PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 15 (29) 1973, pp. 55—73

TOPOLOGICAL STRUCTURES ON CLASSES I

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(Communicated October 10, 1972)

1. Introduction.

This paper represents a further step in realization of our program initiated by the paper [2]. In that paper we proposed an axiomatized system Σ as a foundation for mathematics. Our program has been then to show that this system provides an adequate framework for (all) mathematics. To show this we adopted two ways. The one is to organize the universe $\mathcal{U}$, being a model for the limit system $\Sigma_{\rightarrow}$ of Σ, in a whole with specified internal relationships among its objects subjected to certain laws, a whole which will allow certain reasonable creations in itself and the other to formalize such an organization in a system falling, in general features, under the scheme of the Σ. The structure of the whole that we intend to form will be, in a certain extent, a reflection of the structure of the real world. We regard that the real world is organized in a perfect manner in that, the attainment of such an organization ought to be our aim. If we attain this aim, then the formal system will display, in symbolic form, a structure of the world. Moreover, it will contain in itself all existing logical systems. Thus, all this considerations may be also regarded as a way towards systematization and unification of (all) mathematics.

With the paper [3] we began to consider the structural organization of $\mathcal{U}$. As we saw there, for a complete organization of $\mathcal{U}$ we have to organize it horizontally and vertically. Since the organization of $\mathcal{U}$ is rather complex then we shall be concerned with it a little more. Certainly, there are various ways to organize $\mathcal{U}$. However, at any organization one thing is always present. It is its fruitfulness. An organization in $\mathcal{U}$ will be fruitful if it is such to form wholes which will be capable of certain creations. Thus we shall differ two types of organizations in $\mathcal{U}$. The one without creative aspects and the other with these aspects. The first type of organizations we shall call simply *plane organizations* and the second *spatial organizations* in $\mathcal{U}$. Hence we shall also have terms — plane and spatial structures in $\mathcal{U}$. It is obvious that plane organizations in $\mathcal{U}$ are horizontal and spatial are both horizontal and vertical. They are always horizontal if the classes are not initial, i.e. if they are not universes.

Types of possible organizations in $\mathcal{U}$, both plane and spatial, are many and varied, but they all go to make a universal mathematical structure, which provided with dynamics will make a universal mathematical organism. If we are able to form such an organism then it will represent, with a certain degree of accuracy, an abstract image of the world, if not biological, then at least physical one. On the way towards this aim we shall deal with some particular types of possible organizations in $\mathcal{U}$.

In this paper we shall deal with a type of spatial organization in $\mathcal{U}$. Namely, we shall organize a class on a level with defined plane organization over a class which is strictly dominated by this one to obtain a spatial structure in $\mathcal{U}$ that we shall call a topological structure in $\mathcal{U}$. For that purpose we shall regard two classes s_{i+1} and t_{i+1} of certain mathematical i-objects such that $s_{i+1} \prec t_{i+1}$, where $\prec$ denotes the strict domination.

Let s_{i+1} be a discrete class and let on t_{i+1} be defined a class of rules* p_{i+1} such that $p_{i+1}(t_{i+1})$ is a fundamental semigroupoid. These two notions s_{i+1} and $p_{i+1}(t_{i+1})$ are starting concepts for our further intentions. From them we shall form a spatial structure in $\mathcal{U}$, called a topological structure, in which relationships between objects of s_{i+1} and subclasses of $p_{i+1}(t_{i+1})$ will be its essence. More precisely, the essence consists in insertion of objects of s_{i+1} into $p_{i+1}(t_{i+1})$ as certain distinguished objects for some subclasses of it. At this we assume that subclasses of $p_{i+1}(t_{i+1})$ are organized to be filters. Then inserted objects of s_{i+1} are d-limit objects for certain filters of $f_{i+1}(p_{i+1})$, where $f_{i+1}(p_{i+1})$ denotes the class of all filters on $p_{i+1}(t_{i+1})$. Thus, we assign to each object $s_i \in s_{i+1}$ a filter of $f_{i+1}(p_{i+1})$ in such a way that the insertions of objects of s_{i+1} in $p_{i+1}(t_{i+1})$ are d-limit objects of the filters assigned to these objects. Certainly, this assignment is not arbitrary but controled in a sense. The conditions which regulate it are: conditions of separation, of cardinality and of compactness. Thus, besides assigning the way in which it is to be done is also important. Hence as a conclusion we have that, the basic features of topological structures are: relationships of objects of s_{i+1} and certain organized subclasses of $p_{i+1}(t_{i+1})$ and the conditions which regulate these relatioships.

In the paper we shall first define topological structures and then deal with various ways of their introducing. Afterwards we shall deal with the conditions which regulate formations of these structures. We shall devote the most part of the paper just to these conditions. Many well-known results from general topology concerning these questions will be generalized by this paper. To recognize this one can compare the famous books on the field by Kelley [4], Kowalsky [5], Dugundji [1], and so on. Some new results will also be given.

Since we have finished the general discussion about the spatial organization of $\mathcal{U}$, particularly about formations of topological structures on a level in $\mathcal{U}$ we shall explain our further plan concerning this question. In several subsequent papers under the title „Topological structures on classes“ and some anothers we shall regard classes of topological structures and admissible rules in them, generalization of the notion of topological structure, then discuss formations of spatial structures when is defined a plane structure on the class s_{i+1} and formations of spatial structures on already formed spatial structure. This spatial structure in $\mathcal{U}$ we shall call a hypertopological structure. We shall see that two consecutive universes of $\mathcal{U}$ can be organized in a spatial whole by a hypertopology. At this the universes $\mathcal{U}_0$ and $\mathcal{U}_1$ are to be organized by a topology. Thus, a spatial organization both horizontal and vertical can be realized by a hypertopology. However, we shall see later that all these organizations are only special cases of a general spatial organization in $\mathcal{U}$.

At the end of this section we mention that terms and notations in this paper are taken over from the author's two papers [2] and [3]. However, there are some exceptions. So, a homomorphism between two fundamental semigroupoids we shall call here a *fundamental homomorphism* or a *funhom* for short.

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*) We also use here the term a rule for a connective between two objects which satisfies the conditions emphasized in [3].

Furthermore, a sequent and a presequent of two objects in a fundamental semigroupoid we shall often denote by means of the signs $\vee$ and $\wedge$, respectively. Thus, if $t_i, t'_i \in p_{i+1}(t_{i+1})$ then $t_i \wedge t'_i$ will denote their presequent. We regard further that all *fc* and *lcc* in this papar are unique. New terms and notations will be emphasized throughout the paper. The logical symbols occuring in the paper have the usual meanings.

