ON THE COEFFICIENT STRUCTURE AND A GROWTH THEOREM FOR THE FUNCTIONS \( f(z) \) FOR WHICH \( zf'(z) \) IS SPIRALLIKE

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1. Introduction:

Let \( S(\alpha) \) denote the class of spirallike, regular univalent functions \( f(z) \), for which \( f(0) = 0 \), \( f'(0) = 1 \). This class was introduced by Špacek [6] as early as in 1932. The sharp coefficient structure of this class of functions has been investigated recently in ([8], [4]). As recently as in 1969, Robertson [5] introduced another analogous class which includes the class of convex functions as a proper subfamily. For \( -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \), \( f(z) \) belongs to the class \( C_1(\alpha) \) if

(i) \( f(z) \) is regular in the unit disc \( D \) (ii) \( f'(z) \neq 0 \) in \( D \) and (iii) \( \Re \left\{ e^{i\alpha} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \); \( z \in D \). These three conditions are precisely for the functions \( zf'(z) \) to belong to the class \( S(\alpha) \). It is interesting to note that (see Robertson [5]) \( f(z) \in C_1(\alpha) \) is univalent if \( 0 < \cos \alpha < x_0 \), where \( x_0 \) is a positive root \( 0.231\ldots \) of the equation \( 16x^3 + 16x^2 + x + 1 = 0 \). We denote the class of functions \( f(z) \in C_1(\alpha) \) by \( C(\alpha) \) for which (iv) holds and \( f(0) = 0 \), \( f'(0) = 1 \).

It is known [2] that if \( \alpha = 0 \) and \( f \in C(\alpha) \) then \( |f^n(0)| < n! \) and the result is sharp. In the present paper a sharp coefficient bound for the functions \( f(z) \in C(\alpha) \) is obtained and from which it is deduced that \( |f^n(0)| \) can not exceed \( n! \). Cos \( \alpha \). A similar type of result is also obtained for the functions belonging to class \( S(\alpha) \) in [1].

2. We need the following lemmas:

**Lemma 1.** Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n \), \( a_1 = 1 \) belong to the class \( C(\alpha) \) and let \( F \) be defined by

\[
F(z) = e^{i\alpha} \left[ 1 + \frac{zf''(z)}{f'(z)} \right].
\]
Then, for $0 < r < 1$

$$\frac{1}{2\pi} \int_0^{2\pi} \text{Re} \{F(re^{i\theta})\} |f'(re^{i\theta})|^2 \, d\theta = \cos \alpha \left( \sum_{n=1}^{\infty} n^3 |a_n|^2 r^{2n-2} \right)$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \text{Im} \{F(re^{i\theta})\} |f'(re^{i\theta})|^2 \, d\theta = \sin \alpha \sum_{n=1}^{\infty} n^3 |a_n|^2 r^{2n-2}.$$ 

**Proof:** Fix, $0 < r < 1$. Since $f$ is univalent and belong to class $C(\alpha)$ we have by Parsevall's relation that $e^{i\alpha}[f'(z) + zf''(z)] = f'(z)F(z)$ and

$$\int_0^{2\pi} |f'(z)|^2 \, d\theta + \int_0^{2\pi} f'(z) \cdot zf''(z) \, d\theta = \int_0^{2\pi} e^{-i\alpha} |f'(z)|^2 F(z) \, d\theta$$

$$-2\pi \left( \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} + \sum_{n=1}^{\infty} n^2 (n-1) |a_n|^2 r^{2n-2} \right) = \int_0^{2\pi} e^{-i\alpha} |f'(z)|^2 F(z) \, d\theta$$

or

$$2\pi \sum_{n=1}^{\infty} n^3 |a_n|^2 r^{2n-2} = \int_0^{2\pi} e^{-i\alpha} |f'(z)|^2 F(z) \, d\theta.$$

Equating real and imaginary parts, we have

$$2\pi \sum_{n=1}^{\infty} n^3 |a_n|^2 r^{2n-2} = \cos \alpha \left( \int_0^{2\pi} |f'(z)|^2 \text{Re} \{F(z)\} \, d\theta + \sin \alpha \int_0^{2\pi} |f'(z)|^2 \text{Im} \{F(z)\} \, d\theta \right)$$

$$0 = \cos \alpha \int_0^{2\pi} |f'(z)|^2 \text{Im} \{F(z)\} \, d\theta - \sin \alpha \int_0^{2\pi} |f'(z)|^2 \text{Re} \{F(z)\} \, d\theta.$$

