PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 16 (30), 1973, pp. 5—12

ON THE COEFFICIENT STRUCTURE AND A GROWTH THEOREM FOR THE FUNCTIONS $f(z)$ FOR WHICH $zf'(z)$ IS SPIRALLIKE

S. K. Bajpai and T. J. S. Mehrok

(Received April 19, 1972)

1. Introduction:

Let $S(\alpha)$ denote the class of spirallike, regular univalent functions $f(z)$, for which $f(0)=0$, $f'(0)=1$. This class was introduced by Špacek [6] as early as in 1932. The sharp coefficient structure of this class of functions has been investigated recently in ([8], [4]). As recently as in 1969, Robertson [5] introduced another analogous class which includes the class of convex functions as a proper subfamily. For $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, $f(z)$ belongs to the class $C_1(\alpha)$ if (*i*) $f(z)$ is regular in the unit disc D (*ii*) $f'(z)\neq 0$ in D and (*iii*) Re $\left\{e^{i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0$; $z\in D$. These three conditions are precisely for the functions $zf'(z)$ to belong to the class $S(\alpha)$. It is interesting to note that (see Robertson [5]) $f(z)\in C_1(\alpha)$ is univalent if $0<\cos\alpha\leqslant x_0$, where x_0 is a positive root $0.2315\ldots$ of the equation (*iv*) $16x^3+16x^2+x+1=0$. We denote the class of functions $f(z)\in C_1(\alpha)$ by $C(\alpha)$ for which (*iv*) holds and $f(0)=0$, $f'(0)=1$.

It is known [2] that if $\alpha=0$ and $f\in C(\alpha)$ then $|f^n(0)|\leqslant n!$ and the result is sharp. In the present paper a sharp coefficient bound for the functions $f(z)\in C(\alpha)$ is obtained and from which it is deduced that $|f^n(0)|$ can not exceed $n!$ Cos α. A similar type of result is also obtained for the functions belonging to class $S(\alpha)$ in [1].

2. We need the following lemmas:

Lemma 1. *Let* $f(z)=\sum_{n=1}^{\infty} a_n z^n$, $a_1=1$ *belong to the class* $C(\alpha)$ *and let* F *be defined by*

$$F(z)=e^{i\alpha}\left[1+\frac{zf''(z)}{f'(z)}\right].$$

Then, for $0 \leqslant r < 1$

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\{F(re^{i\theta})\}\,|f'(re^{i\theta})|^2\,d\theta = \cos\alpha\left[\sum_{n=1}^{\infty} n^3\,|a_n|^2\,r^{2n-2}\right]$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Im}\,\{F(re^{i\theta})\}\,|f'(re^{i\theta})|^2\,d\theta = \sin\alpha\sum_{n=1}^{\infty} n^3\,|a_n|^2\,r^{2n-2}.$$

Proof: Fix, $0 \leqslant r < 1$. Since f is univalent and belong to class $C(\alpha)$ we have by Parsevall's relation that $e^{i\alpha}[f'(z) + zf''(z)] = f'(z)\,F(z)$ and

$$\int_0^{2\pi} |f'(z)|^2\,d\theta + \int_0^{2\pi} f'(z)\cdot zf''(z)\,d\theta = \int_0^{2\pi} e^{-i\alpha}\,|f'(z)|^2\,F(z)\,d\theta$$

$$2\pi\left[\sum_{n=1}^{\infty} n^2\,|a_n|^2\,r^{2n-2} + \sum_{n=1}^{\infty} n^2\,(n-1)\,|a_n|^2\,r^{2n-2}\right] = \int_0^{2\pi} e^{-i\alpha}\,|f'(z)|^2\,F(z)\,d\theta$$

or

$$2\pi\sum_{n=1}^{\infty} n^3\,|a_n|^2\,r^{2n-2} = \int_0^{2\pi} e^{-i\alpha}\,|f'(z)|^2\,F(z)\,d\theta.$$

