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ON SOME APPLICATIONS RELATED TO FOX'S H-FUNCTIONS OF TWO VARIABLES

Manilal Shah

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Summary

In a series of papers, the author has made an effort to study the unification, generalization and co-ordination of several interesting results scattered throughout the literature. Here the author presents a set of most general results related to Fox's H-functions of two variables which run as follow:

The Fourier Integration Formula:

$$\text{(i)} \qquad \int_0^c \left(\cos\frac{\pi\rho x}{c}\right)\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} H\begin{bmatrix} u\left(\tan\dfrac{\pi x}{2c}\right)^{2\sigma} \\ v \end{bmatrix} dx;$$

The Fourier Expansion Theorem:

$$\text{(ii)} \qquad \left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} H\begin{bmatrix} u\left(\tan\dfrac{\pi x}{2c}\right)^{2\sigma} \\ v \end{bmatrix}, \text{ and}$$

the solution of partial differential equation associated with a boundary value problem:

$$\text{(iii)} \qquad y(x,t) = \frac{4}{(2\pi)^\sigma}\sum_{\xi=1}^{\infty}\frac{(2\sigma)^{2\xi-1}}{\Gamma(2\xi)} H\left[f \left| \begin{matrix} (a_{p_1}, A_{p_1});\ (b_{q_1}, B_{q_1}) \\ \Psi_3, (c_{p_2}, C_{p_2}),\ \Psi_4;\ \Theta, (d_{q_2}, D_{q_2}) \\ (e_{p_3}, E_{p_3});\ (f_{q_3}, F_{q_3}) \end{matrix} \right| \begin{matrix} u \\ v \end{matrix} \right] \cos\frac{\pi\xi x}{c}\cos\frac{\pi\xi kt}{c}$$

for all positive integral values of σ and in terms of $H\begin{bmatrix} u \\ v \end{bmatrix}$. On appropriately specializing the parameters, the results generalize a large number of important

corollaries for the mathematical functions which arise in analysis and applied problems-the so-called special functions. Results on the Fourier double-integral--expansion analogues have been also recorded.

§ 1. Prerequisites

Fox's H-function of Two Variables:

Munot and Kalla [6, p. 68, (2)] have defined a generalized H-function by means of a double Mellin-Barnes integral in the form

$$(1.1)\qquad H\begin{bmatrix} x \\ y \end{bmatrix} \equiv H\left[\begin{matrix} \begin{bmatrix} m_1, & 0 \\ p_1-m_1, & q_1 \end{bmatrix} \\ \begin{pmatrix} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{pmatrix} \\ \begin{pmatrix} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{pmatrix} \end{matrix} \left| \begin{matrix} (a_{p_1}, A_{p_1});\ (b_{q_1}, B_{q_1}) \\ (c_{p_2}, C_{p_2});\ (d_{q_2}, D_{q_2}) \\ (e_{p_3}, E_{p_3});\ (f_{q_3}, F_{q_3}) \end{matrix} \right| \begin{matrix} x \\ y \end{matrix} \right] =$$

$$= \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi[\xi+\eta]\, \Psi(\xi, \eta)\, x^{\xi} y^{\eta}\, d\xi\, d\eta$$

where

$$\Phi[\xi+\eta] = \frac{\prod_{j=1}^{m_1} \Gamma[a_j + A_j\xi + A_j\eta]}{\prod_{j=m_1+1}^{p_1} \Gamma[1-a_j-A_j\xi-A_j\eta] \prod_{j=1}^{q_1} \Gamma[b_j+B_j\xi+B_j\eta]},$$

$$\Psi(\xi, \eta) =$$

$$= \frac{\prod_{j=1}^{m_2} \Gamma(1-c_j+C_j\xi) \prod_{j=1}^{n_2} \Gamma(d_j-D_j\xi) \prod_{j=1}^{m_3} \Gamma(1-e_j+E_j\eta) \prod_{j=1}^{n_3} \Gamma(f_j-F_j\eta)}{\prod_{j=m_2+1}^{p_2} \Gamma(c_j-C_j\xi) \prod_{j=n_2+1}^{q_2} \Gamma(1-d_j+D_j\xi) \prod_{j=m_3+1}^{p_3} \Gamma(e_j-E_j\eta) \prod_{j=n_3+1}^{q_3} \Gamma(1-f_j+F_j\eta)}$$

with $p_1 \geqslant m_1 \geqslant 0$, $p_2 \geqslant m_2 \geqslant 0$, $p_3 \geqslant m_3 \geqslant 0$, $q_1 \geqslant 0$, $q_2 \geqslant n_2 \geqslant 0$, $q_3 \geqslant n_3 \geqslant 0$, $q_1+q_2 \geqslant$ $\geqslant p_1+p_2$, $q_1+q_3 \geqslant p_1+p_3$ and each of p's, m's and n's is a non-negative integer.

