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ON A SOLUTION OF THE NONLINEAR DIFFERENTIAL EQUATION OF MIKUSIŃSKI OPERATORS

$$W'(z) = -s^{\alpha} a(z) W^{n+1}(z)$$

Marija Skendžić

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In this paper I will give a solution for the first order nonlinear differential equation of Mikusiński operators [1] of this form;

$$W'(z) = -s^{\alpha} a(z) W^{n+1}(z) \tag{1}$$

s is differential operator, $\alpha > 0$ real number, $a(z)$ is a given numerical analytic function, n natural number and $W(z)$ is operational function of the complex variable z. This solution is in a part ω of complex plane, which contains the point $z=0$, a continuous operational function with continuous derivative and satisfy the condition $W(0)=I$ (I is the unit operator). Depending upon the function $a(z)$ the point $z=0$ cannot always be inner point of ω.

This problem was suggested to me by professor B. Stanković. The differential equations, in which these coefficients contain the differential operator are very important for operational calculus, and while the linear cases have been studied in many papers, the nonlinear cases have not been discussed so far. This problem has been one of my basic preoccupations in my thesis.

Theorem 1. *Let $a(z)$ be a numerical analytic function regular in a region ω_0 of complex plane, which contains the point $z=0$, and let be*

$$p(z) = \int_0^z a(u)\, du \text{ for } z \in \omega_0 .$$

If $p(z) \neq 0$ for $z \neq 0$ and if exists a region $\omega_0^* \subseteq \omega_0$ so that the point $z=0$ belongs to $\overline{\omega}_0^*$ ($\overline{\omega}_0^*$ is the adherence of ω_0^*), $|\arg p(z)| \leqslant \frac{\pi}{2}(1-\alpha)$ for $z \in \omega_0^* \setminus 0$ and $0 < \alpha \leqslant 1$, then the nonlinear differential equation (1) has in a part of complex plane $\omega = \{\omega_0 \setminus B(0)) \cup \omega_0^* \cup 0\}$, ($B(0)$ is some neighbourhood of zero) a solution $W(z)$ which is continuous function with continuous derivative and satisfy the condition $W(0)=I$.

This solution is of the form:

$$W(z)=\begin{cases}\sum_{k=0}^{\infty}\binom{-\frac{1}{n}}{k}\dfrac{I^{\alpha k+\frac{\alpha}{n}}}{[np(z)]^{k+\frac{1}{n}}} & z\in\omega\backslash 0\\ I & z=0\end{cases}\tag{2}$$

I is the integral operator.

The proof of the theorem 1 follows from the lemmas 1, 2, 3, 4, 5, 6 which we are just going to formulate and prove.

Lemma 1. *The numerical function of two variables $F(z,t)$ defined in $V=\{(z,t),\ z\in\omega_0\backslash 0,\ t\geqslant 0\}$ by a convergent series*

$$F(z,t)=\sum_{k=0}^{\infty}\frac{\binom{-\frac{1}{n}}{k}t^{\alpha k+\frac{\alpha}{n}+2}}{[np(z)]^{k+\frac{1}{n}}\Gamma\left(\alpha k+\frac{\alpha}{n}+3\right)}\tag{3}$$

$0<\alpha\leqslant 1$, n natural number, $p(z)$ satisfies the conditions of theorem 1; is continuous in V and has continuous partial derivative of z, which is of the form:

$$\frac{\partial F(z,t)}{\partial z}=-\sum_{k=0}^{\infty}\binom{-\frac{1}{n}}{k}\frac{t^{\alpha k+\frac{\alpha}{n}+2}\left(k+\frac{1}{n}\right)a(z)}{n^{k+\frac{1}{n}}p(z)^{k+\frac{1}{n}+1}\Gamma\left(\alpha k+\frac{\alpha}{n}+3\right)}\tag{4}$$

$z,t)\in V$.

