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TOPOLOGICAL STRUCTURES ON CLASSES II

Milan Đurić

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1. Introduction

This paper is a continuation of the paper [1]. We devote it to the study of classes of topological structures in $\mathcal{U}$. To study a class of topological structures we have to make it to be capable of studying. It means that we have to provide it with a class of rules. According to [2] the rules are those ones which are admissible for a class, i.e. the rules which preserve some intrinsic properties of objects in it. Admissible rules for a class of topological spaces we shall call continuous rules. These rules preserve convergence of filters of the spaces. We shall define and characterize them in the paper. We shall also define and characterize some another rules, namely the rules that will preserve those properties of topological spaces which continuous rules do not preserve. These properties are closedness and openness of objects of a space, and rules are closed and open rules. Afterwards we shall deal with the problem of induction and coinduction of topologies and with topological spaces having some local properties.

This paper make a logical whole with the paper [1]. All we have said about the results of that paper is also valid here. Further, all notations, terminology and involved abbreviations in the paper are taken over from [1].

2. Admissible rules for a class of topological spaces

Let us consider a class of topological spaces on the level $i+1$ and denote it by $\mathbf{Top}_{i+1}$. As we have already said a rule in $\mathcal{U}$ will be admissible for $\mathbf{Top}_{i+1}$ if it preserve some intrinsic properties of its objects. The basic property which characterizes a topological space is convergence of its filters. We assume that admissible rules for $\mathbf{Top}_{i+1}$ are those ones which preserve this property. We call them continuous rules. Provided with these rules the class $\mathbf{Top}_{i+1}$ will become a fundamental semigroupoid.

Now we shall proceed to define continuous rules. Let $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ and $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$ be two objects of $\mathbf{Top}_{i+1}$ that we shall denote simply by τ_{i+1} and τ'_{i+1}, respectively. A rule $\Phi_{i+1}:\tau_{i+1}\rightarrow\tau'_{i+1}$ will be continuous if convergence of a filter $\mathfrak{f}_{i+1}\in f_{i+1}(p_{i+1})$ to an object s_i of s_{i+1} implies convergence of the filter $\Phi'_{i+1}(\mathfrak{f}_{i+1})\in f'_{i+1}(p'_{i+1})$ to the object $\Phi''_{i+1}(s_i)$ of s'_{i+1}. At this Φ'_{i+1} is a rule of $p_{i+1}(t_{i+1})$ to $p'_{i+1}(t'_{i+1})$ which assigns, to a filter of $f_{i+1}(p_{i+1})$, a filter of $f'_{i+1}(p'_{i+1})$ and Φ''_{i+1} is a rule of s_{i+1} to s'_{i+1}. Before we give a precise definition of a continuous rule we have to specify the rule Φ'_{i+1}.

Certainly, if $\Phi'_{i+1}:p_{i+1}(t_{i+1})\rightarrow p'_{i+1}(t'_{i+1})$ is a funhom and $q_{i+1}(b_{i+1})$ a filter basis on $p_{i+1}(t_{i+1})$, then $\Phi'_{i+1}(q_{i+1}(b_{i+1}))$ is a filter basis on $p'_{i+1}(t'_{i+1})$. Moreover, if $q_{i+1}(b_{i+1})$ is a filter, then $\Phi'_{i+1}(q_{i+1}(b_{i+1}))$ is not in general a filter but only a filter basis. However $\overline{\Phi'_{i+1}(q_{i+1}(b_{i+1}))}$ is a filter on $p'_{i+1}(t'_{i+1})$. If we define a rule $\Phi'^{*}_{i+i}:p_{i+1}(t_{i+1})\rightarrow p'_{i+1}(t'_{i+1})$ in such a way that $\Phi'^{*}_{i+1}(q_{i+1}(b_{i+1}))=\overline{\Phi'_{i+1}(q_{i+1}(b_{i+1}))}$, then we shall have an induced rule by the rule Φ'_{i+1}. This rule assigns, to each filter of $p_{i+1}(t_{i+1})$, a filter of $p'_{i+1}(t'_{i+1})$. We shall call it a filter rule. It is a filter preserving funhom. To simplify notations we shall write it without the sign $*$, i.e. simply as Φ'_{i+1} and call it a funhom for short. However, whenever the funhom Φ'_{i+1} is applied to a filter it will always mean the filter rule. We take this stipulation for any other funhom. Now we can define continuous rules.