Equating real and imaginary parts, we have

$$2\pi\sum_{n=1}^{\infty} n^3\,|a_n|^2\,r^{2n-2} = \cos\alpha\int_0^{2\pi} |f'(z)|^2\operatorname{Re}\{F(z)\}\,d\theta + \sin\alpha\int_0^{2\pi} |f'(z)|^2\operatorname{Im}\{F(z)\}\,d\theta$$

$$0 = \cos\alpha\int_0^{2\pi} |f'(z)|^2\operatorname{Im}\{F(z)\}\,d\theta - \sin\alpha\int_0^{2\pi} |f'(z)|^2\operatorname{Re}\{F(z)\}\,d\theta.$$

On solving these simultaneous equations we have

$$\int_0^{2\pi} |f'(z)|^2\operatorname{Re}\{F(re^{i\theta})\}\,d\theta = 2\pi\cos\alpha\sum_{n=1}^{\infty} n^3\,|a_n|^2\,r^{2n-2}$$

and

$$\int_0^{2\pi} |f'(z)|^2\operatorname{Im}\{F(re^{i\theta})\}\,d\theta = 2\pi\sin\alpha\sum_{n=1}^{\infty} n^3\,|a_n|^2\,r^{2n-2}.$$

This completes the proof of lemma 1.

Lemma 2. *Let* $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 = 1$, *belong to the class* $C(\alpha)$ *and* $\{S_n\}$

be the sequence of the complex numbers defined by

$$S_n(t) = \sum_{k=1}^{n} k\, a_k e^{ikt} \quad \textit{for } n = 1, 2, 3, \ldots$$

Then

$$\int_0^{2\pi} |S_n(t)|^2 \, d\mu_f(t) = \cos\alpha \sum_{m=1}^{n} m^3 |a_m|^2$$

where $\mu_f(t)$ is a non-decreasing function of t such that

$$\operatorname{Cos}\alpha = \int_0^{2\pi} d\mu_f(t).$$

Proof: Let

$$F(z) = e^{i\alpha}\left[1 + \frac{zf''(z)}{f'(z)}\right] = \frac{1}{2} u_0 + \sum_{n=1}^{\infty} u_n z^n. \tag{2.1}$$

Then there exists a non-decreasing function $\mu_f(t)$ such that

$$F(z) = \int_0^{2\pi} \frac{z + e^{it}}{e^{it} - z} \, d\mu_f(t) + i \sin\alpha$$

where

$$\operatorname{Cos}\alpha = \int_0^{2\pi} d\mu_f(t).$$

Thus, we have [7] for $n \geqslant 1$.

$$\pi u_n r^n = \int_0^{2\pi} e^{-in\theta} \operatorname{Re}\{F(re^{i\theta})\}\, d\theta = \int_0^{2\pi}\int_0^{2\pi} \frac{(1-r^2)\, e^{-in\theta}}{|1 - ze^{-it}|^2} \, d\theta \, d\mu_f(t) = 2\pi r^n \int_0^{2\pi} e^{-int} d\mu_f t).$$

Hence

$$u_n = 2\int_0^{2\pi} e^{-int} \, d\mu_f(t) \quad \text{for } n = 1, 2, 3, \ldots \tag{2.2}$$

From lemma 1, we have

$$\sum_{n=1}^{\infty} \frac{n^3 |a_n|^2 r^{2n-2}}{(1-r^2)} = \frac{\sec\alpha}{2\pi} \int_0^{2\pi} \frac{\operatorname{Re}\{F(re^{i\theta})\} \, |f'(re^{i\theta})|^2 \, d\theta}{(1-r^2)}$$

$$= \frac{\sec\alpha}{2\pi} \int_0^{2\pi}\int_0^{2\pi} \left| \sum_{n=1}^{\infty} e^{it} S_n(t) z^{n-1} \right|^2 d\theta \, d\mu_f(t)$$

$$= \sum_{n=1}^{\infty} \sec\alpha \left\{ \int_0^{2\pi} |S_n(t)|^2 \, d\mu_f(t) \right\} r^{2n-2}$$

Hence on comparing the coefficients of r^{2n-2} from both sides we have

$$\int_0^{2\pi} |S_n(t)|^2 d\mu_f(t) = \cos\alpha \sum_{m=1}^{n} m^3 |a_m|^2$$

This completes the proof of lemma 2.