The contours L_1, L_2, notations and convergence of (1.1) etc. are completely defined in [6] and we avoid such details.

When A's $=B$'s $=C$'s $=D$'s $= \cdots$ etc $=1$, (1.1) reduces to Meijer's G-function of two variables given by Sharma [7, p. 26—40].

The present paper is mainly divided in three parts, namely: (i) the Fourier--integral involving Fox's H-function in two arguments is evaluated in § 2. In § 3 and § 4 it is shown that such integral can be utilized to establish (ii) the Fourier-expansion theorem, and (iii) the method for solving boundary value problems in partial differential equations, with the aid of orthogonality-property of the trigonometric functions. Some interesting corollaries of the basic results have been deduced in § 5. The relations on the double—integral-expansion analogues for the generalized H-functions are also exhibited in § 6.

Notations and formulae used in the investigation:

(i) the L. H. S. of (1.1) is abbreviated as $H\begin{bmatrix} x \\ y \end{bmatrix}$;

(ii) the symbol (a_p, A_p) stands for the set of p-ordered pairs

$$(a_1, A_1),\ (a_2, A_2),\ \ldots,\ (a_p, A_p);$$

(iii) $\Delta(m, n)$ represents for the quantities $\frac{n}{m}, \frac{n+1}{m}, \ldots, \frac{n+m-1}{m}$;

m being a positive integer;

(iv)
$$f = f\begin{pmatrix} m, n \\ p, q \end{pmatrix}; \sigma\Big) = \begin{cases} \begin{bmatrix} m_1 & 0 \\ p_1 - m_1, & q_1 \end{bmatrix} \\ \begin{pmatrix} m_2 + \sigma, & n_2 + 2\sigma \\ p_2 - m_2 + \sigma, & q_2 - n_2 \end{pmatrix} \\ \begin{pmatrix} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{pmatrix}. \end{cases}$$

The integral (1.1) converges, if

(v)
$$\begin{cases} \lambda_1 \equiv \sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j \leq 0; \quad \mu_1 \equiv \sum_1^{p_1} A_j + \sum_1^{p_3} E_j - \sum_1^{q_1} B_j - \sum_1^{q_3} F_j \leq 0, \\ \lambda_2 = \sum_1^{m_1} A_j - \sum_{m_1+1}^{p_1} A_j - \sum_1^{q_1} B_j + \sum_1^{m_2} C_j - \sum_{m_2+1}^{p_2} C_j + \sum_1^{n_2} D_j - \sum_{n_2+1}^{q_2} D_j > 0, \ |\arg x| < \frac{1}{2}\lambda_2 \pi, \\ \mu_2 = \sum_1^{m_1} A_j - \sum_{m_1+1}^{p_1} A_j - \sum_1^{q_1} B_j + \sum_1^{m_3} E_j - \sum_{m_3+1}^{p_3} E_j + \sum_1^{n_3} F_j - \sum_{n_3+1}^{q_3} F_j > 0, \ |\arg y| < \frac{1}{2}\mu_2 \pi. \end{cases}$$

(vi)
$$\begin{cases} \Psi_1 = \Delta\left(\sigma, -\rho + \frac{1}{2}\alpha + 1\right), \ \Psi_2 = \Delta\left(\sigma, -\rho + \frac{1}{2}\alpha + \frac{1}{2}\right), \ \Theta = \Delta(2\sigma, \alpha), \\ \Psi_3 = \Delta\left(\sigma, -\xi + \frac{1}{2}\alpha + 1\right), \ \Psi_4 = \Delta\left(\sigma, -\xi + \frac{1}{2}\alpha + \frac{1}{2}\right). \end{cases}$$

If u is a function of the independent variables x and y, then

(vii) u_x or $u_x(x, y)$ for $\frac{\partial u}{\partial x}$, u_{xx} for $\frac{\partial^2 u}{\partial x^2}$, u_{xy} for $\frac{\partial^2 u}{\partial y \partial x}$ and so on.

