PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 16 (30), 1973, pp. 115—122

SOME INTEGRALS ASSOCIATED WITH THE GENERALIZED LAURICELLA FUNCTIONS*

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(Received April 3, 1973)

Summary

In the present note the author evaluates eight definite integrals, four finite and four infinite, that involve the generalized Lauricella function of several complex variables. These results would extend the integrals associated with the Kampé de Fériet function, or its generalization in two variables, given recently by H. M. Srivastava and others (cf. [4] to [9]; see also [11]). This paper concludes with a brief remark about several trivial variations or obvious particular forms of the Srivastava-Daoust results that are still appearing in the literature.

1. Introduction

In the usual notation for the generalized Lauricella functions, let ([10], p. 454)

$$(1.1)\qquad F\begin{matrix} A:B';\ldots;B^{(r)} \\ C:D';\ldots;D^{(r)}\end{matrix}\begin{pmatrix} z_1 \\ \vdots \\ z_r\end{pmatrix} \equiv F\begin{matrix} A:B';\ldots;B^{(r)} \\ C:D';\ldots;D^{(r)}\end{matrix}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(r)}):\Phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j'+\cdots+m_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b_j')_{m_1\Phi_j'}\cdots\prod_{j=1}^{B^{(r)}}(b_j^{(r)})_{m_r\Phi_j^{(r)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j'+\cdots+m_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d_j')_{m_1\delta_j'}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{m_r\delta_j^{(r)}}}\cdot\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!},$$

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* This work was carried out at the University of Victoria while the author held a post-doctoral fellowship of the National Research Council of Canada under the supervision of Professor H. M. Srivastava. For an abstract of this paper see Notices Amer. Math. Soc. **20** (1973), p. A—434.

where, for convergence, θ's, Φ's, ψ's and δ's are all positive such that

(1.2) $$\Delta_i \equiv 1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\Phi_j^{(i)}\geqq 0.$$
$$(i=1,\ldots,r),$$

the equality holds when $|z_i|<\rho_i$, $i=1,\ldots,r$, with the ρ_i defined by equation (5.3), p. 157 in [12].

For a complete set of conditions of convergence of the multiple hypergeometric series in (1.1), see § 5 of reference [12]. These conditions will be assumed to hold throughout this work.

Here

(1.3) $$(\lambda)_m=\lambda(\lambda+1)\cdots(\lambda+m-1),\ (\lambda)_0=1,$$

and (a) denotes the sequence of A parameters $a_1,\ldots,a_A$; with similar interpretations for (b), (b'), etc. Thus it is understood that there are A of the a parameters, $B^{(i)}$ of the $b^{(i)}$ parameters, $i=1,\ldots,r$, and so on.

When r takes on the values 1 and 2, (1.1) would correspond respectively to the generalized hypergeometric function introduced by Wright ([13] and [14]) and the generalization of Kampé de Fériet's double hypergeometric function, introduced by Srivastava and Daoust ([9], p. 199).

2. Eulerian integrals of the first kind

We recall the well-known formula [2, Vol. I, p. 311 (31)]

(2.1) $$\int_0^1 t^{\alpha-1}(1-t^\lambda)^{\beta-1}\,dt=\lambda^{-1}B(\alpha/\lambda,\beta),$$

where $$\lambda>0,\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,$$

and its generalization, viz.

(2.2) $$\int_0^1 t^{\sigma-1}(1-t^\lambda)^{\beta-1}\,{}_2F_1[-n,\alpha+\beta+n-1;\alpha;t^\lambda]\,dt$$
$$=\frac{1}{\lambda}\,\frac{\Gamma(\sigma/\lambda)\,\Gamma(\alpha+n-\sigma/\lambda)\,\Gamma(\alpha)\,\Gamma(\beta+n)}{\Gamma(\alpha+n)\,\Gamma(\alpha-\sigma/\lambda)\,\Gamma(\beta+n+\sigma/\lambda)},$$
$$\lambda>0,\ \mathrm{Re}(\sigma)>0,\ \mathrm{Re}(\beta)>0,\ n=0,1,2,\ldots,$$

which follows readily from the known integral (2), p. 284 in [2, Vol. II]. Then as immediate consequences of (2.1) and (2.2), we obtain our first two results in the following forms.