On solving these simultaneous equations we have

$$\int_0^{2\pi} |f'(z)|^2 \text{Re} \{F(re^{i\theta})\} \, d\theta = 2\pi \cos \alpha \sum_{n=1}^{\infty} n^3 |a_n|^2 r^{2n-2}$$

and

$$\int_0^{2\pi} |f'(z)|^2 \text{Im} \{F(re^{i\theta})\} \, d\theta = 2\pi \sin \alpha \sum_{n=1}^{\infty} n^3 |a_n|^2 r^{2n-2}.$$ 

This completes the proof of lemma 1.

**Lemma 2.** Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 = 1$, belong to the class $C(\alpha)$ and $\{S_n\}$
be the sequence of the complex numbers defined by

\[ S_n(t) = \sum_{k=1}^{n} k a_k e^{ikt} \text{ for } n = 1, 2, 3, \ldots \]

Then

\[
\int_{0}^{2\pi} |S_n(t)|^2 \, d \mu_f(t) = \cos \alpha \sum_{m=1}^{n} m^3 |a_m|^2
\]

where \( \mu_f(t) \) is a non-decreasing function of \( t \) such that

\[
\cos \alpha = \int_{0}^{2\pi} d \mu_f(t).
\]

**Proof:** Let

\[
F(z) = e^{it} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] = \frac{1}{2} u_0 + \sum_{n=1}^{\infty} u_n z^n.
\]

Then there exists a non-decreasing function \( \mu_f(t) \) such that

\[
F(z) = \int_{0}^{2\pi} \frac{z + e^{it}}{e^{it} - z} \, d \mu_f(t) + i \sin \alpha
\]

where

\[
\cos \alpha = \int_{0}^{2\pi} d \mu_f(t).
\]

Thus, we have \([7]\) for \( n > 1 \).

\[
\pi u_n r^n = \int_{0}^{2\pi} e^{-in\theta} \Re \{ F(re^{i\theta}) \} \, d \theta = \int_{0}^{2\pi} \int_{0}^{2\pi} \frac{(1-r^2) e^{-in\theta}}{|1-z e^{-it}|^2} \, d \theta \, d \mu_f(t) = 2\pi r^n \int_{0}^{2\pi} e^{-int} \, d \mu_f(t).
\]

Hence

\[
u_n = 2 \int_{0}^{2\pi} e^{-int} \, d \mu_f(t) \quad \text{for } n = 1, 2, 3, \ldots
\]

From lemma 1, we have

\[
\sum_{n=1}^{\infty} \frac{n^3 |a_n|^2 r^{2n-2}}{(1-r^2)} = \sec \alpha \int_{0}^{2\pi} \Re \{ F(re^{i\theta}) \} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 \, d \theta
\]

\[
= \sec \alpha \int_{0}^{2\pi} \int_{0}^{\infty} e^{it} S_n(t) z^{n-1} \left| d \mu_f(t) \right|^2
\]

\[
= \sum_{n=1}^{\infty} \sec \alpha \left\{ \int_{0}^{2\pi} |S_n(t)|^2 \, d \mu_f(t) \right\} r^{2n-2}
\]
Hence on comparing the coefficients of $r^{2n-2}$ from both sides we have
\[
\int_0^{2\pi} |S_n(t)|^2 d\nu_f(t) = \cos \alpha \sum_{m=1}^{n} m^3 |a_m|^2
\]
This completes the proof of lemma 2.