(a) The integral:

(1.2)
$$\int_0^c \left(\cos\frac{\pi\rho x}{c}\right)\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} dx$$
$$= \frac{c\,\Gamma(\alpha)\,2^{2\rho-\alpha}\,\Gamma\left(\frac{2\rho-\alpha}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1+\alpha-2\rho}{2}\right)\Gamma(2\rho)},$$

where $\rho > 0$, $\mathrm{Re}(2\rho - \alpha) > 0$, and $\mathrm{Re}(\alpha) > 0$.

The above integral can be obtained from the known result due to MacRobert [5, p. 430, (2)].

(b) The orthogonality-property for cosine functions:

$$\int_{-c}^{c} \cos\left(\frac{n\pi x}{c}\right)\cos\left(\frac{m\pi x}{c}\right)dx = \begin{cases} 0 \text{ for } n \neq m, \\ c \text{ for } n = m. \end{cases} \tag{1.3}$$

(c) Gauss' multiplication theorem:

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)} m^{mz-\frac{1}{2}} \prod_{j=0}^{m-1} \Gamma\left(z + \frac{j}{m}\right). \tag{1.4}$$

Throughout this paper f, λ_1, λ_2, μ_1, μ_2, ψ_1, ψ_2 etc. have the same meaning as referred to earlier.

§ 2. The Fourier Integration Formula

The formula to be proved is

$$\int_0^c \left(\cos\frac{\pi\rho x}{c}\right)\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} H\left[\begin{matrix} u\left(\tan\frac{\pi x}{2c}\right)^{2\sigma} \\ v \end{matrix}\right]dx \tag{2.1}$$

$$= \frac{c\,(2\sigma)^{2\rho-1}\,2^{1-\sigma}}{\pi^\sigma\,\Gamma(2\rho)} H\left[f \left| \begin{matrix} (a_{p_1}, A_{p_1});\ (b_{q_1}, B_{q_1}) \\ \Psi_1, (c_{p_2}, C_{p_2}), \Psi_2;\ \Theta, (d_{q_2}, D_{q_2}) \\ (e_{p_3}, E_{p_3});\ (f_{q_3}, F_{q_3}) \end{matrix} \right| \begin{matrix} u \\ v \end{matrix} \right]$$

where σ is a positive integer, $\rho > 0$ and valid under the conditions of convergence:

$\mathrm{Re}(\alpha) > 0$, $\lambda_1 \leqslant 0$, $\mu_1 \leqslant 0$, $\lambda_2 > 0$, $\mu_2 > 0$, $|\arg u| < \frac{1}{2}\lambda_2\pi$, $|\arg v| < \frac{1}{2}\mu_2\pi$

with $p_1 \geqslant m_1 \geqslant 0$, $p_2 \geqslant m_2 \geqslant 0$, $q_1 \geqslant 0$, $q_2 \geqslant n_2 \geqslant 0$, $p_3 \geqslant m_3 \geqslant 0$, $q_3 \geqslant n_3 \geqslant 0$ and $\mathrm{Re}\left[2\rho - \alpha + \sigma\frac{d_i}{D_i} + \frac{f_j}{F_j}\right] > 0\ (1 \leqslant i \leqslant n_2,\ 1 \leqslant j \leqslant n_3)$.

Proof: On substituting the value of $H\left[\begin{matrix} u \\ v \end{matrix}\right]$ from (1.1) in the integrand of (2.1) and inverting the order of integration, the integral transforms into

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} \Phi[\xi+\eta]\,\Psi(\xi,\eta)\,u^\xi v^\eta \left\{\int_0^c \left(\cos\frac{\pi\rho x}{c}\right)\right.$$

$$\left.\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha+2\sigma\xi-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-2\sigma\xi-1} dx\right\} d\xi\, d\eta.$$

To justify the inversion of the order of integration in the process, we infer that

(i) the x-integral

$$\int_0^c \left(\cos\frac{\pi\rho x}{c}\right)\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha+2\sigma\xi-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-2\sigma\xi-1} dx$$

is absolutely convergent if $\operatorname{Re}(2\rho-\alpha+2\sigma\xi)>0$, $\operatorname{Re}(\alpha-2\sigma\xi)>0$ and σ is a positive integer >0;

(ii) the double contour integral converges absolutely under the conditions referred to earlier in § 1 (v) and convergence of the repeated integral follows from that of integral (2.1); and

(iii) Fox's H-function in two variables is a continuous function of u and v for all finite values of $u\geqslant u_0>0$ and $v\geqslant v_0>0$.