(2.3) $$\int_0^1 t^{\alpha-1}(1-t^\lambda)^{\beta-1}\,F\begin{matrix}A:B';\ldots;B^{(r)}\\ C:D';\ldots;D^{(r)}\end{matrix}\begin{pmatrix}z_1t^{\mu_1}(1-t^\lambda)^{\sigma_1}\\ \vdots\\ z_rt^{\mu_r}(1-t^\lambda)^{\sigma_r}\end{pmatrix}dt$$

$$=\lambda^{-1}B(\alpha/\lambda,\beta)\ F\begin{matrix}A+2:B';\ldots;B^{(r)}\\ C+1:D';\ldots;D^{(r)}\end{matrix}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],\\ \\ [(c):\psi',\ldots,\psi^{(r)}],\end{matrix}\right.$$

$$\begin{aligned}&[\alpha\lambda^{-1}:\mu_1\lambda^{-1},\dots,\mu_r\lambda^{-1}],\ [\beta:\sigma_1,\dots,\sigma_r]:\\&\quad[\alpha\lambda^{-1}+\beta:\mu_1\lambda^{-1}+\sigma_1,\dots,\mu_r\lambda^{-1}+\sigma_r]:\\&\quad\left.\begin{matrix}[(b'):\Phi'];\dots;[(b^{(r)}):\Phi^{(r)}];\\ \\ [(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\quad z_1,\dots,z_r\right),\end{aligned}$$

provided $\lambda>0,\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mu_i>0,$ $\sigma_i>0,\ i=1,\dots,r,$ and the series on the right converges.

(2.4)
$$\int_0^1 t^{\sigma-1}(1-t^\lambda)^{\beta-1}{}_2F_1[-n,\alpha+\beta+n-1;\alpha;t^\lambda]\cdot F\begin{matrix}A:B';\dots;B^{(r)}\\ \\ C:D';\dots;D^{(r)}\end{matrix}\begin{pmatrix}z_1t^{\mu_1}\\ \vdots\\ z_rt^{\mu_r}\end{pmatrix}dt$$

$$=(-)^n\frac{1}{\lambda}\frac{\Gamma(\alpha)\Gamma(\beta+n)\Gamma(\sigma/\lambda)\Gamma(1-\alpha+\sigma/\lambda)}{\Gamma(\alpha+n)\Gamma(\beta+n+\sigma/\lambda)\Gamma(1-\alpha-n+\sigma/\lambda)}$$

$$\cdot F\begin{matrix}A+2:B';\dots;B^{(r)}\\ \\ C+2:D';\dots;D^{(r)}\end{matrix}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}],\\ \\ [(c):\psi',\dots,\psi^{(r)}],\end{matrix}\right.$$

$$\begin{aligned}&[\sigma/\lambda:\mu_1/\lambda,\dots,\mu_r/\lambda],\ [1-\alpha+\sigma/\lambda:\mu_1/\lambda,\dots,\mu_r/\lambda]:\\&[\beta+n+\sigma/\lambda:\mu_1/\lambda,\dots,\mu_r/\lambda],\ [1-\alpha-n+\sigma/\lambda:\mu_1/\lambda,\dots,\mu_r/\lambda]:\\&\left.\begin{matrix}[(b'):\Phi'];\dots;[(b^{(r)}):\Phi^{(r)}];\\ \\ [(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\quad z_1,\dots,z_r\right),\end{aligned}$$

valid within the domain of convergence of the resulting series when n is a nonnegative integer, $\lambda>0,\ \mathrm{Re}(\sigma)>0,\ \mathrm{Re}(\beta)>0$ and $\mu_i>0,\ i=1,\dots,r.$

