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A REMARK ON A GENERALISATION OF MONOTONIC SEQUENCES

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1. A bounded monotonic sequence is convergent. E. T. Copson [1] has generalised this result by proving the following

Theorem C. *If* (a_n) *is a bounded sequence which satisfies the inequality*

$$a_{n+r} \leqslant \sum_{s=1}^{r} k_s a_{n+r-s} \tag{1}$$

where the coefficients k_s *are strictly positive and* $k_1+k_2+\cdots+k_r=1$ *then* (a_n) *is a convergent sequence.*

E. T. Copson also noted that the coefficients k_s need not be all positive. He considered the sequence (a_n) which is bounded and which satisfies

$$a_{n+3} \leqslant -\frac{1}{2} a_{n+2} + \frac{3}{4} a_{n+1} + \frac{3}{4} a_n . \tag{2}$$

That sequence is convergent.

Using a different method of proof, in this section we shall arrive at a result which may be applied to sequences which satisfy (1) whith $k_1+k_2+\cdots+k_r=1$, where all the coefficients k_ν need not be positive. Namely, we shall prove the following

Theorem 1. *Let* (a_n) *be a bounded sequence, which satisfies the inequality* (1) *with* $k_1+k_2+\cdots+k_r=1$. *If* $l_s=1-k_1-\cdots-k_s$ $(s=1,2,\ldots,r-1)$ *and if all roots of the equation*

$$\lambda^{r-1}+l_1\lambda^{r-2}+\cdots+l_{r-1}=0 \tag{3}$$

are distinct and lie in the unit disk, then (a_n) *is a convergent sequence.*

We shall need the following

Lemma. *Let* $(a_{n+p}+q_1 a_{n+p-1}+\cdots+q_p a_n)$ *be a convergent sequence, and let all the roots of the equation*

$$\lambda^p+q_1\lambda^{p-1}+\cdots+q_p=0 \tag{4}$$

be distinct and lie in the unit disk $|\lambda|<1$. *Then* (a_n) *is a convergent sequence*

Proof. Let

(5) $$a_{n+p}+q_1 a_{n+p-1}+\cdots+q_p a_n=x_n.$$

The general solution of the difference equation

(6) $$a_{n+p}+q_1 a_{n+p-1}+\cdots+q_p a_n=0$$

is given by

(7) $$a_n=\sum_{i=1}^{p} C_i \alpha_i^n,$$

where C_i are arbitrary constants and $\alpha_1,\ldots,\alpha_p$ are roots (supposed to be distinct) of the equation (4).

The general solution of equation (5) can be determined starting with (7), by the Lagrange method of variation of parameters. Indeed, it is readily seen that the general solution of (5) can be written in the form

(8) $$a_n=\sum_{i=1}^{p} \alpha_i^n \Big(C_i+D_i \sum_{\nu=1}^{n-1} \frac{x_\nu}{\alpha_i^\nu}\Big),$$

where C_i are arbitrary constants, and D_i are constants which involve $\alpha_1,\ldots,\alpha_p$.

Since (x_n) is a convergent sequence, from (8), applying the well-known result that if $\lim x_n=x$, if $p_\nu>0$ $(\nu=1,2,\ldots)$ and if $P_n=\sum_{\nu=1}^{n} p_\nu \to +\infty$ $(n\to+\infty)$, then $\lim \frac{1}{P_n}\sum_{\nu=1}^{n} p_\nu x_\nu=x$, we conclude that (a_n) is a convergent sequence.

Remark 1. If $|q_1|+|q_2|+\cdots+|q_p|<1$, then by a direct application of Rouché's theorem we see that all the roots of equation (4) lie in the unit disk.

Proof of Theorem 1. Inequality (1) can be written in the form

$$a_{n+r}+l_1 a_{n+r-1}+\cdots+l_{r-1} a_{n+1} \leqslant a_{n+r-1}+l_1 a_{n+r-2}+\cdots+l_{r-1} a_n$$

which means that the sequence $(a_{n+r-1}+l_1 a_{n+r-2}+\cdots+l_{r-1} a_n)$ is bounded and monotonic, i.e. convergent.