2. Definition of basic concepts.

In this section we shall define the concepts which are basic in the paper. They are a topology and a topological space in $\mathcal{U}$. The starting elements for the purpose are a discrete class s_{i+1} and a fundamental semigroupoid $p_{i+1}(t_{i+1})$ such that $s_{i+1} \prec p_{i+1}(t_{i+1})$. We keep them fixed throughout the paper. Our purpose here is to form a spatial structure from these two concepts which will distinguish itself, in a constructive sense, by its beginning and its end and which will have this property at each step of the construction. If we denote the beginning of the construction by o and the end by 1, then all we have just said practically means that, the structure that we mean to build in $\mathcal{U}$ has to contain the objects o and 1 and also the beginning and the end of any its subclass. Certainly, an *l*-semigroupoid formed on $p_{i+1}(t_{i+1})$ will fulfill all these conditions. However, since we want to include the class s_{i+1} in such a construction, then we shall require that $p_{i+1}(t_{i+1})$ be such to allow its inclusion in the construction. The question arises, in which way. The way which we shall follow here is to utilize objects of s_{i+1} in formation of *d*-limits over certain filters of $f_{i+1}(p_{i+1})$. Thus, if I_{1+1} is an injective single-valued rule of s_{i+1} into $p_{i+1}(t_{i+1})$ and τ_{i+1} a many-valued rule of s_{i+1} to $p_{i+1}(t_{1+1})$ which assigns, to each object s_i of s_{i+1}, a filter $\tau_{i+1}(s_i)$ on $p_{i+1}(t_{i+1})$ then the above story means that, there have to exist a rule $\eta_{i+1}: I_{i+1} \to \tau_{i+1}$ such that the images of objects of s_{i+1} under $\mathcal{G}_{i+1} = (I_{i+1}, \eta_{i+1}, \tau_{i+1})$ are cocones in $p_{i+1}(t_{i+1})$. The covertex of such a cocone $\mathcal{G}_{i+1}(s_i)$ for an $s_i \in s_{i+1}$ is a *d*-limit object of the filter $\tau_{i+1}(s_{i+1})$. Certainly, it is not unique. By an $(i+1)$-topology in $\mathcal{U}$ we shall mean a many-valued rule τ_{i+1} for which there exists such a rule η_{i+1}, and by an $(i+1)$-topological space, a triple $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$. To make τ_{i+1} to be reasonable and fruitful in the organizational sense we must impose certain conditions upon it. First, if τ_{i+1} was such that $I_{i+1}(s_i) \in \tau_{i+1}(s_i)$, $s_i \in s_{i+1}$, then for the principial filter $\overline{I_{i+i}(s_i)}$ generated by $I_{i+1}(s_1)$ we would have $\overline{I_{1+1}(s_i)} \vdash$ $\vdash_{i+1} \tau_{i+1}(s_i)$. However, we adopt this as a general requirement upon τ_{i+1}. Namely, if $\neg_{i+1}$ is a many-valued rule of s_{i+1} to $p_{i+1}(t_{i+1})$ which assigns, to each object s_i of s_{i+1}, a principal filter $\neg_{i+1}(s_i)$ generated by $I_{i+1}(s_i)$, then we require that $\neg_{i+1}(s_i) \vdash_{i+1} \tau_{i+1}(s_i)$. To impose the next condition we need some preliminaries. Let us consider an inverse rule of the rule τ_{i+1} defined as follows

$$\tau_{i+1}^{-1}(t_i) = \{s_i \mid s_i \in s_{i+1} \,\&\, t_i \in \tau_{i+1}(s_i)\}.$$

Hence we have that $(\forall s_i \in \tau_{i+1}^{-1}(t_i))(\exists p_i \in p_{i+1})(p_i : I_{i+1}(s_i) \to t_i)$. Thus, the objects $I_{i+1}(s)$ precede the object t_i with respect to certain rules of p_{i+1}. If we denote by $I_{i+1}(s_i) \prec_{p_i} t_i$ the fact that $I_{i+1}(s_i)$ precedes t_i with respect to a rule $p_i \in p_{i+1}$, then the above expression we can write as $(\forall s_i \in \tau_{i+1}^{-1}(t_i))(I_{i+1}(s_i) \prec_{p_i} t_i)$. In that manner we have that objects of a filter may be preceded by many objects of $I_{i+1}(s_{i+1})$. We ensure this by the following requirement upon τ_{i+1}:

$$(\forall s_i \in s_{i+1})(\forall t_i \in \tau_{i+1}(s_i))(\exists t'_i \in \tau_{i+1}(s_i))\,[[(\forall s'_i \in s_{i+1})(I_{i+1}(s'_i) \prec_{p_i} t'_i)] \Rightarrow t_i \in \tau_{i+1}(s'_i)].$$

As we have seen in the above discussion, if τ_{i+1} has the inverse rule τ_{i+1}^{-1} then an object $t_i \in \tau_{i+1}(s_i)$ is preceded by all objects of $I_{i+1}(\tau_{i+1}^{-1}(t_i))$.

The above condition on τ_{i+1} can be expressed purely in terms of filters. Clearly, $t_i \in \tau_{i+1}(s_i') \Rightarrow \tau_{i+1}(s_i') \vdash_{i+1} \overline{t_i}$, where $\overline{t_i}$ denotes a principal filter generated by t_i, and since this holds for all s_i' such that $I_{i+1}(s_i') \leftarrow_{p_i} t_i'$, then the sequent of $\{\tau_{i+1}(s_i') \mid I_{1+1}(s_i') \leftarrow_{p_i} t_i'\}$ denoted by $\vee\{\tau_{i+1}(s_i') \mid I_{i+1}(s_i') \leftarrow_{p_i} t_i'\}$ satisfies the condition

$$\vee\{\tau_{i+1}(s_i') \mid I_{i+1}(s_i') \leftarrow_{p_i} t_i'\} \vdash_{i+1} \overline{t_i},$$

and for the presequent of

$$\{\vee\{\tau_{i+1}(s_i') \mid I_{i+1}(s_i') \leftarrow_{p_i} t_i'\} \mid t_i' \in \tau_{i+1}(s_i)\},$$

denoted by means of the sign $\wedge$ in front of brackets, we obtain

$$\wedge\{\vee\{\tau_{i+1}(s_i') \mid I_{i+1}(s_i') \leftarrow_{p_i} t_i'\} \mid t_i' \in \tau_{i+1}(s_i)\} \vdash_{i+1} \overline{t_i}.$$

Moreover, since this relation holds for all $t_i \in \tau_{i+1}(s_i)$, then we have

$$\wedge\{\vee\{\tau_{i+1}(s_i') \mid I_{i+1}(s_i') \leftarrow_{p_i} t_i'\} \mid t_i' \in \tau_{i+1}(s_i)\} \vdash_{i+1} \tau_{i+1}(s_i).$$

Now we can summarize all the above discussion in the following.

