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EXPONENTIALLY COMPLETE SPACES IV

M. M. Marjanović

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1. Introduction.

This paper is a continuation of our three previous notes [5], [6], [7] and is mostly related to [5]. The main facts that we prove here are: 2.3 (the union mapping is open), 2.6 (if $f: X \to Y$ is open, then $\exp(f): \exp(X) \to \exp(Y)$ is open, too) and the properties of branching and non-branching points of $\{X^{(n)}, u^{(n)}\}$ (3.6, 3.8 and 3.9).

All spaces that we consider are compact Hausdorff and all mappings continuous. The covariant funtor exp associates to each space X, the space $\exp(X) = X^{(1)}$ of all its non-empty closed subsets taken with the finite topology and to each $f: X \to Y$, $\exp(f) = f^{(1)}: X^{(1)} \to Y^{(1)}$ given by $f^{(1)}(F) = f(F)$.

2. Our aim in this section is to prove that all bonding mappings of $\{X^{(n)}, u^{(n)}\}$ are open. Those facts which have a technical or conceptual meaning of their own are formulated as separate statements. We start with the following

2.1. *Let* $\mathcal{B}$ *be any open basis in X. Then, the collection of all sets*

$$\langle B_1, \ldots, B_n \rangle, \; B_i \in \mathcal{B}$$

forms an open basis in $\exp(X) \Leftrightarrow X$ *is compact.*

P r o o f. $\Leftarrow$: Let $\langle U_1, \ldots, U_n \rangle$ be a standard basic open set in $\exp(X)$ and $F_0 \in \langle U_1, \ldots, U_n \rangle$. Since F_0 is compact there exists a finite cover of F_0, say $\{B_1, \ldots, B_m\}$ where $B_i \in \mathcal{B}$ and $B_i \subseteq U_1 \cup \cdots \cup U_n$. Let $x_i \in F_0 \cap U_i$, $i = 1, 2, \ldots, n$ and let $B(x_i) \in \mathcal{B}$ be such that $B(x_i) \subseteq U_i$. Then

$$F_0 \in \langle B_1, \ldots, B_m, B(x_1), \ldots, B(x_m) \rangle \subseteq \langle U_1, \ldots U_n \rangle.$$

This "if" part of 2.1 was proved in [4].

$\Rightarrow$: Suppose X is not compact and let $\mathcal{B}$ be a basis for X such that no finite subcollection of elements of $\mathcal{B}$ covers X. Then, $\{X\} \in \exp(X)$ does not belong to any $\langle B_1, \ldots, B_n \rangle$, $B_i \in \mathcal{B}$. Thus the sets $\langle B_1, \ldots, B_n \rangle$, $B_i \in \mathcal{B}$ do not constitute a basis for $\exp(X)$.

2.2. *Let X be compact and* $\langle U_1, \ldots, U_n \rangle$ *a basic open set of* $\exp(X)$. *Then for every* $F \in \langle U_1, \ldots, U_n \rangle$ *there exist open sets* $V_1, \ldots V_n$ *such that* $\overline{V}_i \subseteq U_i$ *and* $F \in \langle V_1, \ldots, V_n \rangle$.

Proof. For each $x \in F \cap U_i$, let $V_{(i,x)}$ be an open neighborhood of x such that $\overline{V}_{(i,x)} \subseteq U_i$. Then $\{V_{(i,x)} \mid (i, x) \in \{1, \ldots, n\} \times F\}$ covers F and let π be a finite subcover. Take V_i be the union of a $V_{(i,x)}$ and all those members of π having i in their indices. Evidently $\overline{V}_i \subseteq U_i$ and $F \in \langle V_1, \ldots, V_n \rangle$.

The mapping $u: X^{(2)} \to X^{(1)}$ given by $u(F^{(1)}) = \cup\{F \mid F \in F^{(1)}\}$ where $X^{(2)} = \exp(X^{(1)})$ is continuous (Th. 5.7 in [8]) and, when u is considered as a mapping of the second power set of X onto the first, u maps an open set of $\exp(X)$ onto an open set of X (Th. 5, 42. in [3]). Let us prove the following property of u:

2.3. *The mapping* $u: X^{(2)} \to X^{(1)}$ *is open.*

Proof. By 2.1, the sets of the form

$$\langle \langle \sigma_1 \rangle, \ldots, \langle \sigma_n \rangle \rangle,$$

where $\sigma_i = \{U_1^i, \ldots, U_{k_i}^i\}$ is a finite collection of open sets in X and $\langle \sigma_i \rangle = \langle U_1^i, \ldots, U_{k_i}^i \rangle$ are basic open sets in $\exp(X)$, will constitute an open basis in $X^{(2)}$. We prove 2.3 by proving the following relation

$$u(\langle \langle \sigma_1 \rangle, \ldots, \langle \sigma_n \rangle \rangle) = \langle U_1^1, \ldots, U_{k_1}^1, \ldots, U_1^n, \ldots, U_{k_n}^n \rangle$$

and so u maps open basic sets of $X^{(2)}$ onto open sets of $X^{(1)}$.

