PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 16 (30), 1973, pp. 159—168

RADIATION OF SOLIDS WITH THERMALLY CHANGEABLE CONDUCTIVITY

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(Communicated May 9, 1973)

1. Introduction

Many research workers are nowadays dealing with phenomenon of heat transfer. Their goal is to obtain a solution to agree with experimental results. The first mathematical formulation of this process has been given by Fourier [1], but it was immediately noticed that it has some disadvantages. For example, according to solution obtained by Fourier's relation it comes out that thermal disturbance expands with infinite speed what obviously does not correspond to reality. Therefore, many authors [2, 3] have proposed to change the form of Fourier's equation in order to avoid this physically unjustified phenomenon.

However, although the form of equation had been changed to agree, at least slightly, with physics of process, the boundary conditions stayed stationary. Even such simplification which has not a strong physical justificati on, especially when transfer of heat by radiation is in question, solutions have been found for small number of simpler cases [4,5]. The reason for this should be sought in the fact that the radiation of heat, as an aspect of its transfer, is a very complex electromagnetic phenomenon. For its study there are quantmechanical, statistical or phenomenological macroscopic approaches. All listed app roaches clearly point out to the fact that the quantity of heat on the face of solid, radiating the energy, is proportional to the fourth degree of temperature. This physical fact will play an important part in further work, because it shows up as the boundary condition of the task, the linearization of which, from the physical point of view, is senseless when ideally black solid is in question.

In addition to the mentioned difficulty which occurs owing to nonlinear mechanism of heat radiation, in accordance with real processes of heat transfer, the fact to which many experiments point out [6], must be also accepted. The fact is that the thermal properties of material also strictly depend upon the temperature variations, particulary in intensive processes. Therefore, in order to describe completely the temperature distribution in a solid during the radiation in accordance with experimental results, we face the fact that both differential equation and its boundary condition will be strictly of nonlinear character. Having all this in mind, Jaeger [7] has derived an equation of heat radiation in which it is assumed that thermal properties are dependent on temperature and that the speed of temperature disturbance is infinite.

As a consession to the mathematical difficulties which occur in solving thus described physical problem, we have consciously resorted to unreal simplification that thermal coefficients of material are all the time constant. In accordance with a real mechanism of heat thransfer by radiation and many experiments, this paper takes into account the change of conductivity with the variation of temperature. The results obtained under the assumption that the thermal coefficients are constant [8, 9] are obtained as special cases of our solutions.

The rule is to seek some analogy, even formal, with similar problem of classical mechanics, when a strong nonlinear process is studied. However, in this case to seek an analogy is not encouraging, but placing the problem in the scope of classical mechanics formalism is not useless, particularly when one bears in mind that this formalism has been used with a great advantage in miscellaneous fields of modern physics (quantmechanics, menagement, process optimisation, etc.). Beside these difficulties there is also a fact that the existance of solution of linear equations for heat transfer at standard boundary conditions has not been found yet, what directly shows that this whole physical problem is extremely unstandard.

Many mutually different problems of modern physics have been successfully described and unified through the variational principles of Hamilton's type. Direct application of variational principles in heat transfer is very complex for two reasons: 1. even for linear parabolic equation of heat transfer the exact Lagrange density does not exist. and so does not the integral of action which would be stationary during the process; 2. variational principles are developed to describe the problems of conservative mechanics. However, heat transfer as the most outstanding representative of irreversible processes is not suitable for direct applications of variational principles.

To overcome this difficulty, during last ten years, several variational approaches were proposed which are specially adapted for the application and obtaining of approximative solutions in heat transfer. We list some more important ones: Glansdorff-Prigogine principle of local potential [10, 11, 12], Biot's principle [13, 14, 15], Bateman's principle of conjugated functions, etc.

In studying the stated problem, chosen was a variational principle of Hamilton's type developed by B. Vujanović [9, 16—19] which is physically based on using a generalized equation of heat transfer with finite speed of thermal disturbance.

Modern approach to approximative solution of such complex physical problems requires to have as little number of parameters as possible in the very approximative solution, but sufficient for obtaining the solutions of satisfactory accuracy. Briefly, in choosing the form of an approximative solution, every information on the character of the process in the moment plays a decisive role in choosing the form of the solution.

