, \overline{\xi})\, \rho[\xi, x_{n+k}] +$$
$$+ \sum_{i=1}^{k} q_i(\overline{x_n}, \xi)\, \rho[x_{n+i-1}, \xi] \leqq$$
$$\leqq \alpha\, \rho[x_n, x_{n+k}] + \beta\{\rho[\xi, x_{n+k}] + \rho[x_{n+k}, T(\xi, \ldots, \xi)]\} +$$

$$+\gamma\{\rho[x_n, x_{n+k}]+\rho[x_{n+k}, T(\xi, \ldots, \xi)]\}+\delta\rho[\xi, x_{n+k}]+$$
$$+\sum_{i=1}^{k} q_i\,\rho[x_{n+i-1}, \xi]$$

and further, similarly as in the above considerations,

$$\rho[x_{n+k}, T(\xi, \ldots, \xi)] \leqq$$
$$\leqq \lambda\left\{2\rho[x_n, x_{n+1}]+2\rho[\xi, x_{n+k}]+\sum_{i=1}^{k}\rho[x_{n+\ -1}, \xi]\right\}.$$

Hence,

$$\xi=\lim_{n\to\infty} x_{n+k}=T(\xi, \ldots, \xi)\overset{\text{def}}{=}\mathcal{T}(\xi)$$

which was to be proved.

We will prove, finally, that the fixed point ξ is unique. Let us suppose that $\xi^*\neq\xi$ is a fixed point too. Then

$$\rho[\xi, \xi^*]=\rho[T(\xi, \ldots, \xi), T(\xi^*, \ldots, \xi^*)]\leqq\alpha\rho[\xi, T(\xi, \ldots, \xi)]+$$
$$+\beta\rho[\xi^*, T(\xi^*, \ldots, \xi^*)]+\gamma\rho[\xi, T(\xi^*, \ldots, \xi^*)]+$$
$$+\delta\rho[\xi^*, T(\xi, \ldots, \xi)]+\sum_{i=1}^{k} q_i\,\rho[\xi, \xi^*]$$

and consequently

$$(1-\gamma-\delta)\,\rho[\xi, \xi^*]\leqq\rho[\xi, \xi^*]\sum_{i=1}^{k} q_i$$

i.e. since $\rho(\xi, \xi^*)>0$

$$1\leqq\gamma+\delta+\sum_{i=1}^{k} q_i$$

which contradicts (1). This contradiction proves our assertion.

Making $n\to\infty$ in (6), one gets (III).

Corollaries. If we specify the dimension $k\,(=1)$ and the parameters α, β, γ and q_i in the previous theorem, we can obtain two reciprocally incomparable results, due to Banach [1] and to Kannan [3] characterized by the inequalities

$$\rho[Tx, Ty]\leqq\alpha\rho[y, x]\ (x, y\in X;\ \alpha\in[0, 1))$$
$$\rho[Tx, Ty]\leqq\alpha\{\rho[x, Tx]+\rho[y, Ty]\}\ (x, y\in X,\ \alpha\in[0, {}^1/_2))$$

respectively.

In the same manner we get the following

Corollary. ([6], p. 29). Let (X, ρ) be a complete metric space and $T\colon X\to X$ an application with the property

$$\rho[Tx, Ty]\leqq\alpha(\rho[x, y])\,\rho[x, Tx]+\beta(\rho[x, y])\,\rho[y, Ty]+$$
$$+\gamma(\rho[x, y])\,\rho[x, y]$$

α, β, γ being nonnegative functions defined on $[0, +\infty)$ and satisfying

$$\sup_{t \geqq 0} [\alpha(t) + \beta(t) + \gamma(t)] < 1$$

then T has the unique fixed point.

This corollary comprises a result of *D. W. Boyd* and *S. W. Wang* [6] and also some results of *M. Edelstein* [7], *R. Rakotch* [8] and *Lj. Ćirić* [4]..

Some remarks

1. ([4]) For $k=1$, if (1) is replaced by the condition

$$\gamma = \delta, \ \sup_{x, y \in X} (\alpha + \beta + 2\gamma + q) = \lambda < 1$$

the assertions (I) and (II) are valid too, and the (III) is then estimated by

$$\rho[x_n, \xi] \leqq \frac{\lambda^n}{1-\lambda} \rho[x_1, x_2] \ (n \in N).$$

2. The following example shows that for $k=1$ our theorem is effectively more general than other theorems concerning this case.