Let $F_0^{(1)} \in \langle \langle \sigma_1 \rangle, \ldots, \langle \sigma_n \rangle \rangle$. Then each $F \in F_0^{(1)}$ belongs to some $\langle \sigma_i \rangle$. Thus $F \subseteq |\sigma_i|$ $(|\sigma_i| = U_1^i \cup \cdots \cup U_{k_i}^i)$. This implies $u(F_0^{(1)}) = \cup\{F \mid F \in F_0^{(1)}\} \subseteq |\sigma_1| \cup \cdots \cup |\sigma_n|$. On the other hand for every i: $F_0^{(1)} \cap \langle \sigma_i \rangle \neq \varnothing$, what implies the existence of an $F \in F_0^{(1)}$ such that $F \in \langle \sigma_i \rangle$. Thus $F \cap U_j^i \neq \varnothing$, $j = 1, \ldots k_i$ and we also have $u(F_0^{(1)}) \cap U_i^j \neq \varnothing$ for every i and every $j = 1, \ldots, k_i$. Hence $u(F_0^{(1)}) \in \langle U_1^1, \ldots, U_{k_1}^1, \ldots, U_1^n, \ldots, U_{k_n}^n \rangle$ and we have proved that

$$u(\langle \langle \sigma_i \rangle, \ldots, \langle \sigma_n \rangle \rangle) \subseteq \langle U_1^1, \ldots, U_{k_1}^1, \ldots, U_1^n, \ldots, U_{k_n}^n \rangle.$$

Now let $F_0 \in \langle U_1^1, \ldots, U_{k_1}^1, \ldots, U_1^n, \ldots, U_{k_n}^n \rangle$. Then, evidently $F_0 \in \langle |\sigma_1|, \ldots, |\sigma_n| \rangle$, where $\sigma_i = \{U_1^i, \ldots, U_{k_i}^i\}$. By 2.2, there exist $V_1, \ldots, V_n$ such that $\overline{V}_i \subseteq |\sigma_i|$ and $F_0 \langle V_1, \ldots, V_n \rangle$. Put

$$F_i = (F_0 \cap \overline{V}_i) \cup \{x_1^i, \ldots, x_{ki}^i\},$$

where $x_j^i \in F \cap U_j^i$, $j = 1, \ldots, k_i$. Then, $F_i \in \langle \sigma_i \rangle$ and

$$F^{(1)} = \{F_1, \ldots, F_n\} \in \langle \langle \sigma_1 \rangle, \ldots, \langle \sigma_n \rangle \rangle.$$

Since $F_0 \subseteq \overline{V}_1 \cup \cdots \cup \overline{V}_n$, we have

$$F_0 \subseteq (F_0 \cap \overline{V}_1) \cup \cdots \cup (F_0 \cap \overline{V}_n) \subseteq F_1 \cup \cdots \cup F_n \subseteq F_0.$$

Thus $u(F^{(1)}) = F_1 \cup \cdots \cup F_n = F_0$, what proves that $F_0 \in u(\langle \langle \sigma_1 \rangle, \ldots, \langle \sigma_n \rangle \rangle)$.

2.4. *Let* $f: X \to Y$ *be a continuous and open mapping of the compact space* X *into the compact space* Y. *Let* $U \subseteq X$ *be open and* $H \subseteq f(U)$ *closed in* Y. *Then, there exists a closed subset* F *of* $\overline{U}$ *such that* $f(F) = H$.

Proof. First we prove the existence of a set $\tilde{F} \subseteq U$ for which $f(\tilde{F}) \supseteq H$. For each $y \in H$, let $x_y \in f^{-1}(y) \cap U$ and let U_y be a neighborhood of x_y such that $\overline{U_y} \subseteq U$. Since $y \in f(U_y)$ and the mapping f is open, $\{f(U_y) \mid y \in H\}$ is an open cover of H. Being H compact, there is a finite subcover $\{f(U_{y_1}), \dots, f(U_{y_n})\}$. Hence,

$$H \subseteq f(U_{y_1}) \cup \cdots \cup f(U_{y_n}).$$

Put $\tilde{F} = \overline{U}_{y_1} \cup \cdots \cup \overline{U}_{y_n}$. Since $\overline{U}_{y_i} \subseteq U$, $\tilde{F} \subseteq U$ and we also have

$$f(\tilde{F}) = f(\overline{U}_{y_1}) \cup \cdots \cup f(\overline{U}_{y_n}) \supseteq H.$$

Being f continuous, $f^{-1}(H)$ is closed and let $F_0 = f^{-1}(H) \cap \tilde{F}$. Now

$$f(F_0) \subseteq f(f^{-1}(H)) \cap f(\tilde{F}) = H \cap f(\tilde{F}) = H,$$

for $H \subseteq f(\tilde{F})$.