The second imortant characteristic variational approximative method is in the fact that the process is describable through the finite number of time dependent generalized coordinantes which have strictly defined plysical sense.

2. Stating of the Problem

The main place in studying the heat transfer phenomenon takes the equation

$$\rho C(\theta)\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}\left[K(\theta)\frac{\partial\theta}{\partial x}\right], \tag{1}$$

where

$\theta(x, t)$ temperature of solid during the process

$c(\theta)$ thermal capacity of solid

$k(\theta)$ heat conductivity

with the initial condition

$$\theta(x, 0)=\theta_0 \tag{2}$$

where θ_0 is an arbitrary initial temperature of solid.

The choise of the boundary conditions is connected to the process under consideration and they may be stationary, nonstationary, dependent on the temperature gradient etc., but also depend on the configuration of the observed solid (finity slab, semi-infinity slab, etc.).

With regard that this paper studies the temperatures in a semi-infinite slab exposed to radiation, the conductivity of which depends on temperature, the boundary condition is taken

$$k(\theta)\frac{\partial\theta}{\partial x}=-h\left[(\theta+\theta_0)^m-T_0^m\right],\ for\ x=0 \tag{3}$$

where h is a constant, and T_0 is an initial ambient temperature. This problem with boundary condition (3), taken like this, have been studied by many authors [8, 9, 15, 19], but under the assumption that thermal characteristics of the material during the process are constant [8, 9], or that thermal capacity of solid depends on temperature, while conductivity stays constant [19].

This paper considers the possibility to obtain an approximate solution of equation (1) with the initial condition (2) and boundary condition (3), when thermal capacity of semi-infinite slab is constant and heat conductivity is function of temperature. To obtain this solution a Hamilton's variational principle developed by B. Vujanović [9, 16—19] has been chosen.

3. Approximate solution

The variational method by B. Vujanović prescribes to find such a Lagrangian so that the variation of action integral, in accordance with Hamilton's principle, and equalized with zero under specified conditions, gives the process equation (1).

Lagrangian which occurs in the action integral to obtain equation (1) is of the form

$$\mathscr{L}=\left[\frac{k^2(\theta)}{2}\left(\frac{\partial\theta}{\partial x}\right)^2-\frac{\tau}{2}\rho\,c(\theta)\,k(\theta)\left(\frac{\partial\theta}{\partial x}\right)^2\right]e^{\frac{t}{\tau}}$$

where τ is relaxation time, so action integral is

$$(4)\qquad I=\int_{t_1}^{t_2}\int_{x}\left[\frac{k^2(\theta)}{2}\left(\frac{\partial\theta}{\partial x}\right)^2-\frac{\tau}{2}\rho\, c(\theta)\, k(\theta)\left(\frac{\partial\theta}{\partial t}\right)^2\right]e^{\frac{t}{\tau}}\,dx\,dt$$

Recent measurements [6] have shown that thermal coefficients depend on temperature and that dependance is of the form

$$(5)\qquad a+b\,\theta$$

where a and b are constants characterising material. For example they are for copper

$$k=241-0{,}019\,\theta$$

$$\rho\, c=\ 50+0{,}0086\,\theta$$

With regard that in this paper we observe a solid the thermal capacity of which is constant, $c(\theta)=c_0=\text{const}$, and owing to experimental data (5) we accept the linear temperature dependances of conductivity with form

$$(6)\qquad k(\theta)=k_0\left(1+\beta\frac{\theta}{\theta_0}\right)$$

Let the approximative solution of the problem have the form

$$(7)\qquad \theta=-(\theta_0-q_1)\left(1-\frac{x}{q_2}\right)^n$$

$$\theta=T-\theta_0$$

where q_1 and q_2 are the generalized coordinates depending only on time t, and are: q_1 – temperature of the solid on a free surface, q_2 — penetration depth, while T is the temperature variation of the ambient.

Such form of an approximate solution has not been taken unintentionally. The researches have shown that the approximative solutions in which penetration depth is taken as a generalized coordinate are of polynominal form according to that coordinate. That solution has been adopted by Lardner [15], and Rafalsky and Zyskowsky [8], but for the case when $n=2$ (quadratic distribution). A general case, when n has no special value, has been considered by B. Vujanović and Đ. Đukić [19] who showed that their solutions very well agree with those of Rafalsky and Zyskowski.