The result then follows directly from the Lemma.

Remark 2. From Remark 1 follows that all roots of equation (3) lie in $|\lambda|<1$ if $|l_1|+|l_2|+\cdots+|l_{r-1}|<1$.

Remark 3. For $r=2$ Copson's theorem reads: If (a_n) is a bounded sequence and if $a_{n+2}\leqslant(1-k)a_{n+1}+ka_n$, where $0<k<1$, then (a_n) is a convergent sequence. We obtain the following result: If (a_n) is a bounded sequence and if $a_{n+2}\leqslant(1-k)a_{n+1}+ka_n$, with $|k|<1$, then (a_n) is a convergent sequence. This result is a direct generalisation of Theorem C (for $r=2$).

Example 1. A bounded sequence which satisfies (2) is convergent, because both roots of the equation $\lambda^2+\frac{3}{2}\lambda+\frac{3}{4}=0$ lie in the unit disk.

Example 2. A bounded sequence which satisfies

$$a_{n+5} \leqslant \frac{13}{12} a_{n+4} - \frac{1}{2} a_{n+3} + \frac{1}{3} a_{n+2} + \frac{1}{4} a_{n+1} - \frac{1}{6} a_n$$

is convergent, since

$$\left| -\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} \right| + \left| \frac{1}{3} + \frac{1}{4} - \frac{1}{6} \right| + \left| \frac{1}{4} - \frac{1}{6} \right| + \left| -\frac{1}{6} \right| = \frac{3}{4} < 1.$$

Remark 4. The condition that all roots of (3) are distinct is dictated by the Lemma. If we do not request the roots of (4) to be distinct, the general solution of (5) becomes more complicated. It seems, however, that the same conclusions will be valid if equation (3) has multiple roots.

2. In this section we shall give a generalisation of Theorem 1.

We first quote a result which follows from a theorem proved in [2].

Theorem K. *Let* $x_{n+k} = f_n(x_{n+k-1}, x_{n+k-2}, \ldots, x_n)$ *and let*

$$|f_n(u_1, u_2, \ldots, u_k) - f_{n-1}(u_2, u_3, \ldots, u_{k+1})| \leqslant q_1 |u_1 - u_2| + q_2 |u_2 - u_3| + \cdots + q_k |u_k - u_{k+1}| + y_{n-1},$$

where q_i, y_i *are nonnegative real numbers such that* $q_1 + q_2 + \cdots + q_k < 1$ *and the series* $\sum_{\nu=1}^{\infty} y_\nu$ *is convergent.*

Then (x_n) *is a convergent sequence.*

As a generalisation of Theorem 1 we now prove the following

Theorem 2. *Let* (a_n) *be a bounded sequence which satisfies the inequality*

$$a_{n+r} + g(a_{n+r-1}, \ldots, a_{n+1}) \leqslant a_{n+r-1} + g(a_{n+r-2}, \ldots, a_n) \tag{9}$$

and let

$$|g(v_1, v_2, \ldots, v_{r-1}) - g(v_2, v_3, \ldots, v_r)| \leqslant q_1 |v_1 - v_2| + q_2 |v_2 - v_3| + \cdots + q_{r-1} |v_{r-1} - v_r|, \tag{10}$$

with $q_\nu \geqslant 0$ $(\nu - 1, 2, \ldots, r-1)$ *and* $q_1 + q_2 + \cdots + q_{r-1} < 1$.

Then (a_n) *is a convergent sequence.*

Proof. Inequality (9) implies that the sequence $(a_{n+r-1} + g(a_{n+r-2}, \ldots, a_n))$ is monotonic.