The third integral, which involves the associated Legendre function of the first kind, follows similarly from Barnes' integral [3, p. 172 (24)]

(2.5)
$$\int_0^1 t^{\lambda-1}(1-t^2)^{-\mu/2}P_\nu^\mu(t)\,dt=\frac{\sqrt{\pi}\,2^{\mu-\lambda}\Gamma(\lambda)}{\Gamma[(\lambda-\mu-\nu+1)/2]\Gamma[(\lambda-\mu+\nu+2)/2]},$$

$\mathrm{Re}(\lambda)>0,\ \mathrm{Re}(\mu)<1,$ and we obtain

(2.6)
$$\int_0^1 t^{\lambda-1}(1-t^2)^{-\mu/2}P_\nu^\mu(t)\ F\begin{matrix}A:B';\dots;B^{(r)}\\ \\ C:D';\dots;D^{(r)}\end{matrix}\begin{pmatrix}z_1t^{2\sigma_1}\\ \vdots\\ z_rt^{2\sigma_r}\end{pmatrix}dt$$

$$=\frac{\sqrt{\pi}\,2^{\mu-\lambda}\Gamma(\lambda)}{\Gamma[(\lambda-\mu-\nu+1)/2]\Gamma[(\lambda-\mu+\nu+2)/2]}\ F\begin{matrix}A+2:B';\dots;B^{(r)}\\ \\ C+2:D';\dots;D^{(r)}\end{matrix}$$

$$\left.\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}],\ [\lambda/2:\sigma_1,\ldots,\sigma_r],\ [(\lambda+1)/2:\sigma_1,\ldots,\sigma_r]: \\ [(c):\psi',\ldots,\psi^{(r)}],\ [(\lambda-\mu-\nu+1)/2:\sigma_1,\ldots,\sigma_r],\ [(\lambda-\mu+\nu+2)/2:\sigma_1,\ldots,\sigma_r]: \end{matrix}\right.$$

$$\left.\begin{matrix} [(b'):\Phi'];\ldots;[(b^{(r)}):\Phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}\ z_1,\ldots,z_r\right),$$

provided that the multiple series on the right-hand side converges, $\operatorname{Re}(\lambda)>0$, $\operatorname{Re}(\mu)<1$, and $\sigma_i>0$, $i=1,\ldots,r$.

In a similaer manner, by applying the formula [2, Vol. II, p. 399 (4)]

$$\int_0^1 t^{\lambda-1}(1-t)^{\gamma-1}\,{}_2F_1[\alpha,\beta;\gamma;1-t]\,dt = \frac{\Gamma(\gamma)\Gamma(\lambda)\Gamma(\gamma+\lambda-\alpha-\beta)}{\Gamma(\gamma+\lambda-\alpha)\Gamma(\gamma+\lambda-\beta)}, \tag{2.7}$$

$$\operatorname{Re}(\gamma)>0,\ \operatorname{Re}(\lambda)>0,\ \operatorname{Re}(\gamma+\lambda-\alpha-\beta)>0,$$

we can obtain a slight variant of (2.6) in the form

$$\int_0^1 t^{\lambda-1}(1-t)^{\gamma-1}\,{}_2F_1[\alpha,\beta;\gamma;1-t]\,F\begin{matrix} A:B';\ldots;B^{(r)} \\ C:D';\ldots;D^{(r)} \end{matrix}\begin{pmatrix} z_1 t^{\mu_1} \\ \vdots \\ z_r t^{\mu_r} \end{pmatrix} dt \tag{2.8}$$

$$= \frac{\Gamma(\gamma)\Gamma(\lambda)\Gamma(\gamma+\lambda-\alpha-\beta)}{\Gamma(\gamma+\lambda-\alpha)\Gamma(\gamma+\lambda-\beta)}\,F\begin{matrix} A+2:B';\ldots;B^{(r)} \\ C+2:D';\ldots;D^{(r)} \end{matrix}\left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}], \\ [(c):\psi',\ldots,\psi^{(r)}], \end{matrix}\right.$$