Proof: The terms of series (3) and (4) are continuous functions in V and both series are uniformly convergent in every compact subset of V. Therefore the proposition of lemma 1 follows.

Lemma 2. *The Wright's function*

$$\Phi(a,b,z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)\Gamma(a+bk)}\tag{5}$$

has for $a>0$ and $b>0$ in every finite part of complex plane the form.

$$\Phi(a,b,z)=\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}w^{-a}e^{w+zw^{-b}}\,dw\tag{6}$$

$x_0>0$, w^{-b} is the principal branch.

Proof. We denote by

$$G(w)=w^{-a}e^{w+zw^{-b}}\tag{7}$$

If we start from the identity

$$\int_C G(w)\, dw = 0 \tag{8}$$

where the contour C is given in fig. 1, we obtain

$$\int_{C_R} G(w)\, dw + \int_{BD} G(w)\, dw + \int_{DE} G(w)\, dw + \int_{EF} G(w)\, dw + \int_{FG} G(w)\, dw + \tag{9}$$
$$+ \int_{GA} G(w)\, dw = 0$$

C_R is the part of contour C which starts from A, passes round the origin and goes to B. The limit of all integrals in (9) except the limit of integrals $\int_{EF} G(w)\, dw$ and $\int_{C_R} G(w)\, dw$ is 0, for $R \to \infty$.

Indeed, from

$$\left| \int_{BD} G(w)\, dw \right| \leqslant \int_{\frac{\pi}{2}}^{\pi} R^{1-a} e^{R \cos t + |z| R^{-b} \cos(-bt + \arg z)}\, dt \tag{10}$$

we see that the integral $\int_{BD} G(w)\, dw \to 0$ when $R \to \infty$.

In the case of integral $\int_{GA} G(w)\, dw$ we have the same.

For the integral $\int_{DE} G(w)\, dw$ we have the following relation

$$\left| \int_{DE} G(w)\, dw \right| \leqslant \int_0^{x_0} (x^2 + R^2)^{-\frac{a}{2}} e^{x + |z| (\sqrt{x^2 + R^2})^{-b} \cos[-b \arg(x + iR) + \arg z]}\, dx$$

a is positive, so it is obvious that $\int_{DE} G(w)\, dw \to 0$, $R \to \infty$.

For the integral $\int_{GF} G(w)\, dw$ we have the same.

When $R \to \infty$ the contour C_R transforms in the contour C_1, which is given in fig. 2.

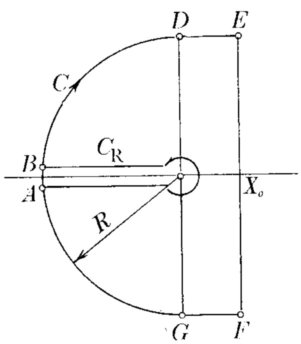

Fig. 1

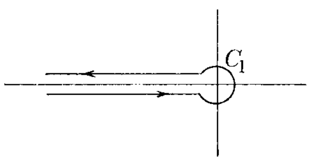

Fig. 2

Finally when $R \to \infty$, from (9) we have

$$\int_{C_1} G(w)\,dw = \int_{x_0-i\infty}^{x_0+i\infty} G(w)\,dw \tag{11}$$

Wright has proved [4] that the function $\Phi(a,b,z)$ can be writ n the form:

$$\Phi(a,b,z) = \frac{1}{2\pi i} \int_{C_1} G(w)\,dw \tag{12}$$

So from (11) follows the proof of lemma 2.