On the other hand, let $y \in H$. Then, there is an $x \in \tilde{F}$ such that $f(x) = y$. Thus $x \in \tilde{F}$ and $x \in f^{-1}(y) \subseteq f^{-1}(H)$, what shows that $x \in F_0$ and $f(x) = y$. Hence, $f(F_0) \supseteq H$ and so $f(F_0) = H$.

2.5. *Let $f: X \to Y$ be continuous and open. Then*

$$\exp(f)(\langle U_1, \dots, U_n \rangle) = \langle fU_1, \dots, fU_n \rangle.$$

Proof. Let $H \in \exp(f)(\langle U_1, \dots, U_n \rangle)$. Then there is an $F \in \langle U_1, \dots, U_n \rangle$ such that $f(F) = H$. Since $F \cap U_i \neq \varnothing$ and $F \subseteq U_1 \cup \cdots \cup U_n = U_0$, we have

$$f(F) \cap f(U_i) \neq \varnothing \text{ and } f(F) \subseteq f(U_1) \cup \cdots \cup f(U_n) = f(U_0).$$

Hence,

$$f(F) = H \in \langle fU_1, \dots, fU_n \rangle.$$

Conversely, let $H \in \langle fU_1, \dots, fU_n \rangle$. Then, $H \cap f(U_i) \neq \varnothing$ and $H \subseteq fU_1 \cup \cdots \cup fU_n = fU_0$. By 2.4, there is an $F \subseteq U_0$ such that $f(F) = H$. Let $x_i \in f^{-1}(H) \cap U_i$, then

$$F_0 = F \cup \{x_1, \dots, x_n\} \in \langle U_1, \dots, U_n \rangle$$

and $f(F_0) = H$. what shows that $H \in f(\langle U_1, \dots, U_n \rangle)$.

2.6. *Let $f: X \to Y$ be a continuous and open mapping of the compact space X into the compact space Y. Then, the mapping*

$$\exp(f): \exp(X) \to \exp(Y)$$

is continuous and open. If, in addition, the mapping f is onto the sets

$$\langle fU_1, \dots, fU_n \rangle,$$

where U_i are open in X, form an open basis in $\exp(Y)$.

Proof. By 2.5, $\exp(f)$ maps the basic open sets in $\exp(X)$ onto open sets in $\exp(Y)$. Thus $\exp(f)$ is open.

If f is onto, then for $\langle V_1, \dots, V_n \rangle$ (V_i open in Y), let $U_i = f^{-1}(V_i)$. Then, $V_i = f(U_i)$ and

$$\langle V_1, \dots, V_n \rangle = \langle f(U_1), \dots, f(U_n) \rangle.$$

Being f open, the sets of the form $\langle fU_1, \ldots, fU_n\rangle$, where U_i are arbitrary open sets in X, are also open basic sets in $\exp(Y)$.

Note that 2.5 and 2.6 were proved in [1], when $\exp(X)$ and $\exp(Y)$ are all subsets of X and Y.

Denote $u: X^{(2)} \to X^{(1)}$ by $u^{(1)}$ and let $u^{(n)} = \exp(u^{(n-1)})$. Then, $\{X^{(n)}, u^{(n)}\}$ is an inverse sequence, and by 2.3 and 2.6, we get

2.7. *All bonding mappings of* $\{X^{(n)}, u^{(n)}\}$. *are open.*

3. This section is devoted to the study of branching and non-branching points of $\{X^{(n)}, u^{(n)}\}$. Some of the more general relevant facts are also formulated and proved.

