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ON THE CONVERGENCE OF CERTAIN SEQUENCES IV

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1. Starting with the results of S. B. Prešić ([1], [2], [3]), we have in three papers ([4], [5], [6]) examined the convergence of sequences which are defined by certain difference equations. In particular, in paper [5] we considered sequences $(x_n^{(1)}), \ldots, (x_n^{(k)})$ defined by

$$(1.1) \qquad x_{n+p}^{(i)} = f_i(x_n^{(1)}, \ldots, x_n^{(k)}, x_{n+1}^{(1)}, \ldots, x_{n+1}^{(k)}, \ldots, x_{n+p-1}^{(1)}, \ldots, x_{n+p-1}^{(k)})$$

where $i = 1, \ldots, k$.

The same sequences were studied in [7] by E. Udovičić. The main result of paper [7] is stated as a generalisation of Prešić's result from [1].

In this note we shall show that Udovičić's results from [7] are not generalisations, but consequences of Prešić's result [1]. Furthermore, in section 3 we shall connect our results from [5] with the mentioned results of Udovičić.

2. In paper [1] S. B. Prešić proved the following result.

Theorem P. *Let E be a complete metric space and let $f: E^k \to E$ satisfy the condition*

$$d(f(u_1, u_2, \ldots, u_k), f(u_2, u_3, \ldots, u_{k+1})) \leqslant \sum_{\nu=1}^{k} a_\nu d(u_\nu, u_{\nu+1})$$

with $a_\nu \geqslant 0$ $(\nu = 1, \ldots, k)$ and $\sum_{\nu=1}^{k} a_\nu < 1$.

Any sequence (x_n) in E which satisfies the equality

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}) \qquad (n = 1, 2, \ldots)$$

is convergent and $\lim x_n = x$ *is the unique solution of the equation* $x = f(x, x, \ldots, x)$.

Let $(x_n^{(1)}), \ldots, (x_n^{(k)})$ be sequences in a complete metric space E which satisfy (1.1) and let

$$(2.1) \qquad d(f_i(y_{11}, \ldots, y_{1k}, y_{21}, \ldots, y_{2k}, \ldots, y_{p1}, \ldots, y_{pk}), \\ f_i(y_{21}, \ldots, y_{2k}, y_{31}, \ldots, y_{3k}, \ldots, y_{p+1,1}, \ldots, y_{p+1,k})) \\ \leqslant \sum_{\nu=1}^{k} \sum_{\mu=1}^{p} a_{\mu\nu}^{(i)} d(y_{\mu\nu}, y_{\mu+1,\nu}) \qquad (i = 1, \ldots, k),$$

with $a_{\mu\nu}^{(i)} \geqslant 0$.

Introduce the notations

$$z_n=(x_n^{(1)},\dots,x_n^{(k)})$$

and

$$F(z_n,\dots,z_{n+p-1})=(f_1(z_n,\dots,z_{n+p-1}),\dots,f_k(z_n,\dots,z_{n+p-1})),$$

where we agree to identify the expressions

$$f_i(z_n,\dots,z_{n+p-1}) \text{ and } f_i(x_n^{(1)},\dots,x_n^{(k)},\dots,x_{n+p-1}^{(1)},\dots,x_{n+p-1}^{(k)})$$

for $i=1,\dots,k$.

System of equations (1.1) can therefore be written in the form

$$z_{n+p}=F(z_n,z_{n+1},\dots,z_{n+p-1}),$$

where $F:(E^k)^p\to E^k$.

Let $Q:(R^+)^k\to R^+$, where R^+ is the set of nonnegative reals, be a function with the following properties:

(i) $Q(w_1,\dots,w_k)=0$ if and only if $w_1=\dots=w_k=0$;

(ii) Q is an increasing function with respect to each argument;

(iii) $Q(v_1+w_1,\dots,v_k+w_k)\leqslant Q(v_1,\dots,v_k)+Q(w_1,\dots,w_k)$;