Lemma 3. *We have*

(2.3)
$$1 + 2^3\cos^2\alpha + \sum_{i=3}^{k} \frac{4i\cos^2\alpha}{(i-1)^2} \prod_{j=2}^{i-1} \left\{1 + j\left(\frac{2\cos\alpha}{j-1}\right)^2\right\} = \prod_{m=2}^{k}\left[1 + m\left(\frac{2\cos\alpha}{m-1}\right)^2\right] \textit{for } k = 4, 5, 6, \dots .$$

Proof: First of all we will verify the lemma for $k=3$. Thus for $k=3$ we have

$$1 + 2^3\cos^2\alpha + \sum_{i=3}^{3} \frac{4i\cos^2\alpha}{(i-1)^2} \prod_{j=2}^{i-1} \left\{1 + j\left(\frac{2\cos^2\alpha}{j-1}\right)^2\right\}$$
$$= 1 + 2^3\cos^2\alpha + \frac{4\cdot 3\cos^2\alpha}{2^2}\left\{1 + 2\cdot\left(\frac{2\cos\alpha}{2-1}\right)^2\right\}$$
$$= \prod_{j=2}^{3}\left[1 + j\left(\frac{2\cos\alpha}{j-1}\right)^2\right].$$

This verifies the lemma for $k=3$. Now assume that the lemma is true for $k = 4, 5, \dots, n$. Then

$$1 + 2^3\cos^2\alpha + \sum_{i=3}^{k+1} \frac{4i\cos^2\alpha}{(i-1)^2} \prod_{j=2}^{i-1} \left\{1 + j\left(\frac{2\cos\alpha}{j-1}\right)^2\right\}$$
$$= \prod_{j=2}^{k}\left\{1 + j\left(\frac{2\cos\alpha}{j-1}\right)^2\right\} + \frac{4(k+1)\cos^2\alpha}{k^2}\prod_{j=2}^{k}\left\{1 + j\left(\frac{2\cos\alpha}{j-1}\right)^2\right\}$$
$$= \prod_{j=2}^{k}\left\{1 + j\left(\frac{2\cos\alpha}{j-1}\right)^2\right\}\left\{1 + (k+1)\left(\frac{2\cos\alpha}{k}\right)^2\right\}$$
$$= \prod_{j=2}^{k+1}\left\{1 + j\left(\frac{2\cos\alpha}{j-1}\right)^2\right\}.$$

Thus by induction the proof of the lemma is complete.
Next we state a lemma due to Hayman ([3], p. 40) and derive some of its consequences which we shall need.

Lemma 4. *Suppose that*

(2.4)
$$\psi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n = u + iv$$

is regular in the disc $|z|<1$ *and* $u>0$ *there. Then the limit*

$$(2.5)\qquad A(\theta)=\lim_{r\to 1}\left(\frac{1-r}{1+r}\right)\psi(re^{i\theta})$$

exists. The set of distinct values $\theta=\theta_\nu$ *in* $0\leqslant\theta<2\pi$ *for which* $\alpha_\nu=A(\theta_\nu)\neq 0$ *is countable and* $\alpha_\nu>0$, $\Sigma\,\alpha_\nu\leqslant 1$.