Definition 1. By an $(i+1)$-*topology* or simply a *topology* on the pair $(s_{i+1}, p_{i+1}(t_{i+1}))$ in $\mathcal{U}$ we mean a many-valued rule $\tau_{i+1}: s_{i+1} \to p_{i+1}(t_{i+1})$ which assigns, to each object s_i of s_{i+1}, a filter $\tau_{i+1}(s_i)$ of $p_{i+1}(t_{i+1})$ in such a way that, there exists a rule $\eta_{i+1}: I_{i+1} \to \tau_{i+1}$, where I_{i+1} is an injective single-valued rule of s_{i+1} into $p_{i+1}(t_{i+1})$, such that for every $s_i \in s_{i+1}$, $\mathcal{G}_{i+1}(s_i) = (I_{i+1}, \eta_{i+1}, \tau_{i+1})(s_i)$ is a cocone in $p_{i+1}(t_{i+1})$ and which for every $s_i \in s_{i+1}$ satisfies the following conditions:

T 1: $\neg_{i+1}(s_i) \vdash_{i+1} \tau_{i+1}(s_i)$.

T 2: $\wedge\{\vee\{\tau_{i+1}(s_i') \mid I_{i+1}(s_i') \leftarrow_{p_i} t_i'\} \mid t_i' \in \tau_{i+1}(s_i)\} \vdash_{i+1} \tau_{i+1}(s_i)$.

The filter $\tau_{i+1}(s_i)$ we shall call the η_{i+1}-*neighborhoods filter* of s_i, and its elements, η_{i+1}-neighborhoods of the object s_i. For a filter $\mathfrak{f}_{i+1}$ of $p_{i+1}(t_{i+1})$ we shall say that τ_{i+1}-*converges* to the object s_i if $\mathfrak{f}_{i+1} \vdash_{i+1} \tau_{i+1}(s_i)$.

The triple $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ we shall call an $(i+1)$- or simply a *topological space* in $\mathcal{U}$.

The convergence in the above definition means the convergence to objects of the class s_{i+1}. If a filter $\mathfrak{f}_{i+1}$ satisfies the condition $\mathfrak{f}_{i+1} \vdash_{i+1} \tau_{i+1}(s_i)$ and if $I_{i+1}(s_i')$ is its d-limit object in $I_{i+1}(s_{i+1})$, then we have the existence of a p_{i+1}-rule $p_i: I_{i+1}(s_i') \to I_{i+1}(s_i)$, where $I_{i+1}(s_i)$ is the d-limit object $\tau_{i+1}(s_i)$. Since the class $I_{i+1}(s_{i+1})$ is discrete, then the only possibility is that p_i is the identity of $I_{i+1}(s_i)$, i.e. that $I_{i+1}(s_i') = I_{i+1}(s_i)$. Hence, $\mathfrak{f}_{i+1}$ has the same limit as the filter $\tau_{i+1}(s_i)$, i.e. it converges to the object s_i. For this convergence we say that it is along the filter $\tau_{i+1}(s_i)$ and call it a τ_{i+1}-convergence.

In one of the following sections we shall specify particular types of topologies on $(s_{i+1}, p_{i+1}(t_{i+1}))$ by separation axioms. Here not entering into these axioms we shall mention some types of possible topologies on $(s_{i+1}, p_{i+1}(t_{i+1}))$, those which will be extreme in the class of all topologies on $(s_{i+1}, p_{i+1}(t_{i+1}))$. If the rule τ_{i+1} is such that $\tau_{i+1}(s_i) = \neg_{i+1}(s_i)$, then such a topology we shall call the *discrete topology*. Under this topology the only filters which converge fo s_i are the first filter $\mathfrak{o}_{i+1}$ of $f_{i+1}(p_{i+1})$ and the principal filter $\rightarrow_{i+1}(s_i)$. The first filter because it converges to every object of s_{i+1}, namely for every $s_i \in s_{i+1}$ we have $\mathfrak{o}_{i+1} \vdash_{i+1} \tau_{i+1}(s_i)$. If the rule τ_{i+1} is such that for all $s_i \in s_{i+1}$, $\tau_{i+1}(s) = \mathfrak{l}_{+1}$, where $\mathfrak{l}_{i+1}$ is the last filter in $f_{i+1}(p_{i+1})$, then we have the *trivial topology*. Under this topology all filters converge to every object of the class s_{i+1}.

In the remainder of this section we mention that, in future, always when there will be no possibilty of confusion we shall write s_i instead of $I_{i+1}(s_i)$ or simply consider that s_{i+1} is strictly contained in $p_{i+1}(t_{i+1})$. Moreover, to simplify terminology, instead of an η_{i+1}-neighborhoods filter of an object $s_i \in s_{i+1}$ we shall simply say a neighborhoods filter of s_i.

3. Reformulation of basic concepts.

In the previous section we have defined the concept of a topological structure on a level in $\mathcal{U}$. In this section we shall study other ways of introducing this structure and also their connections with already given definition. By the way we shall introduce some new concepts. They are openness and closedness of objects in a topological space and also the concept of a basis for topology.

First we shall reformulate the Definition 1 by specifying the semigroupoid $p_{i+1}(t_{i+1})$. We have already said that $p_{i+1}(t_{i+1})$ is to be such to allow inclusions of objects of s_{i+1} in formation of a structural whole in $\mathcal{U}$. The way which we have employed consists in inserting these objects as *d*-limit objects for certain filters on $p_{i+1}(t_{i+1})$. Now we shall specify these filters. We assume that they are those filters whose ranks are greater or equal to a fixed cardinal c_β. By the rank of a filter we mean the smallest cardinality among all the cardinalities of the possible bases of it. Let us denote by $p_{i+1}(t_{i+1})^{c_\beta}$ a subclass of $p_{i+1}(t_{i+1})$, not neccessarily full, which consists of all objects and rules of neighborhoods filters $\tau_{i+1}(s_i), s_i \in s_{i+1}$. Then, for every object $t_i \in p_{i+1}(t_{i+1})^{c_\beta}$ there exist $s_i \in s_{i+1}$ and a rule $p_i \in p_{i+1}$ such that $I_{i+1}(s_i) \leftarrow_{p_i} t_i$. The objects of the class $p_{i+1}(t_{i+1})^{c_\beta}$ we shall call open objects. These objects have the following property: if $t_i \in p_{i+1}(t_{i+1})^{c_\beta}$ and $s_i \in s_{i+1}$, then the existence of a rule $p_i \in p_{i+1}$ such that $I_{i+1}(s_i) \leftarrow_{p_i} t_i$ implies $t_i \in \tau_{i+1}(s_i)$. We can ultilize this property to define the open objects.