1.1. *Let $f: X \to Y$ be a continuous and open mapping of a compact space X onto a compact space Y. If x' and x'' are two different points of $f^{-1}(y)$, then there exists a neighborhood U of x' such that $f(X \backslash U) = Y$.*

Proof. Let U' and U'' be two disjoint open neighborhoods of x' and x''. Since f is open, fU' and fU'' are open neighborhoods of y and let V be a neighborhood of y such that $V \subseteq fU' \cap fU''$. Since f is continuous, choose a neighborhood U of x' such that $U \subseteq U'$ and $fU \subseteq V$. Then,

$$fU \subseteq V \subseteq fU'' \subseteq f(X \backslash U)$$

and

$$Y = f(X) = f(X \backslash U) \cup f(U) = f(X \backslash U).$$

Now let $f: X \to Y$ be a mapping of a set X onto a set Y and let

$$\underline{X} = \{x \mid x = f^{-1}(f(x))\}, \quad \overline{Y} = \{f(x) \mid x \in \underline{X}\}$$
$$= \{y \mid f^{-1}(y) \text{ is a singleton}\}.$$

Then, the mapping $f: \underline{X} \to \overline{Y}$ defined by f is *one-to-one and onto* and $f(X \backslash \underline{X}) = Y \backslash \overline{Y}$.

In case X and Y are two compact topological spaces, let $\langle \underline{X} \rangle$ and $\langle \overline{Y} \rangle$ be the sets of all closed in X subsets of $\underline{X}$ and of all closed in Y subsets of $\overline{Y}$ respectively, and let

$$f^{(1)}: \langle \underline{X} \rangle \to \langle \overline{Y} \rangle.$$

be the mapping defined by $f^{(1)}: X^{(1)} \to Y^{(1)}$

3.2. *Let $f: X \to Y$ be a continuous and open mapping of a compact space X onto a compact space Y. Then,*

$$\underline{X^{(1)}} = \langle \underline{X} \rangle \text{ and } \overline{Y^{(1)}} = \langle \overline{Y} \rangle.$$

Proof. Let $F \in \langle \underline{X} \rangle$, then $f(F) \subseteq \overline{Y}$ and since $f: \underline{X} \to \overline{Y}$ is one-to-one and onto, we have $(f^{(1)})^{-1}(f^{(1)}(F)) = F$. Thus $F \in \underline{X^{(1)}}$. Let now $F \notin \langle \underline{X} \rangle$, then $F \cap (X \backslash \underline{X}) \neq \varnothing$ and choose an $x_0 \in F \cap (X \backslash \underline{X})$. Then the set $f^{-1}(f(x_0))$ contains at least two points x_0 and $x' \neq x_0$. Being f open there exists a closed neighborhood of x_0, let it be U_0, such that $f(X \backslash U_0) = Y$ and this is possible according to 3.1. Since $f(F) \subseteq f(X \backslash U_0)$, then by 2.4, there exists an $F_1 \subseteq X \backslash U_0$

such that $f(F_1)=f(F)$ and the two sets F and F_1 are different. This proves that $F\notin \underline{X^{(1)}}$. Hence, $\langle X\rangle = \underline{X^{(1)}}$.

Let $H\in\langle \overline{Y}\rangle$, then $f^{-1}(\underline{H})\subseteq \underline{X}$ and $f(f^{-1}(H))=H$. Since $f^{-1}(H)\in\langle \underline{X}\rangle$, it follows that $H\in\overline{Y^{(1)}}$. On the other hand, being $f(\underline{X})=\overline{Y}$,

$$\overline{Y^{(1)}}=f^{(1)}(\underline{X^{(1)}})=f^{(1)}(\langle \underline{X}\rangle)\subseteq\langle \overline{Y}\rangle.$$

Thus, $\overline{Y^{(1)}}=\langle\overline{Y}\rangle$.

Let $j_X: X\to X^{(1)}$ be given by $j_X(x)=\{x\}$, and let $j_X^{(1)}=\exp(j_X),\ldots, j_X^{(n)}=\exp(j_X^{(n-1)})$. For $F\in X^{(1)}$, we have $j_X^{(1)}(F)=\{j_X(x)\mid x\in F\}=\{\{x\}\mid x\in F\}$, what is obviously a set different from the set $j_{X^{(1)}}(F)=\{F\}$.

For a set A, let the comprehension for its elements be: $\{x\mid x\in A\}$. Then, an element of $X^{(n)}$ can be written as

$$F^{(n-1)}=\{\{\cdots\{\{x\mid x\in F\}\mid F\in F^{(1)}\}\cdots\}\mid F^{(n-2)}\in F^{(n-1)}\}$$

and we have

$$j_X^{(n)}(F^{(n-1)})=\{\{\cdots\{\{x\}\mid x\in F\}\mid F\in F^{(1)}\}\cdots\}\mid F^{(n-2)}\in F^{(n-1)}\}.$$

Thus we get a sequence of spaces and mappings (index X dropped)

$$X\xrightarrow{j} X^{(1)}\xrightarrow{j^{(1)}} X^{(2)}\to\cdots\to X^{(n)}\xrightarrow{j^{(n)}} X^{(n+1)}\to\cdots$$