(iv) there exist functions $\varphi_i:(R^+)^{k^2}\to R^+$ $(i=1,\dots,p)$ such that

$$Q\left(\sum_{j=1}^{k}\sum_{i=1}^{p}a_{ij}^{(1)}w_{ij},\dots,\sum_{j=1}^{k}\sum_{i=1}^{p}a_{ij}^{(k)}w_{ij}\right)$$

$$\leqslant\sum_{i=1}^{p}\varphi_i(a_{11}^{(i)},\dots,a_{1k}^{(i)},a_{21}^{(i)},\dots,a_{2k}^{(i)},\dots,a_{k1}^{(i)},\dots,a_{kk}^{(i)})\,Q(w_{i1},\dots,w_{ik}).$$

Introduce into the set E^k the metric function by the following definition: If $u=(u_1,\dots,u_k)\in E^k$ and $v=(v_1,\dots,v_k)\in E^k$, then

$$d(u,v):=Q(d(u_1,v_1),\dots,d(u_k,v_k)), \tag{2.2}$$

where d on the right hand side of (2.2) is the metric function in the space E.

It is easily verified that E^k with the metric function defined by (2.2) is a complete metric space.

Let $u_i=(u_{i1},\dots,u_{ik})\in E^k$ for $i=1,\dots,p+1$. Then, we get

$$d(F(u_1,\dots,u_p),F(u_2,\dots,u_{p+1}))$$

$$=d((f_1(u_1,\dots,u_p),\dots,f_k(u_1,\dots,u_p)),(f_1(u_2,\dots,u_{p+1}),\dots,f_k(u_2,\dots,u_{p+1})))$$

$$=Q(d(f_1(u_1,\dots,u_p),f_1(u_2,\dots,u_{p+1})),\dots,d(f_k(u_1,\dots,u_p),f_k(u_2,\dots,u_{p+1})))$$

$$\leqslant Q\left(\sum_{j=1}^{k}\sum_{i=1}^{p}a_{ij}^{(1)}d(u_{ij},u_{i+1,j}),\dots,\sum_{j=1}^{k}\sum_{i=1}^{p}a_{ij}^{(k)}d(u_{ij},u_{i+1,j})\right)$$

$$\leqslant\sum_{i=1}^{p}\varphi_i(a_{11}^{(i)},\dots,a_{1k}^{(i)},\dots,a_{k1}^{(i)},\dots,a_{kk}^{(i)})\,Q(d(u_{i1},u_{i+1,1}),\dots,d(u_{ip},u_{i+1},p))$$

$$\leqslant\sum_{i=1}^{p}\varphi_i(a_{11}^{(i)},\dots,a_{1k}^{(i)},\dots,a_{k1}^{(i)},\dots,a_{kk}^{(i)})\,d(u_i,u_{i+1}).$$

A direct application of Theorem P yields the result:

Let $f_i: E^{pk} \to E$ $(i=1, \ldots, k)$ where E is a complete metric space and let the inequalities (2.1) be valid for all $y_{\mu\nu} \in E$ $(\mu = 1, \ldots, p;\ \nu = 1, \ldots, k)$. Let the sequences $(x_n^{(1)}), \ldots, (x_n^{(k)})$ be defined by (1.1) where the elements $x_j^{(i)}$ $(i=1, \ldots, k;\ j=1, \ldots, p)$ are arbitrary. If the function $Q: (R^+)^k \to R^+$ satisfies (i), (ii), (iii) and (iv), with

$$\sum_{i=1}^{p} \varphi_i (a_{11}^{(i)}, \ldots, a_{1k}^{(i)}, \ldots, a_{k1}^{(i)}, \ldots, a_{kk}^{(i)}) < 1,$$

then the sequences $(x_n^{(1)}), \ldots, (x_n^{(k)})$ converge respectively to $x_1, \ldots, x_k$, where $(x_1, \ldots, x_k)$ is the unique solution of the system

$$x_i = f_i (x_1, \ldots, x_k, x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k) \quad (i=1, \ldots, k).$$

In the special case when $Q(w_1, \ldots, w_k) = (w_1^2 + \ldots + w_k^2)^{1/2}$, $\varphi_1 = \ldots = \varphi_p = \varphi$, where

$$\varphi (u_{11}, \ldots, u_{1k}, \ldots, u_{k1}, \ldots, u_{kk}) = \left(\sum_{j=1}^{k} \sum_{i=1}^{k} u_{ij}^2 \right)^{1/2},$$

we obtain Theorem 1 from [7].