Definition 2. For an object $t_i \in p_{i+1}(t_{i+1})$ we shall say to be *open* if the existence of a $p_i \in p_{i+1}$ such that $I_{i+1}(s_i) \leftarrow_{p_i} t_i$ for an $s_i \in s_{i+1}$ implies that $t_i \in \tau_{i+1}(s_i)$. For a filter we shall say to be open if it has a basis consiting of open objects.

Since we want to give a new definition of topology by means of the class $p_{i+1}(t_{i+1})^{c_\beta}$, then we have to investigate its properties. They are given in the following

Proposition 1. *The class* $p_{i+1}(t_{i+1})^{c_\beta}$ *possesses the following properties*:

i) *It contains the strictly first object* o_i^s *and the last object* 1_i,

ii) *it allows fc formation on each subclass and lcc formation on each* $<c_\beta$-*subclass of it*.

Proof. Let $q_{i+1}(b_{i+1})$ be *a* $<c_\beta$-subclass of $p_{i+1}(t_{i+1})^{c_\beta}$. This class is certainly contained in a neighborhoods filter $\tau_{i+1}(s_i)$, for an $s_i \in s_{i+1}$. Then from the definition of filters [3], in which are now specified cones and cocones to be *fc* and *lcc* we have that, together with $q_{i+1}(b_{i+1})$, $\tau_{i+1}(s_i)$ also contains

its *lcc*. Let us consider now *fc* φ_{i+1} over an arbitrary subclass $r_{i+1}(a_{i+1})$ of $p_{i+1}(t_{i+1})^{c_\beta}$ with the vertex b_i. Since all rules of $p_{i+1}(t_{i+1})^{c_\beta}$ belong to neighborhoods filters, then so do the rules of φ_{i+1}. Thus, for an $a_i \in r_{i+1}(a_i)$ there exists a $\varphi_i \in \varphi_{i+1}$ such that $\varphi_i : a_i \to b_i$ and φ_i belongs to a neighborhoods filter. On the other side we have the existence of an $s_i \in s_{i+1}$ and a rule $p_i \in p_{i+1}$ such that $s_i \leftarrow_{p_i} a_i$. Hence, $a_i \in \tau_{i+1}(s_i)$ and because of the antiresiduality property of filters also that $b_i \in \tau_{i+1}(s_i)$. The property i) follows from ii).

The property i) in the above proposition is not independent from ii). However, we write it separately to emphasize that the strictly first object o_i^s and the last object 1_i exist in $p_{i+1}(t_{i+1})^{c_\beta}$. To mention here that filter generated by the object o_i^s we shall call the null filter. It is obviously the first object $\mathfrak{o}_{i+1}$ in the *l*-semigroupoid $f_{i+1}(p_{i+1})$.

Besides the properties given in the above proposition the class $p_{i+1}(t_{i+1})^{c_\beta}$ also possesses the following property:

iii) for every $s_i \in s_{i+1}$ there exists an object $t_i \in p_{i+1}(t_{i+1})^{c_\beta}$ and a rule $p_i \in p_{i+1}$ such that $I_{i+1}(s_i) \leftarrow_{p_i} t_i$ and for each pair $t_i,\ t_i' \in p_{i+1}(t_{i+1})^{c_\beta}$ such that $I_{i+1}(s_i') \leftarrow_{p_i'} t_i$ and $I_{i+1}(s_i) \leftarrow_{p_i''} t_i'$ with respect to the rules $p_i',\ p_i'' \in p_{i+1}$, the existence of a rule $p_i \in p_{i+1}$, $p_i : t_i \to t_i'$ or $p_i : t_i' \to t_i$ implies holdness of the formula $\mathcal{C}_{\text{com}}(p_i', p_i; p_i'')$ respectively $\mathcal{C}_{\text{com}}(p_i'', p_i; p_i')$.

The above property is in fact a connection between objects of s_{i+1} and $p_{i+1}(t_{i+1})^{c_\beta}$.

By utilizing the properties of the class $p_{i+1}(t_{i+1})^{c_\beta}$ we can give a new definition of topology. In what follows we shall give this definition and establish its connection with already given definition.

D e f i n i t o n 3. By a c_β-*topology* on the pair $(s_{i+1}, p_{i+1}(t_{i+1}))$ in $\mathcal{U}$ we mean a subclass $p_{i+1}(t_{i+1})^{c_\beta}$ of $p_{i+1}(t_{i+1})$ which possesses the properties i) and ii) of the Proposition 1 and the connection property iii) given above.

Now we shall show that this definition is logically equivalent to the Definition 1. At this we certainly consider that the ranks of neighborhoods filters are $\geqslant c_\beta$.

P r o p o s i t i o n 2. *Definition* 1$\Leftrightarrow$*Definition* 3.