Definition 1. Let $(s_{i+1},\ \tau_{i+1},\ p_{i+1}(t_{i+1}))$ and $(s'_{i+1},\ \tau'_{i+1},\ p'_{i+1}(t'_{i+1}))$ or simpy τ_{i+1} and τ'_{i+1} be two topological spaces. By a *continuous rule* of τ_{i+1} to τ'_{i+1} we mean a pair of rules $(\Phi'_{i+1},\ \Phi''_{i+1})$, where Φ'_{i+1} is a covariant funhom of $p_{i+1}(t_{i+1})$ to $p'_{i+1}(t'_{i+1})$ and Φ''_{i+1} a rule of s_{i+1} to s'_{i+1}, such that the following condition holds:

$$\Phi'_{i+1}(\tau_{i+1}(s_i))\vdash'_{i+1}\tau'_{i+1}(\Phi''_{i+1}(s_i))$$

for all $s_i\in s_{i+1}$.

Thus, continuous rules in a class $\mathbf{Top}_{i+1}$ preserve a property of topological spaces called convergence of filters. However, there are properties of topological spaces which these rules do not preserve. These properties are closedness and openness of objects of a topological space. Because of that we shall define the rules which will preserve these properties. We define first the rules which preserve openness of objects. Such rules we call open rules. Their definition is as follows:

Definition 2. Let $(s_{i+1},\ \tau_{i+1},\ p_{i+1}(t_{i+1}))$ and $(s'_{i+1},\ \tau'_{i+1},\ p'_{i+1}(t'_{i+1}))$ or simply τ_{i+1} and τ'_{i+1} be two topological spaces. A rule $O_{i+1}:\tau_{i+1}\rightarrow\tau'_{i+1}$ we shall say to be *open* if the following condition holds:

$$\tau'_{i+1}(O''_{i+1}(s_i))\vdash'_{i+1}O'_{i+1}(\tau_{i+1}(s_i))$$

for all $s_i\in s_{i+1}$. At this O'_{i+1} is a covariant funhom of $p_{i+1}(t_{i+1})$ to $p'_{i+1}(t'_{i+1})$ and O''_{i+1} a rule of s_{i+1} to s'_{i+1}.

In what follows we shall be concerned with characterizations of continuous and open rules. First with continuous rules. However, before doing it we must explain what we shall understand by an adjoint pair of rule pairs. Let Φ_{i+1}: $:\tau_{i+1}\rightarrow\tau'_{i+1}$ and $\tilde{\Phi}_{i+1}:\tau'_{i+1}\rightarrow\tau_{i+1}$ be two rules for relating topological spaces

τ_{i+1} and τ'_{i+1} which consist of pairs of rules, $\Phi_{i+1}=(\Phi'_{i+1}, \Phi''_{i+1})$ and $\tilde{\Phi}_{i+1}= =(\tilde{\Phi}'_{i+1}, \tilde{\Phi}''_{i+1})$. Then by an adjoint pair $\langle\Phi_{i+1}, \tilde{\Phi}_{i+1}\rangle$ we understand two adjoint pairs $\langle\Phi'_{i+1}, \tilde{\Phi}'_{i+1}\rangle$ and $\langle\Phi''_{i+1}, \tilde{\Phi}''_{i+1}\rangle$. The definition of adjointness is usual and can be found for instance in [3]. In an adjoint situation as it is $\langle\Phi_{i+1}, \tilde{\Phi}_{i+1}\rangle$, Φ_{i+1} is left adjoint to $\tilde{\Phi}_{i+1}$ and $\tilde{\Phi}_{i+1}$ is right adjoint to Φ_{i+1}. The adjointness in the case of $\langle\Phi''_{i+1}, \tilde{\Phi}''_{i+1}\rangle$ is reduced to the statement $s_i= \tilde{\Phi}''_{i+1}(s'_i) \Leftrightarrow \Phi''_{i+1}(s_i)=s'_i$, where $s_i\in s_{i+1}$ and $s'_i\in s'_{i+1}$. Hence, Φ''_{i+1} is left inverse to $\tilde{\Phi}''_{i+1}$ and $\tilde{\Phi}''_{i+1}$ is right inverse to Φ''_{i+1}. If $\langle\Phi_{i+1}, \tilde{\Phi}_{i+1}\rangle$ is an adjoint pair of rules between topological spaces $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ and $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$, then we can prove the following

Proposition 1. *The following statements are pairwise equivalent:*

1) Φ_{i+1} *is a continuous rule.*

2) *The images of open (closed) objects under the rule* $\tilde{\Phi}'_{i+1}$ *are open (closed).*

3) *For every object* t'_i *of* $p'_{i+1}(t'_{i+1})$ *there is a unique rule*

$$\tilde{\Phi}'_{i+1}(\mathbf{O}'_{i+1}(t'_i))\to \mathbf{O}_{i+1}(\tilde{\Phi}'_{i+1}(t'_i)).$$