Lemma 3, *The functions* $F(z,t)$ *and* $\frac{\partial F(z,t)}{\partial z}$ *defined respectively by (3) and (4) can be for* $z \in \omega_0^* \backslash 0$ *and* $t \geqslant 0$ *written in the form*:

$$F(z,t) = \frac{t^{\frac{\alpha}{n}+2}}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} \frac{w^{-\left(3+\frac{\alpha}{n}\right)} e^w}{[np(z) + t^\alpha w^{-\alpha}]^{\frac{1}{n}}}\,dw \tag{13}$$

$x_0 > 0$.

$$\frac{\partial F(z,t)}{\partial z} = -\frac{t^{\frac{\alpha}{n}+2} a(z)}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} \frac{w^{-(3-\alpha)} e^w}{[np(z)\,w^\alpha + t^\alpha]^{1+\frac{1}{n}}}\,dw \tag{14}$$

$x_0 > 0$.

Proof. If we put $\Gamma(r) = \int_0^\infty e^{-x} x^{r-1}\,dx$ and if we exchange the limit processes of series and infinite integral in (3) we obtain:

$$F(z,t) = \int_0^\infty e^{-x} x^{\frac{1}{n}-1} t^{\frac{\alpha}{n}+2} \Gamma^{-1}\left(\frac{1}{n}\right) [np(z)]^{-\frac{1}{n}} \Phi\left(\frac{\alpha}{n}+3, \alpha, \frac{-t^\alpha x}{np(z)}\right) dx \tag{15}$$

(Φ is the Wright's function).

For every $z \in \omega_0^* \backslash 0$ is the function $p(z) \neq 0$. α is positive, so we can in (15) for the Wright's function Φ use the result of Lemma 2. If we exhange the limit processes of two infinite integrals the relation (15) takes the form (13).

If we repeat the same process starting from (4) we can obtain the relation (14).

The exchanges of limit processes in relations (3) and (15) are possible. Indeed. We denote by

$$v_k(x) = \frac{(-1)^k t^{\alpha k + \frac{\alpha}{n} + 2} e^{-x} x^{k + \frac{1}{n} - 1}}{k!\,\Gamma\left(\frac{1}{n}\right) [np(z)]^{k+\frac{1}{n}} \Gamma\left(\alpha k + \frac{\alpha}{n} + 3\right)} \tag{16}$$

$x > 0$, $t \geqslant 0$, $z \neq 0$, $\alpha > 0$, $n \in N$, $k = 0, 1, 2, \ldots$.

We can see that the following conditions are satisfied:

a) For every fixed $t \geqslant 0$ and $z \neq 0$ $v_k(x)$ are continuous functions in every interval $[x_1, x_2]$, $0 < x_1 < x_2 < \infty$.

b) For each $t \geqslant 0$ and $z \neq 0$ the series $\sum_{k=0}^{\infty} v_k(x)$ is uniformly convergent in every interval $[x_1, x_2]$, $0 < x_1 < x_2 < \infty$.

c) The series $\sum_{k=0}^{\infty} \int_0^{\infty} |v_k(x)|\, dx$ is convergent.

The property c) follows from the absolute convergence of series (3). If we use the theorem 1.77 ([3], p. 43), from a, b, c follows the possibility of exchange of limit processes in (3). In the case of relation (15) the exchange of limit processes is possible too. We denote by $H(x, y)$ the function of two variables x and y of the form:

$$H(x, y) = (x_0 + iy)^{-\left(\frac{\alpha}{n}+3\right)} x^{\frac{1}{n}-1} \exp\left[-x + x_0 + iy - \frac{t^\alpha x (x_0 + iy)^{-\alpha}}{np(z)}\right] \tag{17}$$

$x_0 > 0$, $-\infty < y < \infty$, $\alpha > 0$, n is natural number, $x > 0$, $z \neq 0$ and $z \in \omega_0^*$, $t \geqslant 0$. For every fixed t and z we can see that the following conditions are satisfied:

1) $H(x, y)$ is a continuous function in every rectangle $0 < x_1 \leqslant x \leqslant x_2 < \infty$, $-\infty < y_1 \leqslant y \leqslant y_2 < \infty$.

2) $\int_0^{\infty} H(x, y)\, dx$ is uniformly convergent in every interval $[y_1, y_2]$, $-\infty < y_1 < y_2 < \infty$.