Hence, the inversion of the order of integration is readily justified by an application of de la Vallée Poussin's theorem [1, p. 504] for the conditions imposed with the result.

On interpreting the x-integral by means of (1.2) and employing (1.4), this becomes

$$\frac{c\,(2\sigma)^{2\rho-1}\,2^{1-\sigma}}{\pi^{\sigma}\,\Gamma(2\rho)}\,\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi[\xi+\eta]\,\Psi(\xi,\eta)\,\frac{\prod_{j=0}^{\sigma-1}\Gamma\left(\frac{\rho-\frac{1}{2}\alpha+j}{\sigma}+\xi\right)\prod_{j=0}^{2\sigma-1}\Gamma\left(\frac{\alpha+j}{2\sigma}-\xi\right)}{\prod_{j=0}^{\sigma-1}\Gamma\left(\frac{-\rho+\frac{1}{2}\alpha+\frac{1}{2}+j}{\sigma}-\xi\right)}\,u^{\xi}\,v^{\eta}\,d\xi\,d\eta \tag{2.2}$$

where L_1 is in the $\xi-$ plane and runs from $\lambda-i\infty$ to $\lambda+i\infty$ with loops to ensure, if necessary, that the poles of $\Gamma(d_j-D_j\xi)$ $(1\leqslant j\leqslant n_2)$, $\Gamma\left(\frac{\alpha+j}{2\sigma}-\xi\right)$ $\{0\leqslant \leqslant j\leqslant(2\sigma-1)\}$ lie to the right and the poles of $\Gamma(1-c_j+C_j\xi)\,(1\leqslant j\leqslant m_2)$, $\Gamma[a_j+A_j\xi+A_j\eta]\,(1\leqslant j\leqslant m_1)$ and $\Gamma\left(\frac{\rho-\frac{1}{2}\alpha+j}{\sigma}+\xi\right)\{0\leqslant j\leqslant(\sigma-1)\}$ to the left of L_1.

Similarly L_2 in the η — plane consists of the portion of the imaginary axis from $\delta-i\infty$ to $\delta+i\infty$ along with necessary loops so as to ensure that the poles of $\Gamma(f_j-F_j\eta)$ $(1\leqslant j\leqslant n_3)$ lie to the right and the poles of $\Gamma(1-e_j+ +E_i\eta)\,(1\leqslant j\leqslant m_3)$ and $\Gamma[a_j+A_j\xi+A_j\eta]\,(1\leqslant j\leqslant m_1)$ to the left of L_2.

Next, utilizing (1.1) and (2.2), we get (2.1), which completes the proof.

§ 3. The Fourier Expansion Theorem

There exists a considerable literature in the expansion theory for a given "arbitrary" or analytic function in a series or orthogonal functions and in great detail by Churchill [3].

Theorem: *If σ is a positive integer* >0, $0<x<c$, $p_1 \geqslant m_1 \geqslant 0$, $p_2 \geqslant m_2 \geqslant 0$, $q_1 \geqslant 0$, $q_2 \geqslant n_2 \geqslant 0$, $p_3 \geqslant m_3 \geqslant 0$, $q_3 \geqslant n_3 \geqslant 0$, *then*

(3.1)
$$\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} H\left[\begin{matrix} u\left(\tan\frac{\pi x}{2c}\right)^{2\sigma} \\ v \end{matrix}\right]$$

$$=\frac{4}{(2\pi)^{\sigma}}\sum_{\xi=0}^{\infty}\frac{(2\sigma)^{2\xi-1}}{\Gamma(2\xi)} H\left[f\left|\begin{matrix} (a_{p_1}, A_{p_1});\ (b_{q_1}, B_{q_1}) \\ \Psi_3, (c_{p_2}, C_{p_2})\,\Psi_4;\ \Theta, (d_{q_2}, D_{q_2}) \\ (e_{p_3}, E_{p_3});\ (f_{q_3}, F_{q_3}) \end{matrix}\right| \begin{matrix} u \\ v \end{matrix}\right]\cos\frac{\pi\xi x}{c}$$

provided $\rho>0$, $\operatorname{Re}\left[2\rho-\alpha+\sigma\frac{d_i}{D_i}+\frac{f_j}{F_j}\right]>0\ (1\leqslant i\leqslant n_2,\ 1\leqslant j\leqslant n_3)$, $\operatorname{Re}(\alpha)>0$, $\lambda_1\leqslant 0$, $\mu_1\leqslant 0$, $\lambda_2>0$, $\mu_2>0$, $|\arg u|<\frac{1}{2}\lambda_2\pi$, $|\arg v|<\frac{1}{2}\mu_2\pi$.