4) *For every object* t_i *of* $p_{i+1}(t_{i+1})$ *there is a unique rule*

$$\Phi'_{i+1}(\mathbf{C}_{i+1}(t_i))\to \mathbf{C}'_{i+1}(\Phi'_{i+1}(t_i)).$$

Proof. 1) $\Leftrightarrow$ 2). Let t'_i be a τ'_{i+1}-open object and let an s_i of s_{i+1} be such that $s_i \leftarrow_{p_i} \tilde{\Phi}'_{i+1}(t_i)$. Then from the adjointness we have the existence of a unique rule $\Phi''_{i+1}(s_i)\to t'_i$ of $p'_{i+1}(t'_{i+1})$. Hence we have $\tau'_{i+1}(\Phi''_{i+1}(s_i)) \vdash'_{i+1} \overline{t}'_i$. Because of continuity of Φ_{i+1}, i.e. from $\Phi'_{i+1}(\tau_{i+1}(s_i) \vdash'_{i+1} \tau_{i+1}(\Phi''_{i+1}(s_i))$ we have further $\Phi'_{i+1}(\tau_{i+1}(s_i)) \vdash'_{i+1} \overline{t}'_i$. Hence $t'_i\in\Phi'_{i+1}(\tau_{i+1}(s_i))$ and there exist an object t_i and a rule p_i of $\tau_{i+1}(s_i)$ such that $\Phi'_{i+1}(p_i)$: $\Phi'_{i+1}(t'_i)\to t'_i$. Because of adjointness we have the existence of a unique rule $p'_i\in\tau_{i+1}(s_i)$ such that $p'_i: t_i\to\tilde{\Phi}'_{i+1}(t'_i)$. From the antiresiduality property of filters we have that $\tilde{\Phi}'_{i+1}(t'_i)$ is a τ_{i+1}-open object. Let us consider now the case of closed objects. Let t'_i be an τ'_{i+1}-open object, then $\mathscr{C}'_{i+1}(t'_i)$ is closed in $p_{i+1}(t_{i+1})$. From adjointness we have $\mathscr{C}_{i+1}(t'_i)\wedge' t'_i=0^{s'}_i \Rightarrow \tilde{\Phi}'_{i+1}(\mathscr{C}_{i+1}(t'_i))\wedge \tilde{\Phi}'_{i+1}(t'_i)=0^s_i$ and also deduce that $\tilde{\Phi}'_{i+1}(\mathscr{C}_{i+1}(t'_i))\vee\tilde{\Phi}'_{i+1}(t'_i)=1_i$. Hence, $\tilde{\Phi}'_{i+1}(\mathscr{C}'_{i+1}(t'_i))$ is a complement of the object $\tilde{\Phi}'_{i+1}(t'_i)$, i.e. $\tilde{\Phi}'_{i+1}(\mathscr{C}_{i+1}(t'_i))= \mathscr{C}_{i+1}(\tilde{\Phi}'_{i+1}(t'_i))$, and since $\tilde{\Phi}'_{i+1}(t'_i)$ is open then $\tilde{\Phi}'_{i+1}(\mathscr{C}_{i+1}(t'_i))$ is closed.

To prove the converse statement we must reword the definition of continuous rules. According to the Proposition 10 from [2] we have the continuity conditions $\Phi'_{i+1}(\tau_{i+1}(s_i)) \vdash'_{i+1} \tau'_{i+1}(\Phi''_{i+1}(s_i))$ is equivalent to: for every object t'_i of $\tau'_{i+1}(\Phi''_{i+1}(s_i))$ there exist an object t_i and a rule p_i of $\tau_{i+1}(s_i)$ such that $\Phi'_{i+1}(p_i):\Phi'_{i+1}(t_i)\to t'_i$. If $t'_i\in\tau'_{i+1}(\Phi''_{i+1}(s_i))$ then $\tilde{\Phi}'_{i+1}(t'_i)\in\tau_{i+1}(s_i)$. Hence, there are an object t_i and a rule p_i of $\tau_{i+1}(s_i)$ such that $p_i: t_i\to\tilde{\Phi}'_{i+1}(t'_i)$. Thus, there exists a (unique) rule $\Phi'_{i+1}(p_i):\Phi'_{i+1}(t_i)\to t'_i$.