Indeed. For $z \in \omega_0^* \backslash 0$ and $y \in [y_1, y_2]$ we have

$$|\arg p(z) + \alpha \arg (x_0 + iy)| \leqslant \frac{\pi}{2} \tag{18}$$

therefore

$$|H(x, y)| = \left(\sqrt{x_0^2 + y^2}\right)^{-\left(\frac{\alpha}{n}+3\right)} x^{-1+\frac{1}{n}} \exp\left\{-x + x_0 - \frac{t^\alpha x}{n} Re\left[\frac{1}{p(z)} (x_0 + iy)^{-\alpha}\right]\right\}$$

and

$$|H(x, y)| \leqslant e^{-x + x_0} x^{\frac{1}{n}-1} \left(\sqrt{x_0^2 + a^2}\right)^{-\left(\frac{\alpha}{n}+3\right)} \tag{19}$$

$a = \min(|y_1|, |y_2|)$.

The integral $\int_0^{\infty} e^{-x} x^{\frac{1}{n}-1}\, dx$ is convergent, and so from Weierstrass theorem follows the property 2).

3) $\int_{-\infty}^{\infty} H(x,y)\,dy$ is uniformly convergent in every interval $[x_1, x_2]$, $0<x_1<x_2<\infty$. This follows from the following inequality:

$$|H(x,y)| \leq x_1^{\frac{1}{n}-1} \left(\sqrt{x_0^2+y^2}\right)^{-\left(\frac{\alpha}{n}+3\right)} e^{x_0} \tag{20}$$

$0<x_1<x<x_2<\infty$, $-\infty<y<\infty$, and from the convergence of the integral

$$\int_{-\infty}^{\infty} \left(\sqrt{x_0^2+y^2}\right)^{-\left(\frac{\alpha}{n}+3\right)} dy$$

4) The repeated integral $\int_{-\infty}^{\infty} dy \int_{0}^{\infty} |H(x,y)|\,dx$ is convergent. Indeed. The relation (18) is true too for $z \in \omega_0{}^*\backslash 0$ and $-\infty<y<\infty$.
So from (19) we have:

$$\int_{-\infty}^{\infty} dy \int_{0}^{\infty} |H(x,y)|\,dx \leq \int_{-\infty}^{\infty} dy \int_{0}^{\infty} \left(\sqrt{x_0^2+y^2}\right)^{-\left(\frac{\alpha}{n}+3\right)} e^{x_0-x} x^{\frac{1}{n}-1} dx \leq$$

$$\Gamma\left(\frac{1}{n}\right) e^{x_0} \int_{-\infty}^{\infty} \left(\sqrt{x_0^2+y^2}\right)^{-\left(\frac{\alpha}{n}+3\right)} dy$$

The integral

$\int_{-\infty}^{\infty} \left(\sqrt{x_0^2+y^2}\right)^{-\left(\frac{\alpha}{n}+3\right)} dy$ is convergent, therefore the property 4) is satisfied.

If we now use the theorem 1.85 ([3], p. 53), then from 1). 2). 3), 4), follows the possibility of exchange of limit processes in (15).

Lemma 4. *The functions* $F(z,t)$ *and* $\frac{\partial F(z,t)}{\partial z}$ *defined respectively by (3) and (4) have the limit value in the point* $(0, t_0)$ $t_0 \geq 0$, *if the points* (z,t) *approach to* $(0,t_0)$ *so that* $t \geq 0$ *and* $z \in \omega_0{}^*\backslash 0$. *The limit values are respectively:*

$$\lim_{\substack{z\to 0\\ t\to t_0}} F(z,t) = \frac{t_0^2}{\Gamma(3)} = \bar{F}(0,t_0) \tag{21}$$

$$\lim_{\substack{z\to 0\\ t\to t_0}} \frac{\partial F(z,t)}{\partial z} = -a(0)\frac{t_0^{2-\alpha}}{\Gamma(3-\alpha)} \tag{22}$$