$$\left.\begin{matrix} [\lambda:\mu_1,\ldots,\mu_r],\ [\gamma+\lambda-\alpha-\beta:\mu_1,\ldots,\mu_r]:[(b'):\Phi'];\ldots;[(b^{(r)}):\Phi^{(r)}]; \\ [\gamma+\lambda-\alpha:\mu_1,\ldots,\mu_r],\ [\gamma+\lambda-\beta:\mu_1,\ldots,\mu_r]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}\ z_1,\ldots,z_r\right),$$

valid when $\operatorname{Re}(\lambda)>0$, $\operatorname{Re}(\gamma)>0$, $\operatorname{Re}(\lambda+\gamma-\alpha-\beta)>0$, $\mu_i>0$, $i=1,\ldots,r$, and the multiple series on the right-hand side converges.

3. The infinite integrals

Making use of the known formulas ([2], Vol. II, p. 292)

$$\int_0^\infty t^{\beta-1}e^{-t}L_n^{(\alpha)}(t)\,dt = \frac{\Gamma(\alpha-\beta+n+1)\Gamma(\beta)}{n!\,\Gamma(\alpha-\beta+1)},\quad \operatorname{Re}(\beta)>0, \tag{3.1}$$

and

$$\int_0^\infty t^{\alpha}e^{-t}\{L_n^{(\alpha)}(t)\}^2\,dt = \frac{\Gamma(\alpha+n+1)}{n!},\quad \operatorname{Re}(\alpha)>-1, \tag{3.2}$$

where $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomial of order α and degree n in x, we get the following two Eulerian integrals of the second kind.

(3.3)
$$\int_0^\infty t^{\beta-1} e^{-t} L_n^{(\alpha)}(t)\, F\begin{matrix} A:B';\dots;B^{(r)} \\ \\ C:D';\dots;D^{(r)}\end{matrix}\left(\begin{matrix} z_1 t^{\mu_1} \\ \vdots \\ z_r t^{\mu_r}\end{matrix}\right) dt$$

$$= \frac{(-)^n \Gamma(\beta)\Gamma(\beta-\alpha)}{n!\,\Gamma(\beta-\alpha-n)} F\begin{matrix} A+2:B';\dots;B^{(r)} \\ \\ C+1:D';\dots;D^{(r)}\end{matrix}\left(\begin{matrix} [(a):\theta',\dots,\theta^{(r)}], \\ \\ [(c):\psi',\dots,\psi^{(r)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[\beta:\mu_1,\dots,\mu_r],\ [\beta-\alpha:\mu_1,\dots,\mu_r]:[(b'):\Phi'];\dots;[(b^{(r)}):\Phi^{(r)}]; \\ \\ [\beta-\alpha-n:\mu_1,\dots,\mu_r]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\dots z_r\right),$$

where $n=0, 1, 2, \dots$, and which holds true if $\mathrm{Re}(\beta)>0$, $\mu_i>0$, $i=1,\dots,r$, and the resulting multiple series converges.

(3.4)
$$\int_0^\infty t^{\alpha} e^{-t} \{L_n^{(\alpha)}(t)\}^2 F\begin{matrix} A:B';\dots;B^{(r)} \\ \\ C:D';\dots;D^{(r)}\end{matrix}\left(\begin{matrix} z_1 t^{\mu_1} \\ \vdots \\ z_r t^{\mu_r}\end{matrix}\right) dt$$

$$= \frac{\Gamma(\alpha+n+1)}{n!} F\begin{matrix} A+1:B';\dots;B^{(r)} \\ \\ C:D';\dots;D^{(r)}\end{matrix}\left(\begin{matrix} [\alpha+n+1:\mu_1,\dots,\mu_r], \\ \\ \end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[(b'):\Phi'];\dots;[(b^{(r)}):\Phi^{(r)}]; \\ \\ [(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\dots,z_r\right),$$

where $n=0, 1, 2, \dots$, $\mathrm{Re}(\alpha)>-1$, $\mu_i>0$, $i=1,\dots,r$, provided the series on the right-hand side is convergent.