2) $\Leftrightarrow$ 3). Let t_i' be an object of $p_{i+1}'(t_{i+1}')$ and $\mathbf{O}_{i+1}'(t_i')$ its interior. Then $\tilde{\Phi}_{i+1}'(\mathbf{O}_{i+1}'(t_i'))$ is open. From the definition of the operator $\mathbf{O}_{i+1}$ there is a unique rule $\tilde{\Phi}_{i+1}'(\mathbf{O}_{i+1}'(t_i)) \to \mathbf{O}_{i+1}(\tilde{\Phi}_{i+1}'(t_i'))$ making commutative the diagram

$$\begin{array}{ccc} & \tilde{\Phi}_{i+1}'(t_i') & \\ \nearrow & & \nwarrow \\ \tilde{\Phi}_{i+1}'(\mathbf{O}_{i+1}'(t_i')) & \longrightarrow & \mathbf{O}_{i+1}(\tilde{\Phi}_{i+1}'(t_i')). \end{array}$$

Conversely, if t_i' is open then $\mathbf{O}_{i+1}'(t_i') = t_i'$. Hence $\tilde{\Phi}_{i+1}'(\mathbf{O}_{i+1}'(t_i')) = \tilde{\Phi}_{i+1}'(t_i')$; then from the diagram follows that $\mathbf{O}_{i+1}(\tilde{\Phi}_{i+1}'(t_i')) \approx \tilde{\Phi}_{i+1}'(\mathbf{O}_{i+1}'(t_i')) = \tilde{\Phi}_{i+1}'(t_i')$. Thus $\tilde{\Phi}_{i+1}'(t_i')$ is open.

2) $\Leftrightarrow$ 4). Let t_i be an object of $p_{i+1}(t_{i+1})$. Then $\mathbf{C}_{i+1}'(\Phi_{i+1}'(t_i))$ is closed in $p_{i+1}'(t_{i+1}')$. According to 2) $\tilde{\Phi}_{i+1}'(\mathbf{C}_{i+1}'(\Phi_{i+1}'(t_i)))$ is closed in $p_{i+1}(t_{i+1})$. From the definition of the operator $\mathbf{C}_{i+1}$ there is a unique rule

$$\mathbf{C}_{i+1}(t_i) \to \tilde{\Phi}_{i+1}'(\mathbf{C}_{i+1}'(\Phi_{i+1}'(t_i)))$$

making commutative the diagram

$$\begin{array}{ccc} & \tilde{\Phi}_{i+1}'(\mathbf{C}_{i+1}'(\Phi_{i+1}'(t_i))) & \\ \nearrow & & \nwarrow \\ t_i & \longrightarrow & \mathbf{C}_{i+1}(t_i). \end{array}$$

Hence because of adjointness there is a unique rule

$$\Phi_{i+1}'(\mathbf{C}_{i+1}(t_i)) \to \mathbf{C}_{i+1}'(\Phi_{i+1}'(t_i)).$$

Conversely, let t_i' be a closed object of $p_{i+1}'(t_{i+1}')$. Then $\mathbf{C}_{i+1}'(t_i') = t_i'$. If we put $\Phi_{i+1}'(t_i')$ instead of t_i in 4) then we have a unique rule $\Phi_{i+1}'(\mathbf{C}_{i+1}(\tilde{\Phi}_{i+1}'(t_i'))) \to \mathbf{C}_{i+1}'(\Phi_{i+1}'(\tilde{\Phi}_{i+1}'(t_i')))$. Hence there are unique rules $\Phi_{i+1}'(\mathbf{C}_{i+1}(\tilde{\Phi}_{i+1}'(t_i'))) \to \to \mathbf{C}_{i+1}(\Phi_{i+1}'(\tilde{\Phi}_{i+1}'(t_i'))) \to \mathbf{C}_{i+1}'(t_i') = t_i'$. From the adjointness there is a unique rule $\mathbf{C}_{i+1}(\tilde{\Phi}_{i+1}'(t_i')) \to \tilde{\Phi}_{i+1}'(t_i')$. Hence $\mathbf{C}_{i+1}(\tilde{\Phi}_{i+1}'(t_i')) \approx \tilde{\Phi}_{i+1}'(t_i')$. Thus $\tilde{\Phi}_{i+1}'(t_i')$ is closed. ∎

A continuous rule obviously preserves compactness properties of spaces.

P r o p o s i t i o n 2. *If a space* $(s_{i+1}', \tau_{i+1}', p_{i+1}'(t_{i+1}'))$ *is a continuous image of a completely compact* (c_γ*-compact*) *space* $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ *then it is also completely compact* (c_γ*-compact*).

Proof. Let $\mathfrak{f}'_{i+1}$ be a filter on $p'_{i+1}(t'_{i+1})$ distinct from $\mathfrak{o}'_{i+1}$ (with $Rk(\mathfrak{f}'_{i+1}) \leqslant c_Y$), then $\tilde{\Phi}'_{i+1}(\mathfrak{f}'_{i+1})$ is a filter on $p_{i+1}(t_{i+1})$ different from $\mathfrak{o}_{i+1}$ (and $Rk(\tilde{\Phi}'_{i+1}(\mathfrak{f}'_{i+1})) \leqslant c_Y$). Hence we have $\mathfrak{o}_{i+1} \neq \tau_{i+1}(s_i) \wedge \tilde{\Phi}'_{i+1}(\mathfrak{f}'_{i+1})$. Furthermore we have

$$\mathfrak{o}_{i+1} \neq \Phi'_{i+1}(\tau_{i+1}(s_i)) \wedge' \tilde{\Phi}'_{i+1}(\mathfrak{f}'_{i+1}) \vdash'_{i+1} \Phi'_{i+1}(\tau_{i+1}(s_i)) \wedge'$$

$$\Phi'_{i+1}\tilde{\Phi}'_{i+1}(\mathfrak{f}'_{i+1}) \vdash'_{i+1} \tau'_{i+1}(\Phi''_{i+i}(s_i)) \wedge' \mathfrak{f}'_{i+1}. \blacksquare$$