Let $u_X: X^{(2)}\to X^{(1)}$ be given by

$$u_X(F^{(1)})=\{x\mid x\in F\wedge F\in F^{(1)}\}.$$

We denote u_X by $u_X^{(1)}$ and $\exp(u_X^{(n-1)})$ by $u_X^{(n)}$. For $F^{(2)}\in X^{(3)}$, we have

$$u_X^{(2)}(F^{(2)})=\{\{x\mid x\in F\wedge F\in F^{(1)}\}\mid F^{(1)}\in F^{(2)}\}$$

what differs from

$$u_{X^{(1)}}(F^{(2)})=\{F\mid F\in F^{(1)}\wedge F^{(1)}\in F^{(2)}\}.$$

It is easy to see that

$$u^{(n)}(F^{(n)})=\{\{\cdots\{u^{(1)}(F^{(1)})\mid F^{(1)}\in F^{(2)}\}\cdots\}\mid F^{(n-1)}\in F^{(n)}\}.$$

If $1_X: X\to X$ is the identity, then $1_X^{(n)}=1_{X^{(n)}}$ and it is evident that $u^{(1)}\circ j^{(1)}=1_{X^{(1)}}$ what implies

3.3. $u^{(n)}\circ j^{(n)}=1_{X^{(n)}}$, $n=1, 2, \ldots$

Let $\{X_n, f_n\}$ be an inverse system and X_∞ its limit space. A point $x\in X_\infty$ will be called *non-branching* if there exists an integer $n_0(x)$ such that for every n, m and $n>m\geq n_0(x)$ the relation $x_n=\overset{-1}{f}_{nm}(x_m)$ holds. All other points of X_∞ will be called *branching*.

If all bonding mappings are on, then for each $n>1$, we have two subsets of X_n, $\underline{X_n}$ related to f_{n-1} and $\overline{X_n}$ related to f_n. Now we are going to determine these subsets in the case of the inverse system

$$X^{(1)}\xleftarrow{u^{(1)}} X^{(2)}\leftarrow\cdots\leftarrow X^{(n)}\xleftarrow{u^{(n)}} X^{(n+1)}\leftarrow\cdots$$

For $u^{(1)}: X^{(2)} \to X^{(1)}$ we have $\overline{X^{(1)}} = \{\{x\} \mid x \in X\}$ and $\underline{X^{(2)}} = \{\{\{x\}\} \mid x \in X\}$ and these subsets are closed in their spaces. In view of 3.2,

$$\overline{X^{(2)}} = (\overline{X^{(1)}})^{(1)}, \quad \underline{X^{(3)}} = (\underline{X^{(2)}})^{(1)}.$$

If A is a closed subset of X and B of Y and $f: X \to Y$ is continuous, then for the mapping $g: A \to B$ defined by f, the mapping $g^{(1)}: A^{(1)} \to B^{(1)}$ is defined by $f^{(1)}$. We use this in proving the following statement, where the mappings defined by a mapping f, will be again denoted by f and their domains and codomains will serve to distinguish one from the other.

3.4. (a) $\overline{X^{(n)}} = (\overline{X^{(1)}})^{(n-1)}, \quad \underline{X^{(n+1)}} = (\underline{X^{(2)}})^{(n-1)}$

(b) $\overline{X^{(n)}} \underset{j^{(n)}}{\overset{u^{(n)}}{\rightleftarrows}} \underline{X^{(n+1)}}$ *are two homeomorphisms.*

(c) $j^{(n)} \circ u^{(n)}: \underline{X^{(n+1)}} \to \overline{X^{(n)}} \to \overline{X^{(n+1)}}$ *is an inclusion.*

(d) $j^{(n)}: X^{(n)} \to \overline{X^{(n+1)}}$ *is a homeomorphism and* $u^{(n)}(\overline{X^{(n+1)}}) = X^{(n)}$.

Proof. (a) follows from 3.2 and 2.3.

To prove (b), consider $\overline{X^{(1)}} \underset{j^{(1)}}{\overset{u^{(1)}}{\rightleftarrows}} \underline{X^{(2)}}$. Obviously $u^{(1)}$ and $j^{(1)}$ are homeomorphisms and $j^{(1)} \circ u^{(1)} = 1$ and $u^{(1)} \circ j^{(1)} = 1$. Being exp covariant, we get

$$(\overline{X^{(1)}})^{(n-1)} \underset{j^{(n)}}{\overset{u^{(n)}}{\rightleftarrows}} (\underline{X^{(2)}})^{(n-1)}$$

where $u^{(n)} \circ j^{(n)} = 1$ and $j^{(n)} \circ u^{(n)} = 1$. Now applying (a), we obtain (b).