Finally, we apply the integral formulas ([2], Vol. I, p. 334 (48)) and ([2], Vol. II, p. 409 (41)) to obtain the following results.

(3.5)
$$\int_0^\infty t^{2\lambda-1} K_{2\mu}(\alpha t) K_{2\nu}(\alpha t)\, F\begin{matrix} A:B';\dots;B^{(r)} \\ \\ C:D';\dots;D^{(r)}\end{matrix}\left(\begin{matrix} z_1 t^{2\sigma_1} \\ \vdots \\ z_r t^{2\sigma_r}\end{matrix}\right) dt$$

$$= \frac{2^{2\lambda-3}\Gamma(\lambda+\mu+\nu)\Gamma(\lambda+\mu-\nu)\Gamma(\lambda-\mu+\nu)\Gamma(\lambda-\mu-\nu)}{\alpha^{2\lambda}\Gamma(2\lambda)}$$

$$\cdot F\begin{matrix} A+4:B';\dots;B^{(r)} \\ \\ C+2:D';\dots;D^{(r)}\end{matrix}\left(\begin{matrix}[\lambda+\mu+\nu:\sigma_1,\dots,\sigma_r],\ [\lambda+\mu-\nu:\sigma_1,\dots,\sigma_r], \\ \\ \end{matrix}\right.$$

$$[\lambda-\mu+\nu:\sigma_1,\ldots,\sigma_r],\ [\lambda-\mu-\nu:\sigma_1,\ldots,\sigma_r],\ [(a):\theta',\ldots,\theta^{(r)}]:$$
$$[\lambda:\sigma_1,\ldots,\sigma_r],\ [\lambda+1/2:\sigma_1,\ldots,\sigma_r],\ [(c):\psi',\ldots,\psi^{(r)}]:$$
$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(r)}):\Phi^{(r)}];\\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ \frac{z_1}{\alpha^{2\sigma_1}},\ldots,\frac{z_r}{\alpha^{2\sigma_r}}\right),$$

provided that $\operatorname{Re}(\alpha)>0$, $\operatorname{Re}(\lambda)>|\operatorname{Re}(\mu)|+|\operatorname{Re}(\nu)|$, $\sigma_i>0$, $i=1,\ldots,r$, and the right-hand side series is convergent.

(3.6) $$\int_0^\infty t^{2\rho-1}W_{\lambda,\mu}(\beta t)W_{-\lambda,\mu}(\beta t)\,F\begin{matrix}A:B';\ldots;B^{(r)}\\ C:D';\ldots;D^{(r)}\end{matrix}\begin{pmatrix}z_1t^{2\sigma_1}\\ \vdots\\ z_rt^{2\sigma_r}\end{pmatrix}dt$$

$$=\frac{\Gamma(2\rho+1)\Gamma(\rho+\mu+1/2)\Gamma(\rho-\mu+1/2)}{2\beta^{2\rho}\Gamma(\rho+\lambda+1)\Gamma(\rho-\lambda+1)}\,F\begin{matrix}A+4:B';\ldots;B^{(r)}\\ C+2:D';\ldots;D^{(r)}\end{matrix}$$

$$\Big([\rho+1/2:\sigma_1,\ldots,\sigma_r],\ [\rho+1:\sigma_1,\ldots,\sigma_r],\ [\rho+\mu+1/2:\sigma_1,\ldots,\sigma_r],$$
$$[\rho+\lambda+1:\sigma_1,\ldots,\sigma_r],$$
$$[\rho-\mu+1/2:\sigma_1,\ldots,\sigma_r],\ [(a):\theta',\ldots,\theta^{(r)}]:$$
$$[\rho-\lambda+1:\sigma_1,\ldots,\sigma_r],\ [(c):\psi',\ldots,\psi^{(r)}]:$$
$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(r)}):\Phi^{(r)}];\\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1(2/\beta)^{2\sigma_1},\ldots,z_r(2/\beta)^{2\sigma_r}\right),$$

provided $\operatorname{Re}(\beta)>0$, $\operatorname{Re}(\rho)>|\operatorname{Re}(\mu)|-1/2$, $\sigma_i>0$, $i=1,\ldots,r$, and the series on the right-hand side is convergent.