The proposition which follows will give a characterization of open rules.

Proposition 3. *Let* $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ *and* $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$ *or simply* τ_{i+1} *and* τ'_{i+1} *be two topological spaces and* $\Psi_{i+1} = (\Psi''_{i+1}, \Psi'''_{i+1})$ *a rule of* τ_{i+1} *into* τ'_{i+1}. *Then the following statements are pairwise equivalent:*

1) Ψ_{i+1} *is an open rule.*

2) *The images of open objects under* Ψ_{i+1} *are open.*

3. *For every object* $t_i \in p_{i+1}(t_{i+1})$ *there exists a unique rule* $\Psi'_{i+1}(\mathbf{O}_{i+1}(t_i)) \to \mathbf{O}'_{i+1}(\Psi''_{i+1}(t_i))$.

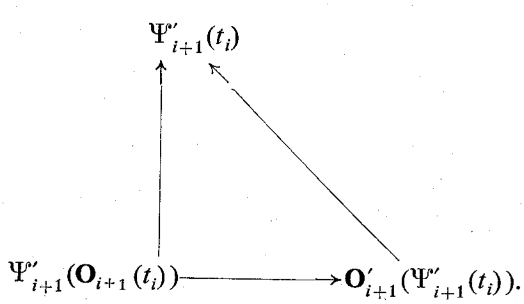
4) *There exists at least one basis of* τ_{i+1} *such that, under the rule* Ψ_{i+1}, *all the members of this basis have open images.*

Proof. 1) $\Rightarrow$ 2). Let t_i be an open object. For any $s'_i \in s'_{i+1}$ such that $s_i \leftarrow \Psi''_{i+1}(t_i)$ there exists an $s_i \in s_{i+1}$ such that $s_i \leftarrow t_i$ and $\Psi'''_{i+1}(s_i) = s'_i$. Certainly we have now $\tau'_{i+1}(s'_i) = \tau'_{i+1}(\Psi'''_{i+1}(s_i)) \vdash'_{i+1} \Psi''_{i+1}(\tau_{i+1}(s_i)) \vdash'_{i+1} \overline{\Psi'_{i+1}(t_i)}$. Thus $\Psi'_{i+1}(t_i)$ is open.

2) $\Leftrightarrow$ 3). Let t_i be an open object of $p_{i+1}(t_{i+1})$, then there exists $lcc\, \mathbf{O}_{i+1}(t_i) \to t_i$. Under Ψ'_{i+1} we have $\Psi'_{i+1}(\mathbf{O}_{i+1}(t_i)) \to \Psi'_{i+1}(t_i)$. On the other side there is $lcc\, \mathbf{O}'_{i+1}(\Psi''_{i+1}(t_i)) \to \Psi''_{i+1}(t_i)$ and hence a unique rule

$$\Psi'_{i+1}(\mathbf{O}_{i+1}(t_i)) \to \mathbf{O}'_{i+1}(\Psi''_{i+1}(t_i))$$

making commutative the following diagram



Conversely, if t_i is open then $\mathbf{O}_{i+1}(t_i) = t_i$ and from the diagram we have $\Psi'_{i+1}(\mathbf{O}_{i+1}(t_i)) = \Psi''_{i+1}(t_i) \approx \mathbf{O}'_{i+1}(\Psi'_{i+1}(t_i))$. Hence $\Psi''_{i+1}(t)$ is open.

2) $\Rightarrow$ 4). Since the objects of every basis are open, then their images are also open.

4) $\Rightarrow$ 1). Let $q_{i+1}(b_{i+1})$ be a basis of τ_{i+1}. Suppose that the images of all objects of $q_{i+1}(b_{i+1})$ are open. If $b_i \in q_{i+1}\, b_{i+1})$ and $s_i \leftarrow b_i$ then we have

$\tau'_{i+1}(\Psi''_{i+1}(s_i)) \vdash'_{i+1} \Psi'_{i+1}(\tau_{i+1}(s_i)) \vdash'_{i+1} \overline{\Psi'_{i+1}(b_i)}$. Since $\mathcal{N}_{i+1}(s_i)$ is a basis of $\tau_{i+1}(s_i)$ then we see that $\tau'_{i+1}(\Psi''_{i+1}(s_i)) \vdash'_{i+1} \Psi'_{i+1}(\tau_{i+1}(s_i))$. ∎

Since under an open rule the image of a closed object need not be closed, then we have to define a rule which will have this property. Such a rule we shall call a closed rule.