Lemma 3. We have
\[
1 + 2^3 \cos^2 \alpha + \sum_{i=3}^{k} \frac{4i \cos^2 \alpha}{(i-1)^2} \prod_{j=2}^{i-1} \left( 1 + j \left( \frac{2 \cos \alpha}{j-1} \right)^2 \right)
\]
(2.3)
\[
= \prod_{m=2}^{k} \left[ 1 + m \left( \frac{2 \cos \alpha}{m-1} \right)^2 \right] \text{ for } k = 4, 5, 6, \ldots
\]
Proof: First of all we will verify the lemma for $k = 3$. Thus for $k = 3$ we have
\[
1 + 2^3 \cos^2 \alpha + \sum_{i=3}^{3} \frac{4i \cos^2 \alpha}{(i-1)^2} \prod_{j=2}^{i-1} \left( 1 + j \left( \frac{2 \cos \alpha}{j-1} \right)^2 \right)
\]
\[
= 1 + 2^3 \cos^2 \alpha + \frac{3 \cdot 3 \cos^2 \alpha}{2^2} \left( 1 + 2 \left( \frac{2 \cos \alpha}{2-1} \right)^2 \right)
\]
\[
= \prod_{j=2}^{3} \left[ 1 + j \left( \frac{2 \cos \alpha}{j-1} \right)^2 \right].
\]
This verifies the lemma for $k = 3$. Now assume that the lemma is true for $k = 4, 5, \ldots, n$. Then
\[
1 + 2^3 \cos^2 \alpha + \sum_{i=3}^{k+1} \frac{4i \cos^2 \alpha}{(i-1)^2} \prod_{j=2}^{i-1} \left( 1 + j \left( \frac{2 \cos \alpha}{j-1} \right)^2 \right)
\]
\[
= \prod_{j=2}^{k} \left[ 1 + j \left( \frac{2 \cos \alpha}{j-1} \right)^2 \right] + \frac{4(k + 1) \cos^2 \alpha}{k^2} \prod_{j=2}^{k} \left( 1 + j \left( \frac{2 \cos \alpha}{j-1} \right)^2 \right)
\]
\[
= \prod_{j=2}^{k} \left[ 1 + j \left( \frac{2 \cos \alpha}{j-1} \right)^2 \right] \left[ 1 + (k + 1) \left( \frac{2 \cos \alpha}{k} \right)^2 \right]
\]
\[
= \prod_{j=2}^{k+1} \left[ 1 + j \left( \frac{2 \cos \alpha}{j-1} \right)^2 \right].
\]
Thus by induction the proof of the lemma is complete.
Next we state a lemma due to Hayman ([3], p. 40) and derive some of its consequences which we shall need.

Lemma 4. Suppose that
\[
(2.4) \quad \psi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n = u + iv
\]
is regular in the disc \(|z|<1\) and \(u>0\) there. Then the limit

\[
A(0) = \lim_{r \to 1} \left( \frac{1-r}{1+r} \right) \psi(re^{i\theta})
\]

exists. The set of distinct values \(\theta = \theta_\nu\) in \(0 < \theta < 2\pi\) for which \(z_\nu = A(\theta_\nu) \neq 0\) is countable and \(\sum \alpha_\nu < 1\).

Further, we have

\[
\lim_{r \to 1} (1-r) \sum_{n=1}^{\infty} |b_n|^2 r^{2n} = 2 \sum \alpha_\nu^2.
\]

In fact, the above lemma implies that

\[
\lim_{r \to 1} (1-r) \sum_{n=1}^{\infty} |b_n|^2 r^n = 4 \sum \alpha_\nu^2.
\]

If instead of (2.4) we choose the function \(F(z) = e^{ia} + \sum_{n=1}^{\infty} u_n z^n\) which is regular in \(|z|<1\) and for which \(\Re \{F(z)\} > 0\), then

\[
A(0) = \lim_{r \to 1} \left( \frac{1-r}{1+r} \right) (F(re^{i\theta}) - e^{ia} + 1) = \lim_{r \to 1} \left( \frac{1-r}{1+r} \right) F(re^{i\theta})
\]

and

\[
\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta}) - e^{ia} + 1|^2 d\theta = 1 + \sum_{n=1}^{\infty} |u_n|^2 r^{2n}
\]

and hence by lemma 4, we have

\[
\lim_{r \to 1} (1-r) \sum_{n=1}^{\infty} |u_n|^2 r^n = 4 \sum \alpha_\nu^2
\]

where \(\alpha_\nu\)'s have the same meaning as in lemma 4.