Further, we have

$$(2.6)\qquad \lim_{r\to 1}(1-r)\sum_{n=1}^{\infty}|b_n|^2 r^{2n}=2\,\Sigma\,\alpha_\nu^2.$$

In fact, the above lemma implies that

$$(2.7)\qquad \lim_{r\to 1}(1-r)\sum_{n=1}^{\infty}|b_n|^2 r^{n}=4\,\Sigma\,\alpha_\nu^2.$$

If instead of (2.4) we choose the function $F(z)=e^{i\alpha}+\sum\limits_{n=1}^{\infty}u_n z^n$ which is regular in $|z|<1$ and for which $\operatorname{Re}\{F(z)\}>0$, then

$$A(\theta)=\lim_{r\to 1}\left\{\left(\frac{1-r}{1+r}\right)(F(re^{i\theta})-e^{i\alpha}+1)\right\}=\lim_{r\to 1}\left\{\left(\frac{1-r}{1+r}\right)F(re^{i\theta})\right\}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}|F(re^{i\theta})|^2\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}|F(re^{i\theta})-e^{i\alpha}+1|^2\,d\theta=1+\sum_{n=1}^{\infty}|u_n|^2 r^{2n}$$

and hence by lemma 4, we have

$$(2.8)\qquad \lim_{r\to 1}(1-r)\sum_{n=1}^{\infty}|u_n|^2 r^{n}=4\,\Sigma\,\alpha_\nu^2$$

where α_ν's have the same meaning as in lemma 4.

3. Theorem 1. *If* $f(z)=\sum\limits_{n=1}^{\infty}a_n z^n$; $a_1=1$ *belong to the class* $C(\alpha)$, *then*

$$|a_2|\leqslant\cos\alpha \text{ and}$$

$$(3.1)\qquad |a_n|\leqslant\frac{2\cos\alpha}{n(n-1)}\prod_{m=2}^{n-1}\left[1+m\left(\frac{2\cos\alpha}{m-1}\right)^2\right]^{\frac{1}{2}}\quad\text{for } n=3,4,5,\ldots.$$

Proof: Define $F(z)$ as in lemma 2 and write the expressions of $f(z)$, $f'(z)$, and $f''(z)$ in terms of their power series in $F(z)$, then on comparing the coefficients of various powers of z, we have

$$n(n+1)\,a_n=e^{-i\alpha}\sum_{m=1}^{n}m\,a_m\,u_{n-m};\ a_1=1.$$

Now by elementary calculations, we have

$$n(n-1)a_n e^{i\alpha} = 2\sum_{m=1}^{n-1} m a_{.n} \int_0^{2\pi} e^{-i(n-m)t} d\mu_f(t)$$

Using lemma 2 and applying Schwarz's inequality, we have

$$(3.2) \quad n^2(n-1)^2 |a_n|^2 \leqslant 4\cos\alpha \int_0^{2\pi} |S_{n-1}(t)|^2 d\mu_f(t) = 4\cos^2\alpha \sum_{m=1}^{n-1} m^3 |a_m|^2$$

for $n = 2, 3, 4, \ldots$.

From (3.2) it is clear that, for $n = 2,3$

$$(3.3) \quad |a_2|^2 \leqslant \frac{4\cos^2\alpha}{4} \cdot 1^3 \cdot |a_1|^2 = \cos^2\alpha$$

and

$$(3.4) \quad |a_3|^2 \leqslant \frac{4\cos^2\alpha}{3^2 \cdot 2^2}\left[1 + 2\left(\frac{2\cos\alpha}{2-1}\right)^2\right].$$