From the assumed approximative solution (7) derivatives which occur in action integral (4) are easily determinable

$$(8)\qquad \frac{\partial\theta}{\partial x}=\frac{n(\theta_0-q_1)}{q_2}\left(1-\frac{x}{q_2}\right)^{n-1}$$

$$(9)\qquad \frac{\partial\theta}{\partial t}=\left(1-\frac{x}{q_2}\right)^{no}\dot{q}_1-\frac{n(\theta_0-q_1)}{q_2^2}\left(1-\frac{x}{q_2}\right)^{n-1}x\,\dot{q}_2$$

Partial derivatives which occur in (14) are easily determinable from (13) so that at the end, after some manipulations. it obtains the form

$$K_0\left(\frac{h}{K_0}\right)^2 \frac{(q_1{}^m - T_0{}^m)^2}{\left[1-\frac{\beta}{\beta_0}(\theta_0-q_1)\right]^2}\left[\frac{1}{2n-1}-\frac{2}{3n-1}\frac{\beta}{\theta_0}(\theta_0-q_1)+\frac{1}{4n-1}\frac{\beta_2}{\theta_0{}^2}(\theta_0-q_1)^2\right]-$$

$$-\rho C_0\left\{-\frac{1}{2n+1}(\theta_0-q_1)\,\dot{q}_1+\frac{2n}{4n^2-1}\frac{(\theta_0-q_1)^2}{q_2}\dot{q}^2-\right.$$

(15)

$$\left.-\frac{\beta}{\theta_0}(\theta_0-q_1)^2\left[-\frac{2}{3}\frac{1}{3n+1}\dot{q}_1+\frac{4}{3}\frac{n}{9n^2-1}\frac{(\theta_0-q_1)}{q_2}\dot{q}_2\right]\right\}=0.$$

Using connection (15) becomes an ordinary first order differential equation

$$-\frac{1}{2n+1}+\frac{2n}{4n^2-1}\left[\frac{2\beta(1-z)-1}{1-\beta(1-z)}-m\frac{z^{m-1}(1-z)}{z^m-z_0{}^m}\right]-$$

$$-\beta(1-z)\left\{-\frac{2}{3}\frac{1}{3n+1}+\frac{4}{3}\frac{n}{9n^2-1}\left[\frac{2\beta(1-z)-1}{1-\beta(1-z)}-m\frac{z^{m-1}(1-z)}{z^m-z_0{}^m}\right]\right\}=$$

(16)

$$=\frac{(z^m-z_0{}^m)^2}{(1-z)[1-\beta(1-z)]^2}\left[\frac{1}{2n-1}-\frac{2}{3n-1}\beta(1-z)+\frac{1}{4n-1}\beta^2(1-z)^2\right]\frac{d\tau}{dz}$$

where

(17) $$z=\frac{q_1}{\theta_0},\; z_0=\frac{T_0}{\theta_0}\text{ and }\tau=\frac{h^2\theta_0^{2(m-1)}}{K_0\rho C_0}$$

Equation (16) is very suitable for numerical integration, because it separates variables, and initial condition given through (2) are

(18) $$t_0=0,\;\tau=0,\;z=1$$

If we put that: $\beta=0$; $m=4$; $n=2$ and $z_0=0$ after integration we obtain

$$(1-70\,\tau)\,z^8+21\,z^2-50\,z+28=0$$

When one takes the same values of parameters the following term may be obtained in Rafalsky and Zyskowsky's paper [8]

$$(1-88{,}2\,\tau)\,z^8+24{,}5\,z^2-57\,z+31{,}5=0$$

what, with regard that the approximative solution is discussed, represents a sufficient equality.

4. Analyses of obtained results

Equation (16) contains four parameters upon which the solution will depend. Parameter z_0 as the ratio of the initial ambient temperature and initial temperature of solid defines the cases: 1. when the solid radiates energy into ambient, $z_0 < 1$, and 2. when the solid is exposed to radiation, $z_0 > 1$. Parameter m defines the model of solid under study, because when $m = 1$, a gray solid is observed, and when $m = 4$, an ideally black solid is in question. Both parameters are closely bound to the physical picture of the process in consideration. Fig. 1 shows a very good agreement between results obtained in this paper (dotted line) through equation (16), and for the case when $n = 2$ and $\beta = 0$, with results obtained by Rafalsky [8] (full line).