P r o o f. From the Proposition 1 and the discussion behind it we have the direct implication. Conversely, let $\mathcal{N}_{i+1}(s_i)$ be a subclass of $p_{i+1}(t_{i+1})^{c_\beta}$, the class of objects of which consists of all those objects $t_i \in p_{i+1}(t_{i+1})^{c_\beta}$ such that $s_i \leftarrow_{p_i} t_i$ for an $s_i \in s_{i+1}$ or more accurate $I_{i+1}(s_i) \leftarrow_{p_i} t_i$, where I_{i+1} denotes an injective rule of s_{i+1} to $p_{i+1}(t_{i+1})$. However, according to our stipulation we write s_i instead of $I_{i+1}(s_i)$. From ii) we have that $\mathcal{N}_{i+1}(s_i)$ is a c_β-filter basis. Then, $\tau_{i+1}(s_i) = \overline{\mathcal{N}_{i+1}(s_i)}$ defines a filter for which s_i is *d*-limit object and which is such that $\overline{s_i} \vdash_{i+1} \tau_{i+1}(s_i)$. Hence $\mathbf{T}_1$ holds. Further, if $t_i \in \mathcal{N}_{i+1}(s_i)$ and $s_i' \leftarrow_{p_i} t_i$, then clearly $t_i \in \mathcal{N}_{i+1}(s_i')$ and thus there exists a $t_i' \in \mathcal{N}_{i+1}(s_i')$ and a $q_i : t_i' \to t_i \in \mathcal{N}_{i+1}(s_i)$ such that the formula $\mathcal{C}_{\text{com}}(p_i'', q_i; p_i')$ holds in $p_{i+1}(t_{i+1})^{c_\beta}$; at this $p_i'' : s_i \to t_i'$ If $t_i \in \tau_{i+1}(s_i)$ is fixed and $s_i' \leftarrow_{p_i} t_i$, then we have $\tau_{i+1}(s_i') \vdash$ $\vdash_{i+1} \overline{t_i}$ and moreover $\bigvee\{\tau_{i+1}(s_i') \mid s_i' \leftarrow_{p_i} t_i\} \vdash_{i+1} \overline{t_i}$. Hence $\bigwedge\{\bigvee\{\tau_{i+1}(s_i') \mid s_i' \leftarrow_{p_i} t_i\}$ $\mid t_i \in \tau_{i+1}(s_i)\} \vdash_{i+1} \bigwedge\{\overline{t_i} \mid t_i \in \tau_{i+1}(s_i)\}$. Denote $\bigwedge\{\overline{t_i} \mid t_i \in \tau_{i+1}(s_i)\}$ by δ_{i+1}. Then obviously $\delta_{i+1} \vdash_{i+1} \tau_{i+1}(s_i)$. On the other side δ_{i+1} has a basis, the objects of which are covertices of *lcc*'s over subclasses of $\mathcal{N}_{i+1}(s_i)$. From the definition of filters we have that these objects belong to $\tau_{i+1}(s_i)$, whence $\tau_{i+1}(s_i) \vdash_{i+1} \delta_{i+1}$. Then, $\tau_{i+1}(s_i) = \delta_{i+1}$ and T 2 also holds. Thus, the rule $\tau_{i+1} : s_{i+1} \to p_{i+1}(t_{i+1})$ such that $\tau_{i+1}(s_i) = \overline{\mathcal{N}_{i+1}(s_i)}$ is a topology on $(s_{i+1}, p_{i+1}(t_{i+1}))$.∎

Now we shall define certain operators on a fundamental semigroupoid, respectively an *l*-semigroupoid $p_{i+1}(t_{i+1})$ and show that a topology can also be involved by means of them. We first define an operator of complementation on $p_{i+1}(t_{i+1})$, i.e. an operation by which we shall be able to distinguish objects in it. Let $p_{i+1}(t_{i+1})$ be a fundamental semigroupoid which contains the strictly first object o_i^s and the last object 1_i and let $\mathcal{C}_{i+1}$ be a funhom of $p_{i+1}(t_{i+1})$ to itself which satisfies the following conditions:

a) $\mathcal{C}_{i+1}$ assings, to each object $t_i \in p_{i+1}(t_{i+1})$, an object $\mathcal{C}_{i+1}(t_i)$ in such a way that the presequent $t_i \wedge \mathcal{C}_{i+1}(t_i)$ of t_i and $\mathcal{C}_{i+1}(t_i)$ is equal to the object o_i^s and the sequent $t_i \vee \mathcal{C}_{i+1}(t_i)$ to the object 1_i;

b) $\mathcal{D}_{0,1}(p_i) = t_i^{0,1} \Rightarrow \mathcal{D}_{1,0}(\mathcal{C}_{i+1}(p_i)) = \mathcal{C}_{i+1}(t_i^{0,1})$;

c) $(\forall p_i \in p_{i+1}(t_{i+1}))(\mathcal{C}_{i+1}\mathcal{C}_{i+1}(p_i) = p_i)$.

A funhom of $p_{i+1}(t_{i+1})$ to itself which satisfies the above conditions we shall call an operator of complementation on $p_{i+1}(t_{i+1})$. The condition b) above means contravariantness of $\mathcal{C}_{i+1}$. Thus, an operator of complementation is a contravariant funhom of $p_{i+1}(t_{i+1})$ to itself satisfying the conditions a) and c). The condition c) concerns both rules and objects. If $p_{i+1}(t_{i+1})$ is an *l*-semigroupoid, then we require moreover that $\mathcal{C}_{i+1}$ is a c_α*-funhom* $c_\alpha \leqslant c_{p(t)}$, i.e. a funhom of the *l*-semigroupoid $p_{i+1}(t_{i+1})$ to itself which preserves *fc* and *lcc* over any its c_α-subclass, where $c_\alpha \leqslant c_{p(t)}$. More correctly, it rewrites *fc* into *lcc* and conversely. Thus, the definition of an operator of complementation is as follows.

Definition 4. By an *operator of complementation* on an *l*-semigroupoid $p_{i+1}(t_{i+1})$ we mean a contravariant funhom $\mathcal{C}_{i+1}$ of $p_{i+1}(t_{i+1})$ to itself which satisfies the following conditions:

$\mathcal{C}1$: $\mathcal{C}_{i+1}$ is a c_α-funhom for every $c_\alpha \leqslant c_{p(t)}$.

$\mathcal{C}2$: $(\forall t_i \in t_{i+1})(t_i \wedge \mathcal{C}_{i+1}(t_i) = o_i^s \;\&\; t_i \vee \mathcal{C}_{i+1}(t_i) = 1_i)$.

$\mathcal{C}3$: $(\forall p_i \in p_{i+1}(t_{i+1}))(\mathcal{C}_{i+1}\mathcal{C}_{i+1}(p_i) = p_i)$.

The pair $\langle p_{i+1}(t_{i+1}); \mathcal{C}_{i+1}\rangle$ we shall call a *complemented l-semigropupoid.*

A unique contravariant funhom $\mathcal{C}_{i+1}$ satisfying the condition $\mathcal{C}2$ is obviously an operator of complementation. In one of subsequent papers we shall be concerned in more details with these and some other funhoms being near to these ones.