Definition 3. By a *closed* rule of a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ into $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$ we mean a rule Θ_{i+1} such that, for every closed object of $p_{i+1}(t_{i+1})$ the image of t_i under Θ_{i+1} is τ'_{i+1}-closed in $p'_{i+1}(t'_{i+1})$.

A characterization of these rules is given by the following

Proposition 4. *A rule* $\Theta_{i+1}:(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) \to (s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$ *is closed iff there exists a unique rule* $\mathbf{C}'_{i+1}(\Theta_{i+1}(t_i) \to \Theta_{i+1}(\mathbf{C}_{i+1}(t_i))$ *for every* $t_i \in p_{i+1}(t_{i+1})$.

Proof. The proof is dual to the proof of 2) $\Leftrightarrow$ 3) in the Proposition 3. ∎

Proposition 5. *Every continuous rule* Φ_{i+1} *of a completely compact space* $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ *into a* T_2*-space* $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$ *is a closed rule.*

Proof. Let t_i be a closed object in $p_{i+1}(t_{i+1})$. Then it is completely compact. According to the Proposition 2, $\Phi'_{i+1}(t_i)$ is also completely compact. Since τ'_{i+1} is a T_2-topology then $\Phi'_{i+1}(t_i)$ is closed. ∎

Now we shall define a special subclass of continuous rules which will allow us to involve an appropriate notion of isomorphism between topological spaces. Such rules in $\mathbf{Top}_{i+1}$ we shall call topological rules or homeomorphisms. Their definition is as follows.

Definition 4. By a *topological rule* or a *homeomorphism* between topological spaces $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ and $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$ we mean a bijective covariant funhom $\Phi_{i+1}=(\Phi'_{i+1}, \Phi''_{i+1})$ such that the following diagram

$$\begin{array}{ccc} p_{i+1}(t_{i+1}) & \xrightarrow{\Phi'_{i+1}} & p'_{i+1}(t'_{i+1}) \\ \uparrow{\scriptstyle \tau_{i+1}} & & \uparrow{\scriptstyle \tau'_{i+1}} \\ s_{i+1} & \xrightarrow[\Phi''_{i+1}]{} & s'_{i+1} \end{array}$$

commutes.

Two topological spaces $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ and $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}, (t'_{i+1}))$ we shall say to be *topologically equivalent* or *homeomorphic*, written symbolically by $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) \cong (s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$, if there exists a topological rule $\Phi_{i+1}:(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) \to (s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$.

A property of a space we shall say to be a *topological invariant* if whenever it is true for one space then it is also true for every space homeomorphic to this one. With this terminology, every topological property of a space is a topological invariant and homeomorphic spaces have the same topological invariants.

Proposition 6. *If* $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ and $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$ *are two topological spaces and* Φ_{i+1} *a rule of* $(s_{i+1}, \tau_{i+1}, p_{i+1}(t'_{i+1}))$ *to* $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$, *then the following statements are pairwise equivalent:*

1) Φ_{i+1} *is a topological rule.*

2) t_i *is* τ_{i+1}*-open* (τ_{i+1}*-closed*) *iff* $\Phi'_{i+1}(t_i)$ *is* τ'_{i+1}*-open* (τ'_{i+1}*-closed*).

3) *We have* $\Phi'_{i+1}(\mathbf{O}_{i+1}(t_i)) \approx \mathbf{O}'_{i+1}(\Phi'_{i+1}(t_i))$ $(\Phi'_{i+1}(\mathbf{C}_{i+1}(t_i)) \approx \mathbf{C}'_{i+1}(\Phi'_{i+1}(t_i)))$.

4) *If* $q_{i+1}(b_{i+1})$ *is a basis of* $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+}))$, *then* $\Phi_{i+1}(q_{i+1}(b_{i+1}))$ *is a basis of* $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$.