Proof. In this proof we use for $F(z,t)$ and $\frac{\partial F(z,t)}{\partial z}$ respectively the forms (13) and (14). After exchange of limit processes in the infinite integrals we obtain without difficulty the limit values (21) and (22). These exchanges of the limit processes are possible because the conditions of the theorem 69 ([2], p 289) are satisfied. To prove this in case of the function $F(z,t)$ we put $w=x_0+iy$ and we consider the function of three variables $G(z,t,y)$ of this form:

$$G(z,t,y)=\frac{t^{\frac{\alpha}{n}+2}(x_0+iy)^{-\left(3+\frac{\alpha}{n}\right)}e^{x_0+iy}}{[np(z)+t^{\alpha}(x_0+iy)^{-\alpha}]^{\frac{1}{n}}} \tag{23}$$

$z\in\omega_0^*\backslash 0$, $t\geqslant 0$, $-\infty<y<\infty$, $0<\alpha\leqslant 1$, $x_0>0$, n is natural number.

Condition I. The limit value $\lim G(z,t,y)=g(y)$, $z\to 0$, $t\to t_0$, exists uniformly in every interval $[y_1,y_2]$, $-\infty<y_1<y<y_2<\infty$. Indeed, for $t_0\neq 0$ we can find a neighbourhood $B(0,t_0)$ which does not contain the points $(z,0)$. For $(z,t)\in K$ where $K=B(0,t_0)\cap\{(z,t),\ z\in\omega_0^*\backslash 0,\ t\geqslant 0\}$, and $y\in[y_1,y_2]$ we have:

$$|G(z,t,y)-g(y)|= \tag{24}$$

$$=\left|\frac{e^{x_0+iy}\{(t^2-t_0^2)+t_0^2[1-(np(z)t^{-\alpha}(x_0+iy)^{\alpha}+1)^{\frac{1}{n}}]\}}{[np(z)t^{-\alpha}(x_0+iy)^{\alpha}+1]^{\frac{1}{n}}(x_0+iy)^3}\right|$$

and

$$|G(z,t,y)-g(y)|\leqslant \tag{25}$$

$$\leqslant\frac{e^{x_0}\left[|t^2-t_0^2|+\dfrac{t_0^2|np(z)t^{-\alpha}(x_0+iy)^{\alpha}|}{\left|\sum\limits_{k=0}^{n-1}[np(z)t^{-\alpha}(x_0+iy)^{\alpha}+1]^{\frac{k}{n}}\right|}\right]}{|np(z)t^{-\alpha}(x_0+iy)^{\alpha}+1|^{\frac{1}{n}}(\sqrt{x_0^2+y^2})^3}$$

Since $z\in\omega_0^*\backslash 0$, $|\arg p(z)|\leqslant\frac{\pi}{2}(1-\alpha)$ we can use the inequality:

$$\left|\sum_{k=0}^{n-1}[np(z)t^{-\alpha}(x_0+iy)^{\alpha}+1]^{\frac{k}{n}}\right|>1$$

and obtain a new inequality for (24)

$$|G(z,t,y)-g(y)|\leqslant$$

$$\leqslant\frac{e^{x_0}\{|t^2-t_0^2|+t_0^2n|p)z)|t^{-\alpha}(\sqrt{x_0^2+a^2})^{\alpha}\}}{(\sqrt{x_0^2+b^2})^3} \tag{26}$$

$a=\max(|y_1|,|y_2|)$, $b=\min(|y_1|,|y_2|)$.

From the inequality (26) and from the property of the function $p(z)$ follows immediately that the condition I is satisfied. In the case $t_0=0$ there are no new difficulties, because $G(z,o,y)=0$ and the limit value in the point $(0,0)$ is zero too.

Condition II. The function $g(y)$ is integrable in Riemann's sense.

