PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 16 (30), 1973, pp. 111—113

A NOTE ON FRACTIONAL INTEGRATION

Branislav Martić

(Received April 14, 1973)

1. In the last two theorems in [1] *S. L. Bora* and *R. K. Saxena* have established connections between *Weyl* fractional integral, *Meijer and Hankel*'s transforms. We give here to these theorems the following forms:

Theorem I. *If*

$$W_\mu \{f(1/t); p\} = h(p; \mu)$$

then

(1.1) $$\mathscr{L}\{t^{2\mu-2} h(1/t^2; \mu); p\} = 2^{\mu+1/2}\pi^{-1/2} p^{-\mu} \mathscr{K}_{\mu-1/2}\{t^{-\mu-2} f(t^2); p\}$$

provided that the Weyl integral of $|f(1/t)|$ *exists,* $Re\,(\mu) > 0$ *and* $Re\,(p) > 0$.

Theorem II. *If*

$$W_\mu \{f(t); p\} = h(p; \mu)$$

then

(1.2) $$H_\nu \{t^{\nu+1/2} h(t^2; \mu); p\} = 2^\mu p^{-\mu} H_{\nu+\mu} \{t^{\mu+\nu+1/2} f(t^2); p\}$$

provided that the Weyl integral of $|f(t)|$ *exists,* $|\,Re\,(\nu) > -1$ *and* $Re\,(\mu) > 0$.

All definitions here are taken from [2]. We list them without many comments. We call

$$h(y; \mu) = W_\mu \{f(x); y\} = \frac{1}{\Gamma(\mu)} \int_y^\infty f(x)(x-y)^{\mu-1}\,dx$$

the *Weyl* fractional integral of order μ of $f(x)$. We call

(1.3) $$\mathscr{K}_\nu \{f(x); y\} = \int_0^\infty f(x) K_\nu(xy)(xy)^{1/2}\,dx.$$

$$H_\nu \{f(x); y\} = \int_0^\infty f(x) \mathscr{J}_\nu(xy)(xy)^{1/2}\,dx, \quad y > 0$$

the *Meijer* and *Hankel*'s transforms of order ν of $f(x)$ respectively. The *Laplace* transform of $f(x)$ has the form

$$\mathscr{L}\{f(x);p\}=\int_0^\infty e^{-pt}f(t)\,dt,\ Re(p)>0. \tag{1.4}$$

In [1] some of these definitions are taken with slight modifications, i. e. on the r. h. s. of (1.3) stands a factor $\sqrt{2\pi^{-1}}\,y$ and on the r. h. s. of (1.4) stands a factor p. Therefore our expression on the r. h. s. of (1.1) differs from the corresponding in [1] in which our factor $\sqrt{2\pi^{-1}}$ is omitted.

The object of the present note is to give very simple proofs of the quoted Theorems which involve minimum technique and different from those in [1].

2. *Proof of Theorem* I. If we start with the connection [2, p. 122]

$$\mathscr{K}_\nu\{f(x);y\}=\frac{\pi^{1/2}\,2^{-\nu}\,y^{\nu+1/2}}{\Gamma(\nu+1/2)}\,\mathscr{L}\left\{\int_0^x (x^2-t^2)^{\nu-1/2}\,t^{1/2-\nu}f(t)\,dt;y\right\},$$

$$Re(\nu)>-1/2$$

and write $\nu-1/2$ for ν and $x^{-\nu-2}f(x^2)$ for $f(x)$ we obtain

$$\mathscr{K}_{\nu-1/2}\{x^{-\nu-2}f(x^2);y\}=\frac{\pi^{1/2}\,2^{-\nu+1/2}\,y^{\nu}}{\Gamma(\nu)}\,\mathscr{L}\left\{\int_0^x (x^2-t^2)^{\nu-1}\,t^{-2\nu-1}f(t^2)\,dt;y\right\},$$

$$Re(\nu)>0. \tag{2.1}$$

Next we notice that

$$\frac{1}{\Gamma(\nu)}\int_0^x (x^2-t^2)^{\nu-1}\,t^{-2\nu-1}f(t^2)\,dt=\frac{1}{2\Gamma(\nu)}\int_x^0 f(t^2)\,x^{2\nu-2}\left(\frac{1}{t^2}-\frac{1}{x^2}\right)^{\nu-1}(-2t^{-3})\,dt=$$

$$=\frac{1}{2\Gamma(\nu)}\,x^{2\nu-2}\int_{1/x^2}^\infty f(1/s)\left(s-\frac{1}{x^2}\right)^{\nu-1}ds=2^{-1}x^{2\nu-2}\,W_\nu\{f(1/s);x^{-2}\}=$$

$$=2^{-1}x^{2\nu-2}\,h(x^{-2};\nu)$$

which combined with (2.1) proves (1.1).

Proof of Theorem II. Take now [2. p. 6 (14)]

$$\int_0^\infty \left\{x^{1/2+\nu}\int_x^\infty \xi^{1/2-\mu-\nu}(\xi^2-x^2)^{\mu-1}f(\xi)\,d\xi\right\}\mathscr{J}_\nu(xy)\,(xy)^{1/2}\,dx=$$

$$=2^{\mu-1}\,\Gamma(\mu)\,y^{-\mu}\int_0^\infty f(x)\,\mathscr{J}_{\nu+\mu}(xy)\,(xy)^{1/2}\,dx,\ Re(\nu+1)>Re(\mu)>0.$$

Writing $x^{1/2+\mu+\nu}f(x^2)$ for $f(x)$ we obtain

$$\int_0^\infty x^{1/2+\nu}\left\{\int_x^\infty \xi(\xi^2-x^2)^{\mu-1}f(\xi^2)\,d\xi\right\}\mathcal{J}_\nu(xy)\,(xy)^{1/2}\,dx$$

$$=2^{\mu-1}\,\Gamma(\mu)\,y^{-\mu}\int_0^\infty x^{\mu+\nu+1/2}f(x^2)\,\mathcal{J}_{\nu+\mu}(xy)\,(xy)^{1/2}\,dx,\ Re\,(\nu+1)>Re\,(\mu)>0.$$

Since

$$\int_x^\infty \xi(\xi^2-x^2)^{\mu-1}f(\xi^2)\,d\xi=2^{-1}\int_{x^2}^\infty (y-x^2)^{\mu-1}f(y)\,dy=2^{-1}\,\Gamma(\mu)\,h(x^2;\mu),$$

it follows

$$H_\nu\{x^{1/2+\nu}h(x^2;\mu);y\}=2^\mu y^{-\mu}H_{\nu+\mu}\{x^{\mu+\nu+1/2}f(x^2);y\},$$

$$Re\,(\nu+1)>Re\,(\mu)>0,$$

i. e. (1.2).

At the end we note that the method presented here may also be applied to prove other two theorems in [1].

REFERENCES

[1] S. L. Bora and R. K. Saxena, *On fractional integration*, Publications de l' Institut Mathématique, Beograd, 11 (25), (1971), 19—22.

[2] A. Erdélyi, et al., *Tables of integral transforms*, Vol. 2, McGraw-Hill New York (1954).

Bjelave 70
Sarajevo
Jugoslavija