3. Theorem 1. If \(f(z) = \sum_{n=1}^{\infty} a_n z^n\; ; \; a_1 = 1\) belong to the class \(C(\alpha)\), then

\[
|a_2| \leq \cos \alpha \text{ and}
\]

\[
(3.1) \quad |a_n| \leq \frac{2 \cos \alpha}{n(n-1)} \prod_{m=2}^{n-1} \left[ 1 + m \left( \frac{2 \cos \alpha}{m-1} \right)^2 \right] \quad \text{for } n = 3, 4, 5, \ldots
\]

Proof: Define \(F(z)\) as in lemma 2 and write the expressions of \(f(z)\), \(f'(z)\), and \(f''(z)\) in terms of their power series in \(F(z)\), then on comparing the coefficients of various powers of \(z\), we have

\[
n(n+1)a_n = e^{-i\alpha} \sum_{m=1}^{n} m a_m u_{n-m} ; \quad a_1 = 1.
\]
Now by elementary calculations, we have
\[ n(n-1) a_n e^{i\alpha} = 2 \sum_{m=1}^{n-1} m a_n \int_0^{2\pi} e^{-i(n-m)t} d\mu_f(t) \]

Using lemma 2 and applying Schwarz’s inequality, we have
\[
(3.2) \quad n^2(n-1)^2 |a_n|^2 \leq 4 \cos \alpha \int_0^{2\pi} |S_{n-1}(t)|^2 d\mu_f(t) = 4 \cos^2 \alpha \sum_{m=1}^{n-1} m^3 |a_m|^2
\]
for \( n = 2, 3, 4, \ldots \).

From (3.2) it is clear that, for \( n = 2, 3 \)
\[
(3.3) \quad |a_2|^2 \leq \frac{4 \cos^2 \alpha}{4} - 1^3 \cdot |a_1|^2 = \cos^2 \alpha
\]
and
\[
(3.4) \quad |a_3|^2 \leq \frac{4 \cos^2 \alpha}{3^2 \cdot 2^2} \left[ 1 + 2 \left( \frac{2 \cos \alpha}{2 - 1} \right)^2 \right].
\]

Hence, the theorem is true for \( n = 2, 3 \). Assume the truth of (3.1) for \( n = k \)
and establish the truth for \( k + 1 \). Thus from (3.2) and the hypothesis, we have
\[
(k+1)^2 k^2 |a_{k+1}|^2 \leq 4 \cos^2 \alpha \left[ (1^3 \cdot |a_1|^2 + 2^3 |a_2|^2 + \cdots + k^3 |a_k|^2) \right]
\]
\[
< 4 \cos^2 \alpha \left[ 1 + 2^3 \cos^2 \alpha + \frac{3 \cdot 4 \cdot \cos^2 \alpha}{2^2} \left( 1 + 2 \left( \frac{2 \cos \alpha}{2 - 1} \right)^2 \right) + \cdots \right]
\]
\[
+ \frac{4 k \cos^2 \alpha k^{k-1}}{(k-1)^2} \left[ 1 + j \left( \frac{2 \cos \alpha}{j - 1} \right)^2 \right].
\]

Now by using lemma 3 we obtain
\[
k^2(k+1)^2 |a_k|^2 \leq 4 \cos^2 \alpha \prod_{j=2}^{k} \left[ 1 + j \left( \frac{2 \cos \alpha}{j - 1} \right)^2 \right]
\]
This establishes the theorem by an appeal to the induction hypothesis, as the result is true actually for \( n = 3 \).

Sharpness of the theorem follows from the function
\[ f(z) = \frac{1}{(1-z) e^{-2i\alpha} - 1}. \]

**Corollary:** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C(\alpha) \) then
\[
(3.5) \quad |a_m| \leq \cos \alpha \left[ \cos^2 \alpha + \frac{4 \sin^2 \alpha}{m^2 (m+1)^2} \right]^{\frac{1}{2}} \text{ for } m = 3, 4, \ldots .
\]
and in particular

\begin{equation}
|a_m| < \cos \alpha.
\end{equation}

**Proof:** That (3.5) holds for \( m = 3 \). Suppose (3.5) holds for \( n = 4, \ldots, m \), then from (3.2) we have

\begin{align}
(3.6) \quad m^2 (m+1)^2 |a_{m+1}|^2 &< 4 \cos^2 \alpha [1^3 + 2^3 |a_1|^2 + 3^3 |a_2|^2 + \cdots + m^3 |a_m|^2] \\
&< 4 \cos^2 \alpha \left[ \left( \frac{m (m+1)}{2} \right)^2 \cos^2 \alpha + \sin^2 \alpha \right] \\
&= m^2 (m+1)^2 \cos^2 \alpha + \sin^2 \alpha.
\end{align}