Proof. Suppose formally that

(3.2)
$$f(x)=\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} H\left[\begin{matrix} u\left(\tan\frac{\pi x}{2c}\right)^{2\sigma} \\ v \end{matrix}\right]$$
$$=\sum_{\xi=0}^{\infty} P_{\xi}\cos\left(\frac{\pi\xi x}{c}\right), \qquad (0<x<c)$$

where the given function f is to be continuous on the interval $(0\leqslant x\leqslant c)$; also $f(0)=f(c)=0$; and assume that f' is at least sectionally continuous on that interval. Under those conditions f is represented by its Fourier cosine series on that interval.

We now seek for the Fourier coefficients P_{ξ} in that series in a purely formal manner.

Multiply (3.2) by $\cos\frac{\pi\rho x}{c}$ and integrate over $(0, c)$. Then (3.2) yields

(3.3)
$$\int_0^c\left(\cos\frac{\pi\rho x}{c}\right)\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} H\left[\begin{matrix} u\left(\tan\frac{\pi x}{2c}\right)^{2\sigma} \\ v \end{matrix}\right]dx$$
$$=\sum_{\xi=0}^{\infty} P_{\xi}\int_0^c\cos\frac{\pi\xi x}{c}\cos\frac{\pi\rho x}{c}\,dx$$

so that all integrals on the right of (3.3) vanish except for the single value $\xi=\rho$.

Hence, invoking (2.1) and (1.3) in (3.3), we find that

$$P_\rho = \frac{4\,(2\,\sigma)^{2\rho-1}}{(2\,\pi)^\sigma\,\Gamma\,(2\,\rho)}\,H\left[\,f\,\middle|\begin{matrix}(a_{p_1},A_{p_1});\ (b_{q_1},B_{q_1})\\ \Psi_1,(c_{p_2},C_{p_2})\,\Psi_2;\ \Theta,(d_{q_2},D_{q_2})\\ (e_{p_3},E_{p_3});\ (f_{q_3},F_{q_3})\end{matrix}\middle|\begin{matrix}u\\ v\end{matrix}\right]. \tag{3.4}$$

Finally, the theorem immediately follows by virtue of (3.2) and (3.4).

§ 4. A Boundary Value Problem

Here, we make an attempt to determine a formal solution of a problem related to the angular displacement or twist $y(x, t)$ in a shaft of circular cross section with its axis along the x-axis between the fixed points $(0, c)$. When the ends $x=0$ and $x=c$ of the shaft are free, the displacement or function $y(x, t)$ is therefore required to satisfy all conditions of the boundary value problem

$$y_{tt}(x, t) = k^2\, y_{xx}(x, t) \text{ where } k \text{ is the constant and } \quad (0<x<c,\ t>0), \tag{4.1}$$

$$y_x(0, t) = 0,\ y_x(c, t) = 0,\ (t \geqslant 0); \text{ and} \tag{4.2}$$

$$y_t(x, 0) = 0,\ y(x, 0) = f(x),\ (0 \leqslant x \leqslant c). \tag{4.3}$$

We employ the principle of superposition of solutions, fundamental to one of the most powerful methods of solving the boundary value problems and orthogonality-property of the trigonometric functions to arrive at a formal solution [3, p. 125, (4)]:

$$y(x, t) = \frac{1}{2} M_0 + \sum_{\xi=1}^{\infty} M_\xi \cos\frac{\pi\xi\, x}{c} \cos\frac{\pi\xi\, kt}{c},\ (0<x<c) \tag{4.4}$$

where $\xi = 1, 2, \ldots$ and k is a constant.

When $t=0$, the series in (4.4) does converge to $f(x)$; that is $y(x, 0) = f(x)$ for $0 \leqslant x \leqslant c$.