Let us define now certain new operators. They are a closure and an interior operator. We first define a closure operator. Let $p_{i+1}(t_{i+1})$ be an *l*-semigroupoid and $\mathbf{C}_{i+1}$ a covariant funhom of $p_{i+1}(t_{i+1})$ to itself. For this homomorphism we shall say to be *idempotent* provided $\mathbf{C}_{i+1}\mathbf{C}_{i+1}(p_i) = \mathbf{C}_{i+1}(p_i)$ for every element p_i of $p_{i+1}(t_{i+1})$. We shall say that it is a *suc-funhom* if there exists a natural rule $\eta_{i+1}: I_{i+1} \to \mathbf{C}_{i+1}$ such that $\eta_{i+1}(t_i): I_{i+1}(t_i) \to \mathbf{C}_{i+1}(t_i)$ is *fc* in $p_{i+1}(t_{i+1})$ with the vertex $\mathbf{C}_{i+1}(t_i)$. Here I_{i+1} denotes the identity funhom of $p_{i+1}(t_{i+1})$. According to the stipulation made in the introduction the mentioned *fc* is unique. We have called $\mathbf{C}_{i+1}$ a suc-funhom because it assigns, to each object of $p_{i+1}(t_{i+1})$, its successor. Furthermore, if the funhom $\mathbf{C}_{i+1}$ is such to preserve *fc* over any $<c_\beta$-subclass, then we shall call it a $fc^{<c_\beta}$*-funhom.* With the above notions we define a closure operator as follows.

D e f i n i t i o n 5. By a *closure operator* on an *l*-semigroupoid $p_{i+1}(t_{i+1})$ we mean a covariant funhom $\mathbf{C}_{i+1}$ of $p_{i+1}(t_{i+1})$ to itself which fulfils the following conditions:

$C\,1: \mathbf{C}_{i+1}$ is a suc-funhom.

$C\,2: \mathbf{C}_{i+1}$ is an idempotent funhom.

$C\,3: \mathbf{C}_{i+1}$ is an $fc^{<c_\beta}$-funhom.

$C\,4: \mathbf{C}_{i+1}$ leaves fixed the first object of $p_{i+1}(t_{i+1})$.

The pair $\langle p_{i+1}(t_{i+1}); \mathbf{C}_{i+1}\rangle$ we shall call a *closure l-semigroupoid.*

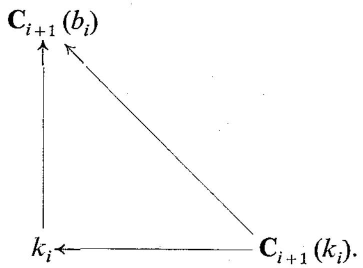
By means of this operator we can define a topology. Let $\langle p_{i+1}(t_{i+1}); \mathbf{C}_{i+1}\rangle$ be a closure *l*-semigroupoid. The image of $p_{i+1}(t_{i+1})$ under $\mathbf{C}_{i+1}$ is a subclass of $p_{i+1}(t_{i+1})$ consisting of the objects $\mathbf{C}_{i+1}(t_i)$, $t_i \in t_{i+1}$ and the rules $\mathbf{C}_{i+1}(p_i)$, $p_i \in p_{i+1}$. This subclass of $p_{i+1}(t_{i+1})$ we shall denote by $\mathbf{C}_{i+1}(p_{i+1})$. If $p_{i+1}(t_{i+1})$ is also a complemented *l*-semigroupoid, then $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$ is a subclass of $p_{i+1}(t_{i+1})$, the class of objects of which consists of complements $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(t_i)$ of the objects $\mathbf{C}_{i+1}(t_i)$ of $\mathbf{C}_{i+1}(p_{i+1})$.In the following proposition we shall show that $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$ fulfils the conditions to be a topology.

P r o p o s i t i o n 3. *Let* $\langle p_{i+1}(t_{i+1}); \mathscr{C}_{i+1}, \mathbf{C}_{i+1}\rangle$ *be a complemented closure l-semigroupoid. Then the image* $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$ *of* $p_{i+1}(t_{i+1})$ *is a topology.*

P r o o f. We have to show that $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$ satisfies the conditions i) and ii) of the Proposition 1. First we show i). Let us consider the object $o_i^s \in p_{i+1}(t_{i+1})$. According to $C\,4$, $\mathbf{C}_{i+1}(o_i^s) = o_i^s$. Hence $o_i^s \in \mathbf{C}_{i+1}(p_{i+1})$ Since $\mathscr{C}_{i+1}(o_i^s) = 1_i$, then $1_i \in \mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$. From the properties of the objects 1_i and $\mathbf{C}_{i+1}(1_i)$ we have $\mathbf{C}_{i+1}(1_i) \approx 1_i$ and hence $1_i \in \mathbf{C}_{i+1}(p_{i+1})$. Since $\mathscr{C}_{i+1}(1_i) = o_i^s$, then $o_i^s \in \mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$.

Now ii). We show first that $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$ allows *lcc* formation on each $<c_\beta$-subclass. Let $r_{i+1}(a_{i+1})$ be a subclass of $p_{i+1}(t_{i+1})$ such that $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(r_{i+1})$ is a c_α-subclass of $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$, where $c_\alpha < c_\beta$. Then from the definition of $\mathscr{C}_{i+1}$ we have that $lcc^{c_\alpha}(\mathscr{C}_{i+1}\mathbf{C}_{i+1}(r_{i+1})) = \mathscr{C}_{i+1}fc^{c_\alpha}(\mathbf{C}_{i+1}(r_{i+1}))$ and because of the property $C\,3$ of $\mathbf{C}_{i+1}$ that $\mathscr{C}_{i+1}fc^{c_\alpha}(\mathbf{C}_{i+1}(r_{i+1})) = \mathscr{C}_{i+1}\mathbf{C}_{i+1}fc^{c_\alpha}(r_{i+1}(a_{i+1}))$. Hence, since $fc^{c_\alpha}(r_{i+1}(a_{i+1}))$ is in $p_{i+1}(t_{i+1})$, then $\mathscr{C}_{i+1}\mathbf{C}_{i+1}fc^{c_\alpha}(r_{i+1}\,a_{i+1}))$ is in $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$. It means that $lcc^{c_\alpha}(\mathscr{C}_{i+1}\mathbf{C}_{i+1}(r_{i+1}))$ is also in $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$ and that this one allows *lcc* formation on each $<c_\beta$ – subclass. Let us consider now an arbitrary subclass $q_{i+1}(b_{i+1})$ of $p_{i+1}(t_{i+1})$ and show that *fc* over its image under $\mathscr{C}_{i+1}\mathbf{C}_{i+1}$ is in $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$. Denote $fc(\mathscr{C}_{i+1}\mathbf{C}_{i+1}(q_{i+1}))$ by C_{i+1}.