Example. Let the application $T: X \to X$, where $X = [0,10]$ be given by $T(x) = \frac{1}{9} x \ (0 \leqq x < 1)$, $T(x) = \frac{3}{5} x \ (1 \leqq x \leqq 10)$. Then the conditions of theorems of *Banach* [1], *R. Kannan* [3] and *S. Reich* [5] are not satisfied, since:

(B) for $x = \frac{999}{1000}$, $y = \frac{1001}{1000}$

$$\rho[Tx, Ty] = \frac{2448}{5000} > 5 \cdot \frac{180}{90.000} = 5\rho[x, y]$$

(K) for $x=0, y=1$ we have

$$\rho[Tx, Ty] = 6 > \alpha(0+4) = \alpha\{\rho[x, Tx] + \rho[y, Ty]\}, \ \alpha \in [0, 1/2)$$

(R) for the same points as in (B),

$$\rho[Tx, Ty] = \frac{2448}{5.000} > \frac{888}{1.000}\alpha + \frac{2002}{5.000}\beta + \frac{180}{90.000} q_1$$

$$= \alpha\rho[x, Tx] + \beta\rho[y, Ty] + q_1 \rho[x, y]; \ \alpha + \beta + q_1 \in [0,1),$$

while our condition of F_λ-contraction is satisfied with

$$\alpha = \frac{1}{9}, \ \beta = \gamma = \frac{1}{20}, \ \delta = q_1 = \frac{1}{3}$$

3. In Theorem 1 the condition (F_λ) of F_λ-contractivity can be replaced by the weaker one:

$$\begin{aligned}(F_\lambda^*)\qquad &\rho(T(x_1,\ldots,x_k),T(x_2,\ldots,x_{k+1}))\leqq\alpha\rho[x_1,T(x_1,\ldots,x_k)]+\\&+\beta\rho[x_2,T(x_2,\ldots,x_{k+1})]+\gamma\rho[x_1,T(x_2,\ldots,x_{k+1})]+\\&+\delta\rho[x_1,T(x_1,\ldots,x_k)]+\sum_{i=1}^{k}q_i\,\rho[x_i,x_{i+1}]\end{aligned}$$

where

$$\sup\left\{\alpha+\left(1+\frac{1}{k}\right)(\beta+\gamma)+\delta+\sum_{i=1}^{k}q_i\right\}=\lambda<1$$

This alteration does not effect essentially the proof of Theorem 1. Theorem 1, under this weaker hypothesis, contains the following two results:

1° (S. Prešić [2]). *Let (X,ρ) be a complete metric space and $T: X^k\to X\,(k\in N)$ an application satisfying the condition*

$$\rho[T(u_1,u_2,\ldots,u_k),T(u_2,\ldots,u_{k+1})]\leqq q_1\,\rho[u_1,u_2]+\cdots+q_k\,\rho[u_k,u_{k+1}]$$

$$(u_1,u_2,\ldots,u_k,u_{k+1}\in X;\sum_{i=1}^{k}q_i<1;\ q_i\geqq 0\,(1\leqq i\leqq k).$$

Then every sequence (x_n) satisfying the condition

$$x_{n+k}=T(x_n,\ldots,x_{n+k-1}),\ (n\in N)$$

is convergent, and $\lim_{n\to\infty}x_n$ is the unique solution of the equation $x=T(x,x,\ldots,x)$.

2° *Let $f: R^k\to R$ satisfy the following condition:*

$$\begin{aligned}&|f(u_1,\ldots,u_k)-f(u_2,\ldots,u_{k+1})|\leqq\alpha|u_1-f(u_1,\ldots,u_k)|\\&+\beta|u_2-f(u_2,\ldots,u_{k+1})|+\gamma|u_1-f(u_2,\ldots,u_{k+1})|+\\&+\delta|u_2-f(u_1,\ldots,u_k)|+\sum_{i=1}^{k}q_i|u_i-u_{i+1}|\quad(u_1,\ldots,u_k,u_{k+1}\in R)\end{aligned}$$

where $\alpha+\left(1+\frac{1}{k}\right)(\beta+\gamma)+\delta+\sum_{i=1}^{k}q_i\leqq\lambda<1$. *Then every sequence (x_n) satisfying the condition*

$$x_{n+k}=f(x_n,\ldots,x_{n+k-1})\,(n\in N;\ x_1,\ldots,x_k\in R)$$

is convergent and $\lim_{n\to\infty}x_n$ is the unique solution of the equation $x=f(x,x,\ldots,x,x)$.

A particular example: the sequence (x_n) of real numbers defined by

$$6\,x_{n+2}=\frac{1}{1+x_{n+1}}+\frac{1}{1+x_n}\,(n=1,2,\ldots;x_1,x_2,\geqq 0).$$

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