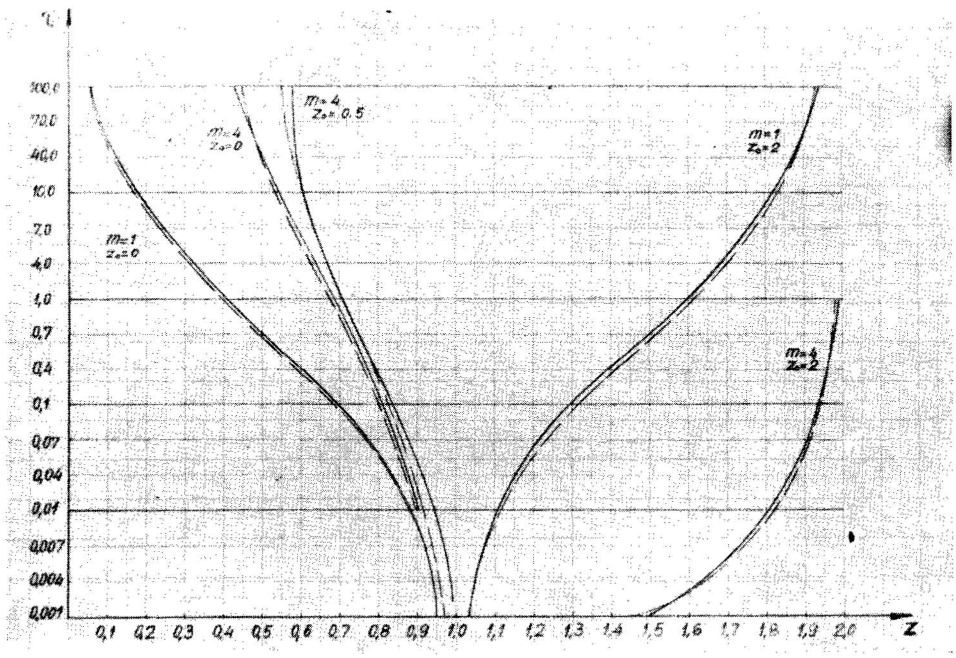


Fig. 1

Besides, it is interesting to study the influence of the exponent n of approximative solution (7), what is given in Figs. 2 and 3 for the case $\beta = 0$.

We can immediately see that the exponent n does not influence essentially to the approximative solution either gray or ideal black solid is in question.

Parameter β defines the material of solid under consideration and its influence to the change of surface temperature is given in Figs. 4 and 5.