Certainly, $C_{i+1}=\mathscr{C}_{i+1}\,lcc\,(\mathbf{C}_{i+1}(q_{i+1}))$, and hence $\mathscr{C}_{i+1}(C_{i+1})=lcc\,(\mathbf{C}_{i+1}(q_{i+1}))$ being in $p_{i+1}(t_{i+1})$. Let k_i be the covertex of this *lcc*. $\mathbf{C}_{i+1}(lcc\,(\mathbf{C}_{i+1}(q_{i+1})))$ is obviously a cocone over $\mathbf{C}_{i+1}(q_{i+1})$ with the covertex $\mathbf{C}_{i+1}(k_i)$. Since $\mathscr{C}_{i+1}(C_{i+1})$ is a unique *lcc* over $\mathbf{C}_{i+1}(q_{i+1})$, then there is a unique rule $\varphi_i:\mathbf{C}_{i+1}(k_i)\to k_r$ such that for every $\mathbf{C}_{i+1}(b_i)$ commutes the diagram



On the other side, from **C** 1 we have the existence of a unique *fc* $k_i\to\mathbf{C}_{i+1}(k_i)$. Hence $\mathbf{C}_{i+1}(k_i)\approx k_i$, respectively $\mathbf{C}_{i+1}\mathscr{C}_{i+1}(C_{i+1})\approx\mathscr{C}_{i+1}(C_{i+1})$. Thus, $\mathscr{C}_{i+1}(C_{i+1})$ is in $\mathbf{C}_{i+1}(p_{i+1})$ and since $\mathscr{C}_{i+1}\mathscr{C}_{i+1}(C_{i+1})=C_{i+1}$ it is also in $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$. Thus, $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(p_{i+1})$ allows *fc* formation on any its subclass.

The next operator that we mean to involve is an interior operator. Its definition is dual to the definition of a closure operator. Because of that we shall write it without any comment.

Definition 6. By an *interior operator* on an *l*-semigroupoid $p_{i+1}(t_{i+1})$ we mean a covariant funhom $\mathbf{O}_{i+1}$ of $p_{i+1}(t_{i+1})$ to itself which fulfils the following conditions:

O 1: $\mathbf{O}_{i+1}$ is a predec-funhom.

O 2: $\mathbf{O}_{i+1}$ is an idempotent funhom.

O 3: $\mathbf{O}_{i+1}$ is an $lcc^{<c_\beta}$-funhom.

O 4: $\mathbf{O}_{i+1}$ leaves fixed the last object of $p_{i+1}(t_{i+1})$.

The pair $\langle p_{i+1}(t_{i+1});\ \mathbf{O}_{i+1}\rangle$ we shall call an *interior l-semigroupoid.*

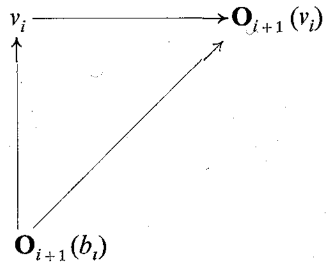
The term, a predec-funhom is dual to the term a suc-funhom. Thus, it is a funhom which assigns, to each object of $p_{i+1}(t_{i+1})$, its predecessor. The remaining terms are quite clear.

Now we shall show that a topology can also be involved by means of this operator. Let $p_{i+1}(t_{i+1})$ be an *l*-semigroupoid and $\mathbf{O}_{i+1}$ an interior operator on it. By $\mathbf{O}_{i+1}(p_{i+1})$ we shall denote a subclass of $p_{i+1}(t_{i+1})$ whose class of objects consists of the objects $\mathbf{O}_{i+1}(t_i)$, $t_i\in t_{i+1}$ and the class of rules of the rules $\mathbf{O}_{i+1}(p_i)$, $p_i\in p_{i+1}$.

Proposition 4. *The subclass* $\mathbf{O}_{i+1}(p_{i+1})$ *of an interior l-semigroupoid* $p_{i+1}(t_{i+1})$ *is a topology.*

Proof. Outline of the proof. Before all $1_i\in\mathbf{O}_{i+1}(p_{i+1})$. From $\mathbf{O}_{i+1}(o_i^s)\to o_i^s$ and $o_i^s\to\mathbf{O}_{i+1}(o_i^s)$ we have $\mathbf{O}_{i+1}(o_i^s)\approx o_i^s$ and hence $o_i^s\in\mathbf{O}_{i+1}(p_{i+1})$. Certainly, $\mathbf{O}_{i+1}(p_{i+1})$ allows *lcc* formations on $<c_\beta$-subclasses. Now we show that it also allows *fc* formation on any subclass of it. Let $\mathbf{O}_{i+1}(q_{i+1}(b_{i+1}))$ be a subclass of $\mathbf{O}_{i+1}(p_{i+1})$. Clearly, *fc* over $\mathbf{O}_{i+1}(q_{i+1})$ is in $p_{i+1}(t_{i+1})$. Denote its vertex

by v_i. Then $\mathbf{O}_{i+1}(fc(\mathbf{O}_{i+1}(q_{i+1})))$ is a cone over $\mathbf{O}_{i+1}(q_{i+1})$ with the vertex $\mathbf{O}_{i+1}(v_i)$. From the definition of fc we have the existence of a rule $v_i \to \mathbf{O}_{i+1}(v_i)$ such that the diagram



commutes for every $\mathbf{O}_{i+1}(b_i) \in \mathbf{O}_{i+1}(q_{i+1})$. Further, from $O1$ and the uniqueness we have $\mathbf{O}_{i+1}(v_i) \approx v_i$ and thus the closeness of $\mathbf{O}_{i+1}(p_{i+1})$ under fc formation on every its subclass.∎

There is an obvious property of the defined operators given in the following

Proposition 5. *The operators $\mathbf{C}_{i+1}$, $\mathbf{O}_{i+1}$ are determined uniquelly up to equivalences.*∎

Hence we have that topologies involved by means of these operators are also unique up to equivalences.

In the next proposition we specify a connection between the operators $\mathbf{C}_{i+1}$ and $\mathbf{O}_{i+1}$ expressed by means of the operator $\mathscr{C}_{i+1}$.