Hence, the theorem is true for $n = 2,3$. Assume the truth of (3.1) for $n = k$ and establish the truth for $k+1$. Thus from (3.2) and the hypothesis, we have

$$(k+1)^2 k^2 |a_{k+1}|^2 \leqslant 4\cos^2\alpha\,[1^3 \cdot |a_1|^2 + 2^3 |a_2|^2 + \cdots + k^3 |a_k|^2]$$

$$\leqslant 4\cos^2\alpha\left[1 + 2^3\cos^2\alpha + \frac{3\cdot 4\cdot\cos^2\alpha}{2^2}\left(1 + 2\left(\frac{2\cos\alpha}{2-1}\right)^2\right) + \cdots + \frac{4k\cos^2\alpha}{(k-1)^2}\prod_{j=2}^{k-1}\left[1 + j\left(\frac{2\cos\alpha}{j-1}\right)^2\right]\right].$$

Now by using lemma 3 we obtain

$$k^2(k+1)^2 |a_k|^2 \leqslant 4\cos^2\alpha \prod_{j=2}^{k}\left[1 + j\left(\frac{2\cos\alpha}{j-1}\right)^2\right]$$

This establishes the theorem by an appeal to the induction hypothesis, as the result is true actually for $n = 3$.

Sharpness of the theorem follows from the function

$$f(z) = e^{2i\alpha}\left[\frac{1}{(1-z)e^{-2i\alpha}} - 1\right].$$

Corollary: *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C(\alpha)$ *then*

$$(3.5) \quad |a_m| \leqslant \cos\alpha\left[\cos^2\alpha + \frac{4\sin^2\alpha}{m^2(m+1)^2}\right]^{\frac{1}{2}} \quad \text{for } m = 3, 4, \ldots .$$

and. in particular

(3.6) $$| a_m | \leqslant \cos \alpha.$$

Proof: That (3.5; holds for $m=3$. Suppose (3.5) holds for $n=4, \ldots, m$, then from (3.2) we have

(3.6) $$\begin{aligned} m^2 (m+1)^2 | a_{m+1} |^2 &\leqslant 4 \cos^2 \alpha [1^3 + 2^3 | a_2 |^2 + 3^3 | a_3 |^2 + \cdots + m^3 | a_m |^2] \\ &\leqslant 4 \cos^2 \alpha [1^3 - \cos^2 \alpha + \cos^2 \alpha + 2^3 \cos^2 \alpha + \cdots + m^3 \cos^2 \alpha] \\ &\leqslant 4 \cos^2 \alpha \left[\left\{ \frac{m(m+1)}{2} \right\}^2 \cos^2 \alpha + \sin^2 \alpha \right] \\ &= m^2 (m+1)^2 \cos^4 \alpha + \sin^2 2\alpha. \end{aligned}$$

Hence

$$| a_{m+1} |^2 \leqslant \cos^2 \alpha \left[\cos^2 \alpha + \frac{4 \sin^2 \alpha}{m^2 (m+1)^2} \right].$$

This establishes (3.5). Further, since $\cos^2 \alpha + \dfrac{4 \sin^2 \alpha}{m^2 (m+1)^2} \leqslant 1$ for $m = 2, 3, \ldots$. (3.5) implies $| a_m | \leqslant \cos \alpha$. This completes the proof of Corollary 1.

4. *A Growth Theorem.*

Theorem . *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \varepsilon C(\alpha)$ *then*

$$\int_0^{2\pi} \log \left| \frac{f'(re^{i\theta})}{r} \right| d\mu_f(\theta) = 2 \cos \alpha \, \Sigma \, \alpha_\nu^2$$

as $r \to 1$, where α_ν's are defined as in lemma 5.

Proof. Using (2.2), we obtain for every r, $0<r<1$

$$\begin{aligned} 2 \int_0^{2\pi} F(re^{i\theta}) \, d\mu_f(\theta) &= 2 \int_0^{2\pi} \frac{u_0}{2} \, d\mu_f(\theta) + 2 \sum_{n=1}^{\infty} u_n \int_0^{2\pi} r^n e^{in\theta} \, d\mu_f(\theta) \\ &= 2 e^{i\alpha} \int_0^{2\pi} d\mu_f(\theta) + \sum_{n=1}^{\infty} | u_n |^2 r^n \\ &= 2 e^{i\alpha} \cos \alpha + \sum_{n=1}^{\infty} r^n | u_n |^2. \end{aligned}$$