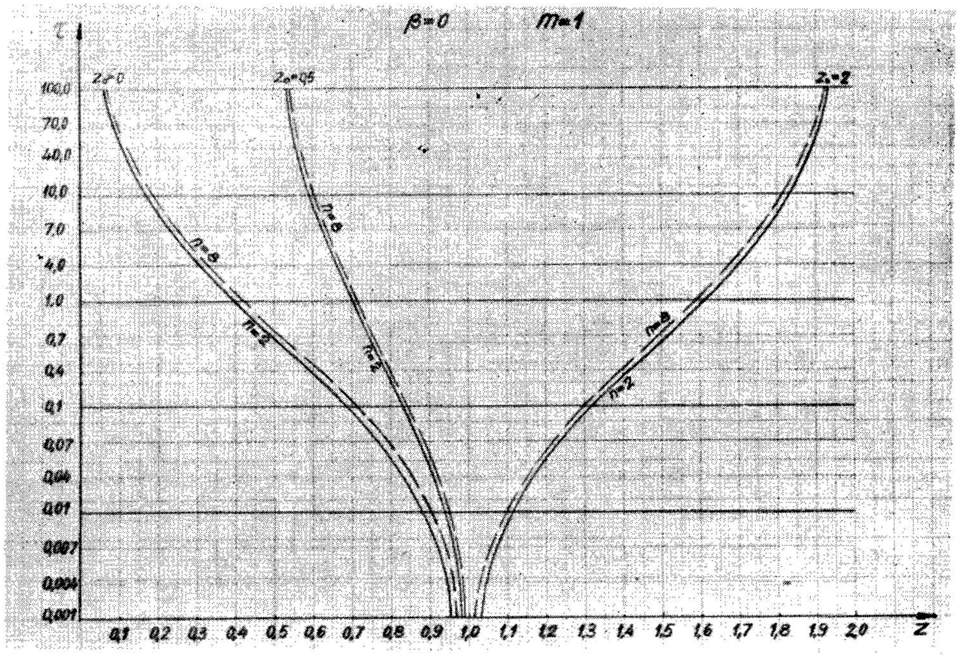
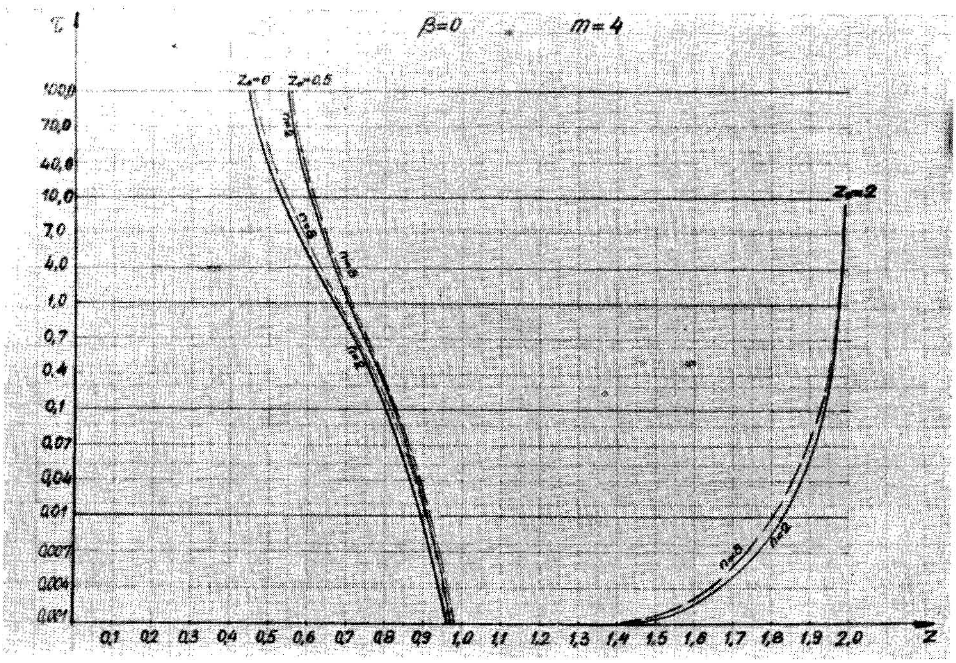

β=0 m=1
τ
Z
Z_0=0
Z_0=0.5
Z_0=2
n=8
n=2
100,0
70,0
40,0
10,0
7,0
4,0
1,0
0,7
0,4
0,1
0,07
0,04
0,01
0,007
0,004
0,001
0,1 0,2 0,3 0,4 0,5 0,6 0,7 0,8 0,9 1,0 1,1 1,2 1,3 1,4 1,5 1,6 1,7 1,8 1,9 2,0

Fig. 2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τ
Z
Z_0=0
Z_0=0.5
Z_0=2
n=8
n=2
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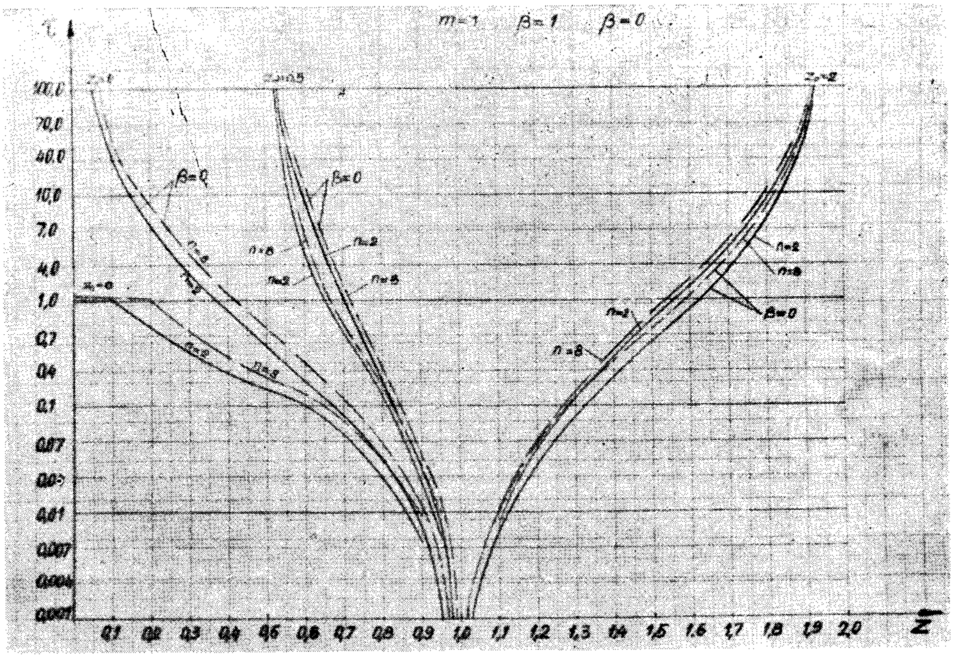
70,0
40,0
10,0
7,0
4,0
1,0
0,7
0,4
0,1
0,07
0,04
0,01
0,007
0,004
0,001
0,1 0,2 0,3 0,4 0,5 0,6 0,7 0,8 0,9 1,0 1,1 1,2 1,3 1,4 1,5 1,6 1,7 1,8 1,9 2,0