Furthermore, if $\mathbf{A} = \| w_{ij} \|_{k \times k}$, we can take $\varphi_1 = \ldots = \varphi_k = \varphi$, with $\varphi (w_{11}, \ldots, w_{ik}, \ldots, w_{k1}, \ldots, w_{kk}) = n(\mathbf{A})$, where $n(\mathbf{A})$ is any norm of $\mathbf{A}$, and then define the corresponding function Q. This result is given in Theorem 2 of [7].

3. In [5] we have given two sufficient conditions which ensure that the sequences (x_n), (y_n), defined by

$$x_{n+2} = f(x_n, y_n, x_{n+1}, y_{n+1}), \quad y_{n+2} = g(x_n, y_n, x_{n+1}, y_{n+1})$$

$(x_1, x_2, y_1, y_2$ arbitrary), where f and g satisfy

$$\text{(3.1)} \quad d(f(u_1, u_2, v_1, v_2), f(v_1, v_2, w_1, w_2)) \leqslant a_1 d(u_1, v_1) + a_2 d(u_2, v_2) + b_1 d(v_1, w_1) + b_2 d(v_2, w_2)$$

$$d(g(u_1, u_2, v_1, v_2), g(v_1, v_2, w_1, w_2)) \leqslant a_3 d(u_1, v_1) + a_4 d(u_2, v_2) + b_3 d(v_1, w_1) + b_4 d(v_2, w_2)$$

$(a_i, b_i \geqslant 0)$ are convergent to x, y respectively, where (x, y) is the only solution of the system $x = f(x, y, x, y)$, $y = g(x, y, x, y)$.

They are:

$$\text{(3.2)} \qquad \max (a_1 + b_1, a_2 + b_2) + \max (a_3 + b_3, a_4 + b_4) < 1$$

and

$$\text{(3.3)} \qquad \max (a_1 + b_1 + a_3 + b_3, a_2 + b_2 + a_4 + b_4) < 1.$$

We have also shown that the condition (3.2) is implied by Prešić's result (Theorem P).

If we adopt the notation used by Udovičić, inequalities (3.1), (3.2) and (3.3) can be written as

$$\left\| \begin{matrix} d(f(u_1, u_2, v_1, v_2), f(v_1, v_2, w_1, w_2)) \\ d(g(u_1, u_2, v_1, v_2), g(v_1, v_2, w_1, w_2)) \end{matrix} \right\| \leqslant \mathbf{A} \left\| \begin{matrix} d(u_1, v_1) \\ d(u_2, v_2) \end{matrix} \right\| + \mathbf{B} \left\| \begin{matrix} d(v_1, w_1) \\ d(v_2, w_2) \end{matrix} \right\|,$$

where $n(\mathbf{A}) + n(\mathbf{B}) < 1, \quad n(\mathbf{A} + \mathbf{B}) < 1$

$$\mathbf{A} = \left\| \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} \right\|, \quad \mathbf{B} = \left\| \begin{matrix} b_1 & b_2 \\ b_3 & b_4 \end{matrix} \right\| \text{ and } n\left(\left\| \begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix} \right\| \right) = \max(\alpha + \gamma, \beta + \delta).$$

Since n is a matrix norm, we have

$$n(\mathbf{A} + \mathbf{B}) \leqslant n(\mathbf{A}) + n(\mathbf{B}),$$

which clearly shows that, with this *special* choice for the matrix norm, our result from [5] gives a better condition for matrices **A** and **B**. (Without any difficulty, our proof for the case of 2×2 matrices is carried over to $k \times k$ matrices).

This suggests the following question:

If $n(\mathbf{A})$ is any norm of the $k \times k$ matrix **A**, is it possible to replace the condition given in Theorem 2 of [7], namely

$$n(\mathbf{A}_1) + \cdots + n(\mathbf{A}_p) < 1$$

(which, in fact, follows from Theorem P) by the better condition

$$n(\mathbf{A}_1 + \cdots + \mathbf{A}_p) < 1?$$

If the norm $n(\mathbf{A})$ of $\mathbf{A} = \| w_{ij} \|_{k \times k}$ is defined by $n(\mathbf{A}) = \max\limits_{1 \leq j \leq n} \sum\limits_{i=1}^{n} w_{ij}$, the answer is affirmative.

4. We shall return to these topics in an other paper.

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