Hence

\begin{equation}
|a_{m+1}|^2 < \cos^2 \alpha \left[ \cos^2 \alpha + \frac{4 \sin^2 \alpha}{m^2 (m+1)^2} \right].
\end{equation}

This establishes (3.5). Further, since \( \cos^2 \alpha + \frac{4 \sin^2 \alpha}{m^2 (m+1)^2} < 1 \) for \( m = 2, 3, \ldots \), (3.5) implies \( |a_m| < \cos \alpha \). This completes the proof of Corollary 1.


**Theorem.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C(\alpha) \) then

\begin{equation}
\int_0^{2\pi} \log \left| \frac{f'(re^{i\theta})}{r} \right| d\mu_f(\theta) = 2 \cos \alpha \sum |x|^2
\end{equation}

as \( r \to 1 \), where \( \alpha \)'s are defined as in lemma 5.

**Proof.** Using (2.2), we obtain for every \( r, 0 < r < 1 \)

\begin{align}
2 \int_0^{2\pi} F(re^{i\theta}) d\mu_f(\theta) & = 2 \int_0^{2\pi} u_0^2 \frac{1}{2} d\mu_f(\theta) + \sum_{n=1}^{\infty} u_n \int_0^{2\pi} r^n e^{i\alpha} d\mu_f(\theta) \\
& = 2 e^{i\alpha} \int_0^{2\pi} d\mu_f(\theta) + \sum_{n=1}^{\infty} u_n |n|^2 r^n \\
& = 2 e^{i\alpha} \cos \alpha + \sum_{n=1}^{\infty} r^n |n|^2.
\end{align}

Equating real and imaginary parts, we obtain

\begin{equation}
2 \int_0^{2\pi} \Re \{F(re^{i\theta}) - \cos \alpha \} d\mu_f(\theta) = \sum_{n=1}^{\infty} |u_n|^2 r^n,
\end{equation}

and

\begin{equation}
2 \int_0^{2\pi} \Im \{F(re^{i\theta}) - \sin \alpha \} d\mu_f(\theta) = 0
\end{equation}
Also, if $G(z) = 1 + \frac{zf''(z)}{f'(z)}$ then

$$\text{Re} \{ F(re^{i\theta}) \} = \cos \alpha \text{Re} \{ G(re^{i\theta}) \} - \sin \alpha \text{Im} \{ G(re^{i\theta}) \}$$

$$\text{Im} \{ F(re^{i\theta}) \} = \cos \alpha \text{Im} \{ G(re^{i\theta}) \} + \sin \alpha \text{Re} \{ G(re^{i\theta}) \}.$$ 

The above simultaneous equations yield,

$$(4.3) \quad \text{Re} \{ G(re^{i\theta}) \} = 1 + \cos \alpha [\text{Re} \{ F(re^{i\theta}) \} - \cos \alpha] + \sin \alpha [\text{Im} \{ F(re^{i\theta}) \} - \sin \alpha]$$

Equations (4.1), (4.2) and (4.3) together yield

$$2 \int_0^{2\pi} \text{Re} \{ G(re^{i\theta}) \} \, d\mu_f(\theta) = 2 \int_0^{2\pi} d\mu_f(\theta) + \cos \alpha \sum_{n=1}^{\infty} |u_n|^2 r^n.$$ 

From this equation, we have

$$2 \int_0^{2\pi} d\mu_f(\theta) + 2 \text{Re} \int_0^{2\pi} \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} \, d\mu_f(\theta) = 2 \int_0^{2\pi} d\mu_f(\theta) + \cos \alpha \sum_{n=1}^{\infty} |u_n|^2 r^n,$$

$$2 \text{Re} \int_0^{2\pi} \log |f'(re^{i\theta})| \, d\mu_f(\theta) = \cos \alpha \sum_{n=1}^{\infty} \frac{|u_n|^{2r^n}}{n} r^n,$$

$$2 \int_0^{2\pi} \log |f'(re^{i\theta})| \, d\mu_f(\theta) = \cos \alpha \sum_{n=1}^{\infty} \frac{|u_n|^2}{n} r^n.$$ 

This completes the proof of theorem.

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REFERENCES