To prove (c), cheque that $j^{(1)} \circ u^{(1)}: \underline{X^{(2)}} \to \overline{X^{(1)}} \to \overline{X^{(2)}}$ is an inclusion. Indeed,

$$j^{(1)} \circ u^{(1)}(\{\{x\}\}) = j^{(1)}(\{x\}) = \{\{x\}\}.$$

Applying $\exp^{(n-1)}$ to the above sequence, it follows that

$$j^{(n)} \circ u^{(n)}: (\underline{X^{(2)}})^{(n-1)} \xrightarrow{u^{(n)}} (\overline{X^{(1)}})^{(n-1)} \xrightarrow{j^{(n)}} (\overline{X^{(2)}})^{(n-1)}$$

is also an inclusion.

To prove (d), start with the homeomorphism $X \xrightarrow{j} \overline{X^{(1)}}$ and we get $X^{(n)} \xrightarrow{j^{(n)}} \overline{X^{(n+1)}}$. Thus $\overline{X^{(n+1)}} = j^{(n)}(X^{(n)})$. By 3.3,

$$u^{(n)}(\overline{X^{(n+1)}}) = u^{(n)} \circ j^{(n)}(X^{(n)}) = X^{(n)}.$$

Let $j^{(m, m)} = j^{(m)}$ and $j^{(0)} = j$ and put

$$j^{(m, n)} = j^{(n)} \circ \cdots \circ j^{(m)}: X^{(m)} \to X^{(n+1)}, \quad n \geqslant m \geqslant 0.$$

Then all these mappings are imbeddings. Let $u^{(m, m)} = u^{(m)}$,

$$u^{(n, m)} = u^{(m)} \circ \cdots \circ u^{(n)}: X^{(n+1)} \to X^{(m)}, \quad n \geqslant m \geqslant 1.$$

If $n>m\geqslant 1$, then by 3.3,

$$u^{(n,m)}\circ j^{(m,n)}=1\,X^{(m)},$$

and for a fixed m and each $n>m$, the mappings $j^{(m,n)}:X^{(m)}\to X^{(n)}$ induce the mapping

$$j^{(m,\omega)}:X^{(m)}\to X^{(\omega)}$$

which is also an imbedding.

3.5. $j^{(n,\omega)}(X^{(n)})\subseteq j^{(n+k,\omega)}(X^{(n+k)})$ *and*

$$X^{(\omega)}=\mathrm{cl}\,(\cup\{j^{(n,\omega)}(X^{(n)})\mid n\in N\}).$$

Proof. If $F^{(n-1)}\in X^{(n)}$, then

$$j^{(n,\omega)}F^{(n-1)})=(F,F^{(1)},\ldots,F^{(n-1)},\ldots,F^{(k)},\ldots)$$

where $F^{(k-1)}=u^{(n-1,k)}(F^{(n-1)})$ for $k<n$ and for $k>n$ $F^{(k-1)}=j^{(n,k-1)}(F^{(n-1)})$. Further on,

$$j^{(n+k,\omega)}j^{(n,n+k-1)}(F^{(n-1)})=j^{(n+k,\omega)}(j^{(n,n+k-1)}(F^{(n-1)}))=j^{(n,\omega)}(F^{(n-1)}).$$

Thus,

$$j^{(n,\omega)}(X^{(n)})=j^{(n+k,\omega)}(j^{(n,n+k-1)}(X^{(n)}))\subseteq j^{(n+k,\omega)}(X^{(n+k)}),$$

since $j^{(n,n+k-1)}:X^{(n)}\to X^{(n+k)}$ is an imbedding.

Let $u^{(\omega,n)}:X^{(\omega)}\to X^{(n)}$ be the natural projection, then the sets of the form $(u^{(\omega,n)})^{-1}(U)$, where U is open in $X^{(n)}$and $n=1,2,\ldots$, constitute a basis in $X^{(\omega)}$ ([2], ch. VIII, 3.12). Now let $F^{(\omega)}\in X^{(\omega)}$ and let $u^{(\omega,n)}(F^{(\omega)})=F^{(n-1)}\in U$, where U is open in $X^{(n)}$. Then $j^{(n,\omega)}(F^{(n-1)})\in(u^{(\omega,n)})^{-1}(U)$ and this proves that

$$\mathrm{cl}\,(\cup\{j^{(n,\omega)}(X^{(n)})\mid n\in N\})=X^{(\omega)}.$$

3.6. $\cup\{j^{(n,\omega)}(X^{(n)})\mid n\in N\}$ *is the set of all non-branching points of* $X^{(\omega)}$.

Proof. If $F^{(\omega)}\in\cup\{j^{(n,\omega)}(X^{(n)})\mid n\in N\}$, then by 3.4 (d),

$$j^{(n)}\circ u^{(\omega,n)}(F^{(\omega)})\in\overline{X^{(n+1)}}$$

and $F^{(\omega)}$ is a non-branching point of $X^{(\omega)}$.