If we take $f(x) = \left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} H\left[\begin{matrix}u\left(\tan\frac{\pi x}{2c}\right)^{2\sigma}\\ v\end{matrix}\right]$

it now follows that

$$f(x) = \left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} H\left[\begin{matrix}u\left(\tan\frac{\pi x}{2c}\right)^{2\sigma}\\ v\end{matrix}\right] \tag{4.5}$$

$$= \frac{1}{2} M_0 + \sum_{\xi=1}^{\infty} M_\xi \cos\frac{\pi\xi\, x}{c}.$$

Evaluation of M_ξ is the same as that of P_ξ and we omit such details.

Therefore, we have

$$M_\rho = \frac{4\,(2\,\sigma)^{2\rho-1}}{(2\,\pi)^\sigma\,\Gamma(2\,\rho)} H\left[f \left| \begin{matrix} (a_{p_1}, A_{p_1});\ (b_{q_1}, B_{q_1}) \\ \Psi'_1, (c_{p_2}, C_{p_2})\,\Psi'_2;\ \Theta, (d_{q_2}, D_{q_2}) \\ (e_{p_3}, E_{p_3});\ (f_{q_3}, F_{q_3}) \end{matrix} \right| \begin{matrix} u \\ v \end{matrix} \right]. \tag{4.6}$$

Next, the use of (4.6) in (4.4) yields the solution as

$$y(x,\,t) = \frac{4}{(2\,\pi)^\sigma} \sum_{\xi=1}^{\infty} \frac{(2\,\sigma)^{2\xi-1}}{\Gamma(2\,\xi)} \cos\frac{\pi\xi\,x}{c} \cos\frac{\pi\xi\,kt}{c} \tag{4.7}$$

$$H\left[f \left| \begin{matrix} (a_{p_1}, A_{p_1});\ (b_{q_1}, B_{q_1}) \\ \Psi'_3, (c_{p_2}, C_{p_2}), \Psi'_4;\ \Theta, (d_{q_2}, D_{q_2}) \\ (e_{p_3}\ E_{p_3});\ (f_{q_3}, F_{q_3}) \end{matrix} \right| \begin{matrix} u \\ v \end{matrix} \right]$$

where $\xi = 1, 2, 3, \ldots$; σ is a positive integer >0, $0<x<c$, $\mathrm{Re}(\alpha)>0$, $p_1 \geqslant m_1 \geqslant 0$, $p_2 \geqslant m_2 \geqslant 0$, $q_1 \geqslant 0$, $q_2 \geqslant n_2 \geqslant 0$, $p_3 \geqslant n_3 \geqslant 0$, $q_3 \geqslant n_3 \geqslant 0$, $\lambda_1 \leqslant 0$, $\mu_1 \leqslant 0$, $\lambda_2 > 0$, $\mu_2 > 0$, $|\arg u| < \frac{1}{2}\lambda_2\pi$ and $|\arg v| < \frac{1}{2}\mu_2\pi$.

§ 5. Corollaries

The general results established in this paper will unify and extend a good number of useful and interesting particular cases by giving special values to the parameters. Some of them are listed below:

(i) If we assign the values $A_j = B_i = \ldots$ etc. $= 1$ $(1 \leqslant j \leqslant p_1,\ 1 \leqslant i \leqslant q_1, \ldots$ etc.,) then we essentially obtain the results associated with Meijer's G-functions in two variables which, in turn, also include most of the commonly used, functions in two arguments, e. g., Kampé de Fériet's double hypergeometric functions incorporating the generalizations of the Appell functions, the Whittaker functions of two variables etc.

(ii) On taking $p_1 = m_1 = q_1 = 0$ and deleting $H^{n_3,\, m_3}_{p_3,\, q_3}[v]$ from both the sides etc., the basic results lead to

$$\int_0^c \left(\cos\frac{\pi\rho\,x}{c}\right)\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} H^{n,\,m}_{p,\,q}\left[u\left(\tan\frac{\pi x}{2c}\right)^{2\sigma}\right]dx \tag{5.1}$$

$$= \frac{c\,(2\,\sigma)^{2\rho-1}\,2^{1-\sigma}}{\pi^\sigma\,\Gamma(2\,\rho)} H^{n+2\sigma,\ m+\sigma}_{p+2\sigma,\ q+2\sigma}\left[u \left| \begin{matrix} \Delta\left(\sigma,\ -\rho+\frac{1}{2}\alpha+1\right),\ (a_p, A_p), \\ \Delta\left(\sigma,\ -\rho+\frac{1}{2}\alpha+\frac{1}{2}\right) \\ \Delta(2\,\sigma,\,\alpha),\ (b_q, B_q) \end{matrix} \right. \right]$$