On the other hand, if b_i are numbers such that $g(b_1, \ldots, b_{r-1}) = B$ is finite, from (10) we find

$$|g(v_1, v_2, \ldots, v_{r-1}) - g(b_1, b_2, \ldots, b_{r-1})| \leqslant |g(v_1, v_2, \ldots, v_{r-1}) - g(v_2, v_3, \ldots, v_{r-1}, b_1)| + |g(v_2, v_3, \ldots, v_{r-1}, b_1) - g(v_3, v_4, \ldots, b_1, b_2)| + \cdots + |g(v_{r-1}, b_1, \ldots, b_{r-2}) - g((b_1, b_2, \ldots, b_{r-1})|$$

$$\leqslant q_1 |v_1 - v_2| + q_2 |v_2 - v_3| + \cdots + q_{r-1} |v_{r-1} - b_1|$$
$$+ q_1 |v_2 - v_3| + q_2 |v_3 - v_4| + \cdots + q_{r-2} |v_{r-1} - b_1| + q_{r-1} |b_1 - b_2|$$
$$+ \cdots + q_1 |v_{r-1} - b_1| + q_2 |b_1 - b_2| + \cdots + q_{r-1} |b_{r-2} - b_{r-1}|$$
$$\leqslant 2(q_1 + \cdots + q_{r-1})(|v_1| + |v_2| + \cdots + |v_{r-1}| + |b_1| + \cdots + |b_{r-1}|).$$

Suppose that $|a_n| \leqslant m$ and $|b_i| \leqslant b\ (i = 1, 2, \ldots, r-1)$. Then

(11) $$|a_{n+r-1} + g(a_{n+r-2}, \ldots, a_n)| \leqslant |a_{n+r-1}| + |g(a_{n+r-2}, \ldots, a_n)|,$$

and

$$|g(a_{n+r-2}, \ldots, a_n) - g(b_1, \ldots, b_{r-1})|$$
$$\leqslant 2(q_1 + \cdots + q_{r-1})(|a_{n+r-2}| + \cdots + |a_n| + (r-1)b)$$
$$\leqslant 2(q_1 + \cdots + q_{r-1})(r-1)(m+b) = M.$$

Hence,

$$B - M \leqslant g(a_{n+r-2}, \ldots, a_n) \leqslant M + B$$

which implies

$$|g(a_{n+r-2}, \ldots, a_n)| \leqslant \max(|M+B|, |B-M|) = A,$$

and inequality (11) then yields

$$|a_{n+r-1} + g(a_{n+r-2}, \ldots, a_n)| \leqslant m + A.$$

Therefore, $(a_{n+r-1} + g(a_{n+r-2}, \ldots, a_n))$ is bounded and monotonic, and hence convergent.

Put

$$a_{n+r-1} + g(a_{n+r-2}, \ldots, a_n) = y_n,$$

i.e.

$$a_{n+r-1} = f_n(a_{n+r-2}, \ldots, a_n),$$

where

$$f_n(u_1, u_2, \ldots, u_{r-1}) = g(u_1, u_2, \ldots, u_{r-1}) + y_n.$$

Then, in virtue of (10) we find

$$|f_n(u_1, u_2, \ldots, u_{r-1}) - f_{n-1}(u_2, u_3, \ldots, u_r)|$$
$$\leqslant q_1 |u_1 - u_2| + q_2 |u_2 - u_3| + \cdots + q_{r-1} |u_{r-1} - u_r| + |y_n - y_{n-1}|,$$

where $q_1 + \cdots + q_{r-1} < 1$ and the series $\sum_{n=1}^{\infty} |y_n - y_{n-1}|$ is convergent, since (y_n) is a convergent monotonic sequence.

According to Theorem K, (a_n) is a convergent sequence.

The proof is complete.

Remark 5. If $g(x_1, x_2, \ldots, x_{r-1}) = 0$, Theorem 2 yields the well-known theorem that a bounded monotonic sequence is convergent.

Remark 6. Theorem 2 suggests that in Theorem 1 we may drop the condition that all the roots of equation (9) are distinct.

Remark 7. Theorem 2 for $r=2$ reads:

Let (a_n) be a bounded sequence which satisfies the inequality

$$a_{n+2}+g(a_{n+1})\leqslant a_{n+1}+g(a_n),$$

where

$$|g(u)-g(v)|\leqslant q\,|u-v|$$

Then (a_n) is a ...

Thi... direct generalisation of Theorem C in the case $r=2$.

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REFERENCES

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[2] J. D. Kečkić, *On the convergence of certain sequences*, Publ. Inst. Math. (Beograd) **9 (23)** (1969), 157—162.