Condition III. The integral $\int_{-\infty}^{\infty} G(z,t,y)\,dy$ is uniformly convergent on K. This condition is satisfied by the inequality

$$\int_{-\infty}^{\infty} |G(z,t,y)|\,dy \leqslant \int_{-\infty}^{\infty} \frac{T^2 e^{x_0}}{(\sqrt{x_0^2+y^2})^3}\,dy, T = \sup_{t \in K} t. \tag{27}$$

So the exchange of limit processes in this case is possible. To verify the relation (22) we verify the same conditions I, II, III. In this case it is very important to see that there exists a neighbourhood $B_p(0,t_0)$ of the point $(0, t_0)$ $t_0 \geqslant 0$, in which $|p(z)|$ is so small, that for $(z,t) \in K_p$, where $K_p = \{(z,t),\ z \in \omega_0{}^*\backslash 0,\ t \geqslant 0\} \cap B_p(0,t_0)$, the following inequality is true:

$$\left| \sum_{k=0}^{n-1} [np(z)(x_0+iy)^\alpha t^{-\alpha} + 1]^{(n+1)\frac{k}{n}} \right| > 1$$

Now let us introduce a function $U(z,t)$ of two variables

$$U(z,t) = \begin{cases} F(z,t), & z \in \omega\backslash 0 \\ & t \geqslant 0 \\ \dfrac{t^2}{(3)}, & z = 0 \\ & t \geqslant 0 \end{cases} \tag{28}$$

($F(z,t)$ is defined by (3)).

Lemma 5. *For $z \in \omega$ and $t \geqslant 0$ the functions $U(z,t)$ and $\frac{\partial U(z,t)}{\partial z}$ are continuous.*

Proof. If we use the lemmas 1 and 4, we must only prove the continuity of $\frac{\partial U(z,t)}{\partial z}$ in the points $(0,t_0)$, $t_0 \geqslant 0$. In the point $(0,0)$ the continuity is evident because $U(z,0)=0$ for $z \in \omega\backslash 0$, and $U(0,0)=0$. In the case of the point $(0,t_0)$, $t_0 \neq 0$ we have by definition

$$\left.\frac{\partial U(z,t)}{\partial z}\right|_{\substack{z=0\\ t=t_0}} = \lim_{z\to 0} \frac{U(z,t_0)-U(0,t_0)}{z}. \tag{29}$$

We can use that $z \to 0$ in $\omega_0{}^*\backslash 0$ and so by representation (13) for $F(z,t)$; (29) takes the form:

$$\lim_{z\to 0} \frac{1}{z} \left\{ \frac{t_0^2}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} \frac{w^{-3} e^w}{[np(z)\, t_0^{-\alpha} w^\alpha + 1]^{\frac{1}{n}}}\,dw - \frac{t_0^2}{\Gamma(3)} \right\} =$$

$$= \lim_{z\to 0} \frac{1}{z} \frac{t_0^2}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} \frac{e^w}{w^3} \frac{1-\sqrt[n]{np(z)\,t_0^{-\alpha}w^\alpha+1}}{\sqrt[n]{np(z)\,t_0^{-\alpha}w^\alpha+1}}\,dw =$$

$$= \lim_{z\to 0} \frac{t_0^{2-\alpha}}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} \frac{e^w\, n\,(p(z)-p(0))}{z\,w^{3-\alpha} \sum\limits_{j=0}^{n-1} [np(z)\,t_0^{-\alpha}w^\alpha+1]^{\frac{j+1}{n}}}\,dw. \tag{30}$$

If we now exchange the limit processes in (30) we have:

$$\left.\frac{\partial U(z,t)}{\partial z}\right|_{\substack{z=0\\ t=t_0}} = -\frac{t_0^{2-\alpha}a(0)}{\Gamma(3-\alpha)} = \lim_{\substack{z\to 0\\ t\to t_0}} \frac{\partial F(z,t)}{\partial z} \tag{31}$$

which proves the continuity of $\frac{\partial U(z,t)}{\partial z}$ in the points $(0, t_0)$, $t_0 \neq 0$.