(5.2)
$$\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} H_{p,\,q}^{n,\,m}\left[u\left(\tan\frac{\pi x}{2c}\right)^{2\sigma}\right]$$

$$=\frac{4}{(2\pi)^{\sigma}}\sum_{\xi=0}^{\infty}\frac{(2\sigma)^{2\xi-1}}{\Gamma(2\xi)} H_{p+2\sigma,\,q+2\sigma}^{n+2\sigma,\,m+\sigma}\left[u\left|\begin{matrix}\Delta\left(\sigma,\ -\xi+\frac{1}{2}\alpha+1\right),(a_p,A_p),\\ \Delta\left(\sigma,\ -\xi+\frac{1}{2}\sigma+\frac{1}{2}\right)\\ \Delta(2\sigma,\alpha),\ (b_q,B_q)\end{matrix}\right.\right]\cos\frac{\pi\xi x}{c},$$

and

(5.3)
$$y(x,t)=\frac{4}{(2\pi)^{\sigma}}\sum_{\xi=1}^{\infty}\frac{(2\sigma)^{2\xi-1}}{\Gamma(2\xi)}\cos\frac{\pi\xi x}{c}\cos\frac{\pi\xi kt}{c}$$

$$H_{p+2\sigma,\,q+2\sigma}^{n+2\sigma,\,m+\sigma}\left[u\left|\begin{matrix}\Delta\left(\sigma,\ -\xi+\frac{1}{2}\alpha+1\right),\ (a_p,A_p),\\ \Delta\left(\sigma,\ -\xi+\frac{1}{2}\alpha+\frac{1}{2}\right)\\ \Delta(2\sigma,\alpha),\ (b_q,B_q)\end{matrix}\right.\right]$$

where $H_{p,\,q}^{n,\,m}[u]$ is the well-known Fox's H-function [4, p. 408] and the conditions of validity for these situations are easily deduciable from their respective main results.

It is remarkable to note that all achievements on Fox's H-functions of two variables are most important since each expression formulated becomes a master or key formula as they generalize a large number of results for various special functions occuring in Physics and Applied Mathematics.

Recently, Saxena, Bora, Kalla, Munot, Shah [2, 8, 9 – 12], Batting and several others have also made an effort for the unification, generalization, extension and correlation of many frequently investigated mathematical functions and their basic properties associated with Fox's H-functions of two variables.

§ 6. Double Fourier-Integral-Expansion Analogues

Lastly, we state without proofs the formulae analogues to our earlier results (2.1) and (3.1). Details are omitted as they follow readily from the preceding sections.

The Fourier-Double-Integration Formula:

(6.1)
$$\int_0^c\int_0^c\left(\cos\frac{\pi\rho x}{c}\right)\left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1}\left(\cos\frac{\pi x}{2c}\right)^{\alpha-1}\left(\cos\frac{\pi\mu y}{c}\right)$$

$$\left(\sin\frac{\pi y}{2c}\right)^{2\mu-\beta-1}\left(\cos\frac{\pi y}{2c}\right)^{\beta-1} H\left[\begin{matrix}u\left(\tan\frac{\pi x}{2c}\right)^{2\sigma}\\ v\left(\tan\frac{\pi y}{2c}\right)^{2\lambda}\end{matrix}\right]dxdy=$$

$$= \frac{c^2 (2\sigma)^{2\rho-1} (2\lambda)^{2\mu-1} 2^{2-\sigma-\lambda}}{\pi^{\sigma+\lambda} \Gamma(2\rho)\Gamma(2\mu)}$$