On the other hand if $F^{(\omega)}$ is a non-branching point of $X^{(\omega)}$, then there exists an n_0 such that $u^{(\omega,n)}(F^{(\omega)})\in\overline{X^{(n)}}$ for $n\geqslant n_0$. Then, by 3.4 (b) and (c),

$$F^{(\omega)}=j^{(n_0,\omega)}(u^{(\omega,n_0)}(F^{(\omega)}))\in j^{(n_0,\omega)}(X^{(n_0)}).$$

Denote $X^{(\omega)}\backslash\cup\{j^{(n,\omega)}(X^{(n)})\mid n\in N\}$ by $\mathrm{br}\,(X^{(\omega)})$ (branching points of $X^{(\omega)}$). Now we will prove that for each $F^{(\omega)}\in\mathrm{br}\,(X^{(\omega)})$ and each $n\in N$ there exists an $F_n^{(\omega)}\in\mathrm{br}\,(X^{(\omega)})$ such that

$$u^{(\omega,n)}(F^{(\omega)})=u^{(\omega,n)}(F_n^{(\omega)})$$

and $F^{(\omega)}\neq F_n^{(\omega)}$. Thus, each element of $\mathrm{br}\,(X^{(\omega)})$ will be branching in this set.

Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be a pair of mappings and $\tilde{X} = \{x \mid f(x) = g(x)\}$. When f and g are continuous (and X and Y compact), then $\tilde{X}$ is a closed subset of X. For the mappings $f^{(1)}: X^{(1)} \to Y^{(1)}$ and $g^{(1)}: X^{(1)} \to Y^{(1)}$, let $\widetilde{X^{(1)}} = \{F \mid f^{(1)}(F) = g^{(1)}(F)\}$.

3.7. *Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be two continuous mappings of a compact space X into a compact space Y. If*

$$f(\tilde{X}) \setminus f(X \setminus \tilde{X}) = \varnothing, \quad g(\tilde{X}) \setminus g(X \setminus \tilde{X}) = \varnothing$$

and

$$f(X \setminus \tilde{X}) \cap g(X \setminus \tilde{X}) = \varnothing$$

then $\widetilde{X^{(1)}} = (\tilde{X})^{(1)}$ *and*

$$f^{(1)}(X^{(1)} \setminus \widetilde{X^{(1)}}) \cap f^{(1)}(\widetilde{X^{(1)}}) = \varnothing$$

$$g^{(1)}(X^{(1)} \setminus \widetilde{X^{(1)}}) \cap g^{(1)}(\widetilde{X^{(1)}} = \varnothing$$

and

$$f^{(1)}(X^{(1)}) \setminus \widetilde{X^{(1)}}) \cap g^{(1)}(X^{(1)} \setminus \widetilde{X^{(1)}}) = \varnothing .$$

Proof. It is obvious that $(\tilde{X})^{(1)} \subseteq \widetilde{X^{(1)}}$. If $F \notin (\tilde{X})^{(1)}$ then $F \cap X \setminus \tilde{X}) \neq \varnothing$ and let $x \in F \cap (X \setminus \tilde{X})$. Then, $f(x) \in f(F)$ and $f(x) \in f(X \setminus \tilde{X})$ Thus $f(x) \notin g(X \setminus \tilde{X})$ and $f(x) \notin f(\tilde{X}) = g(\tilde{X})$, what implies $f(x) \notin g(F)$. Hence, $f(F) \neq g(F)$ and $F \notin \widetilde{X^{(1)}}$. This proves $(\tilde{X})^{(1)} = \widetilde{X^{(1)}}$. If $H \in f^{(1)}(X^{(1)} \setminus \widetilde{X^{(1)}})$, then $H = f(F)$ and $F \notin \widetilde{X^{(1)}} = (\tilde{X})^{(1)}$. Then, again $F \cap (X \setminus \tilde{X}) \neq \varnothing$ and we can conclude that $f(F) \neq g(F)$ what proves the first relation in 3.7. The second relation follows from the first by symmetry of assumptions in 3.7.

If $H \in f^{(1)}(X^{(1)} \setminus \widetilde{X^{(1)}})$, then $H = f(F)$, $F \notin \widetilde{X^{(1)}}$. Let $x \in F \cap (X \setminus \tilde{X})$, then $f(x) \notin f(\tilde{X})$ and $f(x) \in f(X) \setminus f(\tilde{X}) = f(X \setminus \tilde{X})$. But $f(x) \notin g(\tilde{X})$ and $f(x) \in g(X \setminus \tilde{X})$ what implies $f(x) \notin g(X)$. Hence, $f(F) \neq g(K)$ for every $K \in X^{(1)}$. This proves the last relation in 3.7.