Proof. Outline of the proof. From commutativity of the diagram we have that $\tau_{i+1}(s_i) \vdash_{i+1} \overline{t_i} \Rightarrow \tau'_{i+1}(\Phi''_{i+1}(s_i)) \vdash'_{i+1} \overline{\Phi'_{i+1}(t_i)}$. Hence, t_i is open iff $\Phi'_{i+1}(t_i)$ is open and moreover $\Phi'_{i+1}(\mathbf{O}_{i+1}(t_1)) \approx \mathbf{O}'_{i+1}(\Phi'_{i+1}(t_i))$, whence

$$\Phi'_{i+1}(\tau_{i+1}(s_i)) = \tau'_{i+1}(\Phi''_{i+1}(s_i)).$$

The paranthetical contentions follow by taking complements. If t'_i is τ'_{i+1}-open and $\overline{\Phi}'_{i+1}$ the right inverse of Φ'_{i+1}, then $\overline{\Phi}'_{i+1}(t'_i)$ is τ_{i+1}-open. The result is then obvious. The converse is easy to be proved.

The following proposition can serve for establishing when a continuous rule will be a homeomorphism. Since the proof is simple we omit it.

Proposition 7. *A continuous rule* $\Phi_{i+1} : (s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) \to (s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$ *is a homeomorphism if there exists a continuous rule* $\Psi_{i+1} : (s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1})) \to (s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ *such that both* $\Phi_{i+1} \circ \Psi_{i+1} = 1_{\tau'_{i+1}}$ *and* $\Psi_{i+1} \circ \Phi_{i+1} = 1_{\tau_{i+1}}$. |

3. Induction and coinduction of topologies

In this section we shall be concerned with stipulation of topologies by means of rules. Namely, if $\Phi_{i+1} : (s_{i+1}, p_{i+1}(t_{i+1})) \to (s'_{i+1}, p'_{i+1}(t'_{i+1}))$ is a rule between two pairs consisting of a class and a fundamental semigroupoid between which there exists a strict domination and if on one of these pairs is defined a topology then we can transfer this topology to the other pair by means of the rule Φ_{i+1}. If a topology is defined on $(s_{i+1}, p_{i+1}(t_{i+1}))$ its transferring to $(s'_{i+1}, p'_{i+1}(t'_{i+1}))$ by means of Φ_{i+1} we shall call the topology induced by Φ_{i+1}. On the other side, if a topology is defined on $(s'_{i+1}, p'_{i+1}(t'_{i+1}))$ its transferring to $(s_{i+1}, p_{i+1}(t_{i+1}))$ by means of Φ_{i+1} we shall call the topology coinduced by Φ_{i+1}. To define these notions we need some preliminaries.

Certainly, we can involve many different topologies on a pair $(s_{i+1}, p_{i+1}(t_{i+1}))$. Let $\mathbf{T}_{i+1}(s_{i+1}, p_{i+1}(t_{i+1}))$ denote the class of all possible topologies on $(s_{i+1}, p_{i+1}(t_{i+1}))$. We involve rules in it in the following manner. For two objects $\tau_{i+1}, \tau'_{i+1} \in \mathbf{T}_{i+1}(s_{i+1}, p_{i+1}(t_{i+1}))$ we define

$$\tau_{i+1} \vdash_{i+1} \tau'_{i+1} \text{ iff } \tau_{i+1}(s_i) \vdash_{i+1} \tau'_{i+1}(s_i)$$

for all $s_i \in s_{i+1}$. We also define, in a usual manner, an initial and a terminal object in $\mathbf{T}_{i+1}(s_{i+1}, p_{i+1}(t_{i+1}))$. These objects we call an initial and a terminal topology on $(s_{i+1}, p_{i+1}(t_{i+1}))$. By these notions we have

D e f i n i t i o n 5. Let $\Phi_{i+1}:(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) \to (s'_{i+1}, p'_{i+1}(t'_{i+1}))$ be a rule between a topological space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ and a pair $(s'_{i+1}, p'_{i+1}(t'_{i+1}))$. By the topology *induced* on $(s'_{i+1}, p'_{i+1}(t'_{i+1}))$ by the rule Φ_{i+1} we mean an initial topology in $\mathbf{T}_{i+1}(s'_{i+1}, p'_{i+1}(t'_{i+1}))$ for which Φ_{i+1} is continuous.

The space with induced topology has the property of being the vertex of a particular *fc* over the space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$. Nameıy it is the vertex of *fc* with respect to the class of spaces formed on the same underlaying pair $(s'_{i+1}, p'_{i+1}(t'_{i+1}))$.

The definition of coinduced topology is dual to the above one. If $\Psi_{i+1}:(s_{1+1}, p_{i+1}(t_{i+1})) \to (s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$ is a rule then by the topology *coinduced* on $(s_{i+1}, p_{i+1}(t_{i+1}))$ by the rule Ψ_{i+1} we mean a terminal topology in $\mathbf{T}_{i+1}(s_{i+1}, p_{i+1}(t_{i+1}))$ for which Ψ_{i+1} is continuous. The space with this topology has the property of being the covertex of *lcc* over $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$ with respect to the class of spaces on the same pair $(s_{i+1}, p_{i+1}(t_{i+1}))$.