We still have to prove that the exchange of the limit processes when we pass from (30) to (31) is possible.

By the theorem for exchange the limit processes ([2], p 289), the relation:

$$\lim_{z\to a} \int_{-\infty}^{\infty} R(z,y)\,dy = \int_{-\infty}^{\infty} \lim_{z\to a} R(z,y)\,dy \tag{32}$$

is true, if the following three conditions are satisfied:

I. $\lim\limits_{z\to a} R(z,y) = R_a(y)$ exists uniformly in every interval $[y_1, y_2]$, $-\infty < y_1 < y_2 < \infty$.

II. $R_a(y)$ is an integrable function in every interval $[y_1, y]$.

III. $\int_{-\infty}^{\infty} R(z,y)\,dy$ converges uniformly on $B(a)\backslash a$.

$B(a)$ is a neighbourhood of the point $z=a$.

In our case if we put $w = x_0 + iy$, $x_0 > 0$, $R(z,y)$ is:

$$R(z,y) = -\frac{t_0^{2-\alpha}\, e^{x_0+iy}\, np(z)}{z\,(x_0+iy)^{3-\alpha} \sum\limits_{j=0}^{n-1} [np(z)\,t_0^{-\alpha}(x_0+iy)^\alpha+1]^{\frac{j+1}{n}}} \tag{33}$$

and

$$\lim_{z\to 0} R(z,y) = -\frac{t_0^{2-\alpha}\, e^{x_0+iy}\, a(0)}{(x_0+iy)^{3-\alpha}} = R_0(y),\ z \in \omega_0^*\backslash 0 \tag{34}$$

To verify the condition I. we transform $|R(z,y) - R_0(y)|$ for $z \in B(0) \cap (\omega_0^*\backslash 0)$ and $y \in [y_1, y_2]$. $B(0)$ is a neighbourhood of the point 0 using so, that is for $z \in B(0) \cap (\omega_0^*\backslash 0)$

$$\left| \sum_{j=0}^{n-1} [np(z)\,t_0^{-\alpha}(x_0+iy)^\alpha+1]^{\frac{j+1}{n}} \right| > 1 \tag{35}$$

So we have

(36) $$|R(z,y)-R_0(y)|=$$

$$=\left|\frac{t_0^{2-\alpha}e^{x_0+iy}\left\{np(z)-a(0)z\sum_{j=0}^{n-1}[np(z)t_0^{-\alpha}(x_0+iy)^\alpha+1]^{\frac{j+1}{n}}\right\}}{(x_0+iy)^{3-\alpha}z\sum_{j=0}^{n-1}[np(z)t_0^{-\alpha}x_0+iy)^\alpha+1]^{\frac{j+1}{n}}}\right|\leqslant$$

$$\frac{t_0^{2-\alpha}e^{x_0}}{(\sqrt{x_0^2+y^2})^{3-\alpha}}\left\{n\left|\frac{\int_0^z[a u)-a(0)]\,du}{z}\right|+\left|a(0)\left(\sum_{j=0}^{n-1}[np(z)t_0^{-\alpha}(x_0+iy)^\alpha+1]^{\frac{j+1}{n}}-n\right)\right|\right\}$$

$$\leqslant\frac{t_0^{2-\alpha}e^{x_0}}{(\sqrt{x_0^2+y^2})^{3-\alpha}}\left\{n\left|\frac{\int_0^z[a(u)-a(0)]\,du}{z}\right|+\right.$$

$$\left.+|a(0)|\sum_{j=0}^{n-1}\left|[np(z)t_0^{-\alpha}(x_0+iy)^\alpha+1]^{j+1}-1\right|\right\}=$$

$$=\frac{t_0^{2-\alpha}e^{x_0}}{(\sqrt{x_0^2+y^2})^{3-\alpha}}\left\{|a(0)|\sum_{j=0}^{n-1}\left|\sum_{k=1}^{j+1}\binom{j+1}{k}[np(z)t_0^{-\alpha}(x_0+iy)^\alpha]^k\right|+\right.$$

$$\left.+n\left|\frac{\int_0^z[a(u)-a(0)]\,du}{z}\right|\right\}.$$

From (36) it is easy to see that the condition I is satisfied. The condition II is obviously satisfied.