$$H\left[\begin{array}{c|c|c} \begin{array}{c} \begin{bmatrix} m_1, & 0 \\ p_1-m_1, & q_1 \end{bmatrix} \\ \begin{pmatrix} m_2+\sigma, & n_2+2\sigma \\ p_2-m_2+\sigma, & q_2-n_2 \end{pmatrix} \\ \begin{pmatrix} m_3+\lambda, & n_3+2\lambda \\ p_3-m_3+\lambda, & q_3-n_3 \end{pmatrix} \end{array} & \begin{array}{c} (a_{p_1}, A_{p_1});\ (b_{q_1}, B_{q_1}) \\ \Delta\left(\sigma,\ -\rho+\frac{1}{2}\alpha+1\right),\ (c_{p_2}, C_{p_2}), \\ \Delta\left(\sigma,\ -\rho+\frac{1}{2}\alpha+\frac{1}{2}\right);\ \Delta(2\sigma, \alpha),\ (d_{q_2}, D_{q_2}) \\ \Delta\left(\lambda,\ -\mu+\frac{1}{2}\beta+1\right),\ (e_{p_3}, E_{p_3}), \\ \Delta\left(\lambda,\ -\mu+\frac{1}{2}\beta+\frac{1}{2}\right);\ \Delta(2\lambda, \beta),\ (f_{q_3}, F_{q_3}) \end{array} & \begin{array}{c} u \\ v \end{array} \end{array}\right].$$

The Fourier-Double-Expansion Formula:

$$(6.2) \qquad \left(\sin\frac{\pi x}{2c}\right)^{2\rho-\alpha-1} \left(\cos\frac{\pi x}{2c}\right)^{\alpha-1} \left(\sin\frac{\pi y}{2c}\right)^{2\mu-\beta-1} \left(\cos\frac{\pi y}{2c}\right)^{\beta-1}$$

$$H\left[\begin{array}{c} u\left(\tan\frac{\pi x}{2c}\right)^{2\sigma} \\ v\left(\tan\frac{\pi y}{2c}\right)^{2\lambda} \end{array}\right] = \frac{16}{(2\pi)^{\sigma+\lambda}} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(2\sigma)^{2r-1}(2\lambda)^{2s-1}}{\Gamma(2r)\Gamma(2s)} \cos\frac{\pi r x}{c} \cos\frac{\pi s y}{c}$$

$$H\left[\begin{array}{c|c|c} \begin{array}{c} \begin{bmatrix} m_1, & 0 \\ p_1-m_1, & q_1 \end{bmatrix} \\ \begin{pmatrix} m_2+\sigma, & n_2+2\sigma \\ p_2-m_2+\sigma, & q_2-n_2 \end{pmatrix} \\ \begin{pmatrix} m_3+\lambda, & n_3+2\lambda \\ p_3-m_3+\lambda, & q_3-n_3 \end{pmatrix} \end{array} & \begin{array}{c} (a_{p_1}, A_{p_1});\ (b_{q_1}, B_{q_1}) \\ \Delta\left(\sigma,\ -r+\frac{1}{2}\alpha+1\right),\ (c_{p_2}, C_{p_2}), \\ \Delta\left(\sigma,\ -r+\frac{1}{2}\alpha+\frac{1}{2}\right);\ \Delta(2\sigma, \alpha),\ (d_{q_2}, D_{q_2}) \\ \Delta\left(\lambda,\ -s+\frac{1}{2}\beta+1\right),\ (e_{p_3}, E_{p_3}), \\ \Delta\left(\lambda,\ -s+\frac{1}{2}\beta+\frac{1}{2}\right);\ \Delta(2\lambda, \beta),\ (f_{q_3}, F_{q_3}) \end{array} & \begin{array}{c} u \\ v \end{array} \end{array}\right]$$

where the conditions of validity for these results are:

σ, λ are positive integers >0, $0<x<c$, $0<y<c$, $\rho>0$, $\mu>0$, $p_1 \geqslant m_1 \geqslant 0$, $p_2 \geqslant m_2 \geqslant 0$, $q_1 \geqslant 0$, $q_2 \geqslant n_2 \geqslant 0$, $p_3 \geqslant m_3 \geqslant 0$, $q_3 \geqslant n_3 \geqslant 0$, $\mathrm{Re}\left[2\rho-\alpha+\sigma\frac{d_i}{D_i}\right]>0\left(1 \leqslant i \leqslant n_2\right)$, $\mathrm{Re}\left[2\mu-\beta+\lambda\frac{f_j}{F_j}\right]>0\,(1 \leqslant j \leqslant n_3)$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\lambda_1 \leqslant 0$, $\mu_1 \leqslant 0$, $\lambda_2>0$, $\mu_2>0$, $|\arg u|<\frac{1}{2}\lambda_2\pi$, $|\arg v|<\frac{1}{2}\mu_2\pi$.

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Department of Mathematics,
P. M. B. G. Science College,
Sanyogitaganj, Indore (M. P.),
India.