In the same way we define induction and coinduction of topologies from classes of topological spaces. These definitions go by means of classes of rules.

In what follows we shall be concerned with invariantess of properties at the above defined transferrings of topologies. We regard first the case of coinduced topologies.

P r o p o s i t i o n 8. *Every coinduced topology of T_n-topology, $n=1, 2, 3$, by means of an insertion rule is itself a T_n-topology.*

P r o o f. The proof is simple and we illustrate one case only. Let us consider a space $(s'_{i+1}, \tau'_{i+1}, p'_{i+1}(t'_{i+1}))$, a pair $(s_{i+1}, p_{i+1}(t_{i+1}))$ and a rule Φ_{i+1}: $(s_{i+1}, p_{i+1}(t_{i+1})) \to (s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$. If τ'_{i+1} is a T_2-topology then for $s_i, \bar{s}_i \in s_{i+1}$ we have $\Phi''_{i+1}(s_i) \neq \Phi''_{i+1}(\bar{s}_i) \Rightarrow \tau'_{i+1}(\Phi''_{i+1}(s_i) \wedge \tau'_{i+1}(\Phi''_{i+1}(\bar{s}_i)) = \mathfrak{o}_{i+1}$. Let τ_{i+1} be a topology on $(s_{i+1}, p_{i+1}(t_{i+1}))$ for which Φ_{i+1} is continuous. Then we have $\Phi''_{i+1}(s_i) \neq \Phi''_{i+1}(\bar{s}_i) \Rightarrow \Phi'_{i+1}(\tau_{i+1}(s_i)) \wedge \Phi'_{i+1}(\tau_{i+1}(\bar{s}_i)) = \mathfrak{o}_{i+1}$. Since Φ_{i+1} is an insertion rule then we have the result immediately. ∎

P r o p o s i t i o n 9. *Every coinduced topology of a $T_4(T_5)$-topology by means of a closed insertion (insertion) rule is itself a $T_4(T_5)$-topology.* ∎

Much less properties are preserved under induction than coinduction. In a general case we have only.

P r o p o s i t i o n 10. *An induced topology of a completely compact (c_Y--compact) space by a surjective rule is itself completely compact c_Y-compact),*

P r o o f. The proof follows from the Proposition 2. ∎

4. Spaces with local properties

Before we finish this paper we shall briefly regard one more question. It is concerned with properties which possess objects of a space. Namely, certain objects in a space can possess a property and that the space as a whole does not possess it. We shall say then that the space possesses that property only locally. We define such spaces as follows.

Definition 6. For a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ we shall say to possess a property **P** *locally* if for every $s_i \in s_{i+1}$ there exists a neighborhood t_i having this property.

Some other definitions may also be given. However, we employ this one and regard only one property. It is complete compactness. The spaces with this property are locally completely compact spaces. A characterization of these spaces is given by the following

Proposition 11. *A T_2-or a T_3-space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ is locally completely compact iff for every $s_i \in s_{i+1}$ there exists a $t_i \in \tau_{i+1}(s_i)$ such that $\mathbf{C}_{i+1}(t_i)$ is completely compact.*

Proof. If t_i is a neighborhood of s_i then so is $\mathbf{C}_{i+1}(t_i)$. From complete compactness of $\mathbf{C}_{i+1}(t_i)$ we have that the space is locally completely compact. Conversely, let t_i be a completely compact neighborhood belonging to $\tau_{i+1}(s_i)$. If τ_{i+1} is a T_2-topology then by the Proposition 23 of [1] t_i is closed.

Denote $\mathbf{O}_{i+1}(t_i)$ by t_i'. Then $t_i' \in \tau_{i+1}(s_i)$ and there exists a unique rule $\mathbf{C}_{i+1}(t_i') \to t_i$. Hence we have that $\mathbf{C}_{i+1}(t_i)$ is completely compact. Namely, every filter $\mathfrak{f}_{i+1} \neq \mathfrak{o}_{i+1}$ such that $f_{i+1} \vdash_{i+1} \overline{\mathbf{C}_{i+1}(t_i')}$ has an adherent object in t_i which is also in $\mathbf{C}_{i+1}(t_i')$. In the same way we have the case of a T_3-topology. ∎

Local complete compactness together with the axiom (T_2) implies regularity of a space. Therefore it can serve as a link between the axioms (T_2) and (T_3).

Proposition 12. *Every locally completely compact T_2-space is regular.*

Proof. It is easy to show that for every $t_i \in \tau_{i+1}(s_i)$ and a $t_i' \in \tau_{i+1}(s_i)$ there exists a unique $\tau_{i+1}(s_1)$-rule $\mathbf{C}_{i+1}(t_i') \to t_i$ and that by the Proposition 13 of [1] the space is regular. ∎

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