3.8. *If X is not a singleton space, then for each $F^{(\omega)} \in$* br $(X^{(\omega)})$ *and each $n \in N$ there exists an $F_n^{(\omega)} \in$* br $(X^{(\omega)})$ *such that $F^{(\omega)} \neq F_n^{(\omega)}$ and*

$$u^{(\omega, n)}(F^{(\omega)}) = u^{(\omega, n)}(F_n^{(\omega)}).$$

Proof. Consider the mapping $k: X^{(1)} \to X^{(2)}$ defined by $k(F) = \langle F \rangle$. Then for the pair of mappings

$$X^{(1)} \underset{j^{(1)}}{\overset{k}{\rightrightarrows}} X^{(2)},$$

we have $\widetilde{X^{(1)}} = j(X)$ and

$$k(X^{(1)} \setminus \widetilde{X^{(1)}}) \cap j^{(1)}(X^{(1)} \setminus \widetilde{X^{(1)}}) = \varnothing ,$$

because

$$k(F) = \langle F \rangle \supset j^{(1)}(F) = \{\{x\} \mid x \in F\},$$

if $F \in \widetilde{X^{(1)}}$ and when X is not a singleton space. Evidently,

$$k(\widetilde{X^{(1)}}) \cap k(X^{(1)} \backslash \widetilde{X^{(1)}}) = \varnothing \quad \text{and} \quad j^{(1)}(\widetilde{X^{(1)}}) \cap j^{(1)}(X^{(1)} \backslash \widetilde{X^{(1)}}) = \varnothing$$

so that we can apply 3.7. By 3.7,

$$\widetilde{X^{(n)}} = (j(X))^{(n)} = (\overline{X^{(1)}})^{(n-1)} = \overline{X^{(n)}},$$

where the last equality follows from 3.4 (a). Put $k = k^{(1)}$, then it is easy to see that $u^{(1)} \circ k^{(1)} = 1_{X^{(1)}}$, what implies $u^{(n)} \circ k^{(n)} = 1_{X^{(n)}}$. Now, let $F^{(\omega)} \in \mathrm{br}(X^{(\omega)})$ and $n \in N$. Let $F_n^{(\omega)}$ be such that its r-th coordinate $(F_n^{(\omega)})_r = u^{(\omega, r)}(F^{(\omega)})$ for $r \leqslant n$ and $(F_n^{(\omega)})_r = k^{(n, r-1)}(u^{(\omega, n)}(F^{(\omega)}))$ for $r > n$, where $k^{(n, r)} = k^{(r)} \circ \cdots \circ k^{(n)}$ $(k^{(n-1, n-1)} = k^{(n-1)})$. Since $u^{(r-1)}((F_n^{(\omega)})_r) = (F_n^{(\omega)})_{r-1}$ for $r > n$, it follows that $F_n^{(\omega)} \in X^{(\omega)}$. Since $(F_n^{(\omega)})_n \notin \overline{X^{(n)}} = \widetilde{X^{(n)}}$, it follows that $u^{(\omega, n+1)}(F^{(\omega)}) \neq F_n^{(\omega)})_{n+1}$. Since $k^{(n, r)}((F^{(\omega)})_n \notin \widetilde{X^{(r+1)}} = \overline{X^{(n+1)}}$ for $r > n$, then by 3.6, $F_n^{(\omega)}$ is a branching point of $X^{(\omega)}$. This concludes the proof of 3.8.

In view of 3.8, the points of br $(X^{(\omega)})$ are branching in this set and br $(X^{(\omega)}$ is dense in itself in a set theoretic sense. Of course, br $(X^{(\omega)})$ is also dense in itself as a topological space. Thus, no proof of the following corrolary of 3.8 need be given.

3.9. *If* card $(X) > 1$, *then* card $(\mathrm{br}(X^{(\omega)})) \geqslant c$ *and the set* $br(X^{(\omega)})$ *is dense in itself.*

A. Pelczynski proved in [9] that $T(C)$ is a unique compact metric zero--dimensional space having an everywhere dense set of isolated points and a Cantor set of non-isolated points.

Taking X to be a finite discrete space, then all points of $\cup\{j^{(n, \omega)}(X^{(n)}) \mid n \in N\}$ are isolated and $\mathrm{br}(X^{(\omega)}$ is dense in itself as it follows from 3.8 and 3.6. Hence,

3.10. *If X is a finite discrete space and card $(X) > 1$, then* $X^{(\omega)} \approx T(C)$.

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