Equating real and imaginary parts, we obtain

(4.1) $$2 \int_0^{2\pi} \operatorname{Re} \{ F(re^{i\theta}) - \cos \alpha \} \, d\mu_f(\theta) = \sum_{n=1}^{\infty} | u_n |^2 r^n,$$

and

(4.2) $$2 \int_0^{2\pi} \operatorname{Im} \{ F(re^{i\theta}) - \sin \alpha \} \, d\mu_f(\theta) = 0$$

Also, if $G(z)=1+\frac{zf''(z)}{f'(z)}$ then

$$\operatorname{Re}\{F(re^{i\theta})\}=\cos\alpha\operatorname{Re}\{G(re^{i\theta})\}-\sin\alpha\operatorname{Im}\{G(re^{i\theta})\}$$

$$\operatorname{Im}\{F(re^{i\theta})\}=\cos\alpha\operatorname{Im}\{G(re^{i\theta})\}+\sin\alpha\operatorname{Re}\{G(re^{i\theta})\}.$$

The above simultaneous equations yield,

(4.3) $\operatorname{Re}\{G(re^{i\theta})\}=1+\cos\alpha[\operatorname{Re}\{F(re^{i\theta})\}-\cos\alpha]+\sin\alpha[\operatorname{Im}\{F(re^{i\theta})\}-\sin\alpha]$

Equations (4.1), (4.2) and (4.3) together yield

$$2\int_0^{2\pi}\operatorname{Re}\{G(re^{i\theta})\}\,d\mu_f(\theta)=2\int_0^{2\pi}d\mu_f(\theta)+\cos\alpha\sum_{n=1}^{\infty}|u_n|^2r^n.$$

From this equation, we have

$$2\int_0^{2\pi}d\mu_f(\theta)+2\operatorname{Re}\int_0^{2\pi}\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\,d\mu_f(\theta)=2\int_0^{2\pi}d\mu_f(\theta)+\cos\alpha\sum_{n=1}^{\infty}|u_n|^2r^n,$$

$$2\,Re\int_0^{2\pi}\log f'(re^{i\theta})\,d\mu_f(\theta)=\cos\alpha\sum_{n=1}^{\infty}\frac{|u_n|^{2r^n}}{n}$$

$$2\int_0^{2\pi}\log|f'(re^{i\theta})|\,d\mu_f(\theta)=\cos\alpha\sum_{n=1}^{\infty}\frac{|u_n|^2}{n}r^n.$$

This completes the proof of theorem.

In the end authors esxress their sincere thanks to Professor R. S. L. Srivastava for his interest and encouragement in the work.

REFERENCES

[1]. S. K. Bajpai and R. S. L. Srivastava: *On univalent α-spiral functions* (communicated).

[2]. W. K. Hayman: *Multivalent Functions*, Cambridge Univ. Press, London (1958).

[3]. W. K. Hayman: *On functions with positive real part*, J. London Math. Soc. **36** (1961), 36—48.

[4]. R. J. Libera: *Univalent α-spiral functions*, Canadian J. Math. **19**, (1969), 449—456.

[5] M. S. Robertson: *Univalent functions f(z) for which zf′ (z) is spirallike;* Michigan Math. J. **16**, (1969), 97 — 101.

[6] L. Špacek: *Constribution à la théorie des fonctions univalentes* (in Czech). Časop Pest. Mat. — Fys. **62** (1933), 12 — 19.

[7] E. C. Titchmarsh: *The theory of functions.* Second Edition, Oxford Univ. Press (1958).

[8] J. Zamorski: *About the extremal spiral schicht functions*, Ann. Polon. Math **9** (1962), 265 — 273.

Department of Mathematics
Indian Institute of Technology
Kanpur — 16 (U. P.)
INDIA