Fig. 3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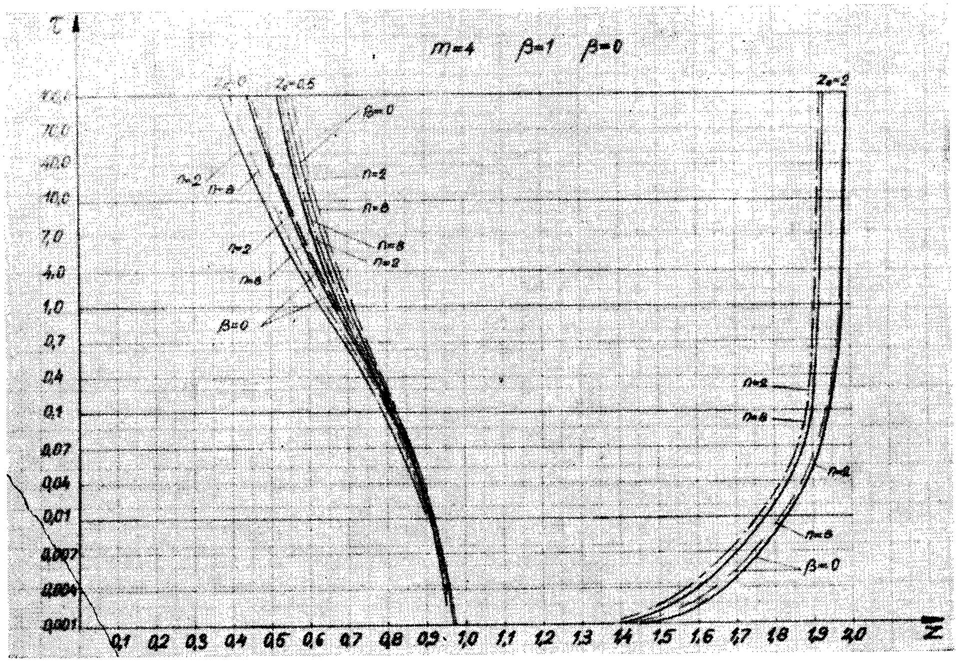
τ
Z
β=0
n=2
n=8
0,1 0,2 0,3 0,4 0,5 0,6 0,7 0,8 0,9 1,0 1,1 1,2 1,3 1,4 1,5 1,6 1,7 1,8 1,9 2,0

Fig. 4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τ
Z
β=0
n=2
n=8
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Fig. 5

5. Conclusion

This paper shows that using the Hamilton's variational principle one may obtain very good results of processes of nonlinear irreversible thermodynamics. The greatest problem is the proper choice of Lagrange density in the action integral, while the boundary conditions are determined according to the shape of solid and of the process studied. Figures given in Section 4 clearly show that the ordinary differential equation obtained by Hamilton's variational principle from the nonlinear second order partial differential equation of parabolic type with nonlinear and nonstationary boundary condition gives very good approximate solutions. Besides, the observed solid is close to the real one, because it was assumed that the conductivity is a function of temperature.

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