We remark in passing that in the integral formulas (3.3), (3.4), (3.5) and (3.6) we have assumed that the behaviour of the generalized Lauricella functions involved at the upper limit of integration is given by

(3.7) $$F\begin{matrix}A:B';\ldots;B^{(r)}\\ C:D';\ldots;D^{(r)}\end{matrix}\begin{pmatrix}z_1t\\ \vdots\\ z_rt\end{pmatrix}=0(t^{\zeta}),$$ when $z_1,\ldots,z_r$ are fixed and $t\to\infty$,

for some value of ζ.

4. Particular cases

First of all, we note that the special cases of formulas (2.4), (2.5) and (2.6), when $r=2$, were given earlier by Srivastava and Daoust [9]. Similar special cases of our results (3.3) and (3.4) are contained in a recent paper of Martić [4].

The integral formula (2.8) provides an elegant generalization of the known result (la), p. 21 of Sevaria [7]. Note that this formula of Sevaria is essentially the same as integral (2.1), p. 22 evaluated subsequently by Saxena [6]. The latter [6, p. 22, Eq. (2.2)] also gave a ve y special case of the formula (2.3), p. 200 of Srivastava and Daoust [9]. On the other hand, a special case of the integral (2.7), p. 201 in the Srivastava-Daoust paper [9] happens to be the main result (2.1), p. 93 of a very recent paper of Shah [8].

Next we recall the fact that our formulas (3.5) and (3.6) are analogous respectively to the known results (2.2), p. 473 and (2.3), p. 474, involving Meijer's G-function of two variables, given earlier by Srivastava and Joshi [11]. Note, in this connection, that formulas (2), p. 104 and (4), p. 105 of Saxena and Vyas [5], which indeed follow from the aforementioned results of Srivastava and Joshi [11], are also special cases of our integrals (3.6) and (3.5), respectively, when $r=2$, $A=C=0$, and the various parameters involved are suitably chosen.

Finally, we remark that the integral* (1), p. 104 in the Saxena-Vyas paper [5], involving Jacobi polynomials, is contained in the Srivastava-Daoust formula (2.4), p. 200 in reference [9], and hence also in our result (2.4) above. Thus it would seem obvious that the eight integral formulas, given in this paper, can be specialized to a large number of scattered results, known or new.

We conclude by mentioning that the method of preceding sections can also be applied to evaluate integrals similar to those given in this paper, but with only one of the arguments of the generalized Lauricella functions involved depending on the variable t of integration. Alternatively, these analogous results can be derived as limiting cases of the integrals (2.3), (2.4), (2.6), (2.8), and (3.3) to (3.6), when all but one of the parameters like $\mu_1, \ldots, \mu_r$ and $\sigma_1, \ldots, \sigma_r$ approach 0. The details may be left as an exercise for the interested reader.

Acknowledgment

The author wishes to thank Professor H. M. Srivastava for kindly suggesting a number of improvements in this paper.

* Incidentally, this result of Saxena and Vyas [5] follows also as a special case of the integral formula (2.5), p. 981 of J. P. Singhal [Proc. Nat. Acad. Sci. India Sect. A **36** (1966), 976—986] which involves the G-function of two variables. Note also that a special case of the Srivastava-Daoust result (2.4), p. 200 in reference [9] happens to be the main integral formula in a very recent paper of F. Singh and L. K. Sharma [Indian J. Pure Appl. Math. **3** (1972), no. 1, 142—147].

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