To verify the condition III, let us take for a given $\varepsilon>0$ $K_\varepsilon=B_\varepsilon(0)\cap(\omega_0^*\backslash 0)$, where $B_\varepsilon(0)$ is a neighbourhood of $z=0$ in which is

$$\left|p'(0)-\frac{p(z)-p(0)}{z}\right|<\varepsilon \text{ and } B_\varepsilon(0)\subset B(0).$$

For $z\in K_\varepsilon$ and $-\infty<y<\infty$, we have:

(37) $$|R(z,y)|\leqslant\frac{t_0^{2-\alpha}e^{x_0}n|p(z)|}{|z|(\sqrt{x_0^2+y^2})^{3-\alpha}}\leqslant\frac{kt_0^{2-\alpha}e^{x_0}n}{(\sqrt{x_0^2+y^2})^{3-\alpha}}$$

$$k=\varepsilon+|a(0)|.$$

As the integral:

$\int_{-\infty}^{\infty}\frac{dy}{\sqrt{x_0^2+y^2})^{3-\alpha}}$ converges, the integral $\int_{-\infty}^{\infty}R(z,y)\,dy$ converges uniformly on K_ε, so the condition III is satisfied.

Lemma 6. *The operator function $W(z)$ defined by relation* (2) *is on ω continuous and has continuous operator derivative.*

Proof. It is not difficult to see, that is for $z \in \omega$ $\{U(z, t)\} = l^3 W(z)$ and $W'(z) = s^3 \left\{\frac{\partial U(z, t)}{\partial z}\right\}$.

In lemma 5. we have proved the continuity of the numerical functions

$$U(z, t) \text{ and } \frac{\partial U(z, t)}{\partial z} \text{ on } \{(z, t),\ z \in \omega,\ t \geqslant 0\}.$$

So by definition of continuity for an operator function and operator derivative the proposition of lemma 6 is a direct consequence of lemma 5.

Proof of Theorem 1. If we use the proposition of lemma 6, to prove Theorem 1 we still have to see that the function $W(z)$ for $z \in \omega$ satisfies the differential equation (1). The condition $W(0) = I$ is obviously satisfied. Using the relations:

$$\sum_{j=0}^{k} \frac{\Gamma(p+j)\,\Gamma(k-j+q)}{\Gamma(j+1)\,\Gamma(p)\,\Gamma(k-j+1)\,\Gamma(q)} = \frac{\Gamma(k+p+q)}{\Gamma(k+1)\,\Gamma(p+q)} \tag{38}$$

$p, q > 0$, and

$$\binom{-p}{k} = (-1)^k \frac{\Gamma(k+p)}{\Gamma(k+1)\,\Gamma(p)} \tag{39}$$

we have for $W^{n+1}(z)$ the relation:

$$W^{n+1}(z) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \binom{-\frac{1}{n}}{k} \frac{\left(k+\frac{1}{n}\right) l^{\alpha k + \alpha\left(1+\frac{1}{n}\right)}}{n^{k+\frac{1}{n}}\, p(z)^{k+1+\frac{1}{n}}} & z \in \omega \backslash 0 \\ I & z = 0. \end{cases} \tag{40}$$

After the multiplication of relation (40) with $-s^{\alpha} a(z)$ we have the relation for $W'(z)$. So the Th orem 1 is proved.

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