Proposition 6. *Provided $\mathbf{C}_{i+1}$ is a closure operator, $\mathbf{O}_{i+1}$ an interior operator and $\mathscr{C}_{i+1}$ an operator of complementation on an l-semigroupoid, there is the following relation*

$$\mathbf{O}_{i+1} = \mathscr{C}_{i+1} \cdot \mathbf{C}_{i+1} \cdot \mathscr{C}_{i+1}.$$

Proof. We have to show that the rule $\mathscr{C}_{i+1} \cdot \mathbf{C}_{i+1} \cdot \mathscr{C}_{i+1}$ prosseses the properties $O1-O4$. It is obviously a covariant funhom. We show first $O1$. Let us consider an object $t_i \in p_{i+1}(t_{i+1})$. Its complement is the object $\mathscr{C}_{i+1}(t_1)$. By applying to this object the funhom $\mathbf{C}_{i+1}$ we obtain fc

$$\rho_{i+1}(\mathscr{C}_{i+1}(t_i)) : \mathscr{C}_{i+1}(t_i) \to \mathbf{C}_{i+1} \mathscr{C}_{i+1}(t_i).$$

The funhom $\mathbf{C}_{i+1}$ rewrites it into $lcc\, \mathscr{C}_{i+1}(\rho_{i+1}(\mathscr{C}_{i+1}(t_i))) : \mathscr{C}_{i+1} \cdot \mathbf{C}_{i+1} \cdot \mathscr{C}_{i+1}(t_i) \to t_i$. Furthermore we have $[\mathscr{C}_{i+1} \mathbf{C}_{i+1} \mathscr{C}_{i+1}]\, [\mathscr{C}_{i+1} \mathbf{C}_{i+1} \mathscr{C}_{i+1}]\, (t_i) = \mathscr{C}_{i+1} \mathbf{C}_{i+1} \mathscr{C}_{i+1}(t_i)$. Thus it fulfils $O2$. The proporties $O3$ and $O4$ are easy to be shown. For instance $\mathscr{C}_{i+1} \mathbf{C}_{i+1} \mathscr{C}_{i+1}(1_i) = \mathscr{C}_{i+1} \mathbf{C}_{i+1}(\overset{s}{o_i}) = \mathscr{C}_{i+1}(\overset{s}{o_i}) = 1_i$.∎

From the above connection between operators $\mathbf{O}_{i+1}$ and $\mathbf{C}_{i+1}$ we have that an object $t_i \in p_{i+1}(t_{i+1})$ is closed (open) iff its complement $\mathscr{C}_{i+1}(t_i)$ is open (closed). To determine closed objects in $p_{i+1}(t_{i+1})$ we define the concept of an adherent object of s_{i+1} to an object of $p_{i+1}(t_{i+1})$.

Definition 7. For an object $s_i \in s_{i+1}$ we shall say to be *adherent* to an object $t_i \in p_{i+1}(t_{i+1})$ iff $\tau_{i+1}(s_i) \wedge \bar{t}_i \neq \mathfrak{o}_{i+1}$.

If we take fc over an object $t_i \in p_{i+1}(t_{i+1})$ together with the class of all its adherent objects, then we have that the vertex $\bar{t}_i$ of this fc is a closure of t_i. Certainly, $\bar{t}_i$ is obtained from t_i by adjoining objects of s_{i+1}, or more accurate of $I_{i+1}(s_{i+1})$, which are adherent to the object t_i. This operation obviously fulfils the conditions to be a closure operation. Hence we have an obvious

Proposition 7. *An object* $t_i \in p_{i+1}(t_{i+1})$ *is closed iff it contains all its adherent objects.*

In the remainder of this section we shall define the concept of a basis for topology and show that this concept completely characterizes a topology.

Let us denote by $q_{i+1}(b_{i+1})$ a subclass of $p_{i+1}(t_{i+1})$ of a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ which consists of all objects and rules of bases of neighborhoods filters $\tau_{i+1}(s_i)$, $s_i \in s_{i+1}$, one for each neighborhoods filter. This class possesses the following property: for every τ_{i+1}-open object t_i and every $s_i \in s_{i+1}$ such that $s_i \leftarrow_{p'_i} t_i$ there exist an object $b_i \in q_{i+1}(b_{i+1})$ and a rule $p_i \in p_{i+1}$ such $s_i \leftarrow_{p''_i} b_i$ and $p_i : b_i \to t_i$ and for which the formula $\mathcal{C}_{\text{com}}(p''_i, p_i; p'_i)$ holds. We utilize the properties of this class to define a basis for topology.

Definition 8. A subclass $q_{i+1}(b_{i+1})$ of $p_{i+1}(t_{i+1})^{c_\beta}$ of a c_β-space $(s_{i+1}, p_{i+1}(t_{i+1})^{c_\beta}, p_{i+1}(t_{i+1}))$ we shall say to be a *basis* of this space if for every object $t_i \in p_{i+1}(t_{i+1})^{c_\beta}$ and every $s_i \in s_{i+1}$ such that $s_i \leftarrow_{p'_i} t_i$ there exist an object $b_i \in q_{i+i}(b_{i+1})$ and a rule $p_i \in p_{i+1}(t_{i+1})$ such that $s_i \leftarrow_{p''_i} b_i$ and $p_i : b_i \to t_i$.

Certainly, from the property iii) of $p_{i+1}(t_{i+1})^{c_\beta}$ we have holdness of the formula $\mathcal{C}_{\text{com}}(p''_i, p_i; p'_i)$.

A space $(s_{i+1}, p_{i+1}(t_{i+1})^{c_\beta}, p_{i+1}(t_{i+1}))$ possesses at least one basis, $p_{i+1}(t_{i+1})^{c_\beta}$ itself. By means of the notion of a basis we can characterize topological space. The characterization is given by the following

Proposition 8 *Let the subclass* $q_{i+1}(b_{i+1})$ *of* $p_{i+1}(t_{i+1})^{c_\beta}$ *possesses the following property*:

(*B*) *For each* $s_i \in s_{i+1}$ *the subclass* $\mathcal{N}_{i+1}(s_i)$ *of* $q_{i+1}(b_{i+1})$, *the class of objects of which consists of all those objects* $b_i \in q_{i+1}(b_{i+1})$ *such that* $s \leftarrow b_i$, *is a filter basis.*

Then the rule $\tau_{i+1} : s_{i+1} \to p_{i+1}(t_{i+1})$ *such that* $\tau_{i+1}(s_i) = \overline{\mathcal{N}_{i+1}(s_i)}$ *is a topology having* $q_{i+1}(b_{i+1})$ *as basis. Conversely, every topology on* $(s_{i+1}, p_{i+1}(t_{i+1}))$ *can be obtained in this way.*

Proof. Analogously to the proof of the Proposition 2 we have that the rule τ_{i+1} such that $\tau_{i+1}(s_i) = \overline{\mathcal{N}_{i+1}(s_i)}$ defines a topology on $(s_{i+1}, p_{i+1}(t_{i+1}))$. If t_i is on object of $p_{i+1}(t_{i+1})^{c_\beta}$, then for each $s_i \in s_{i+1}$ such that $s_i \leftarrow b_i$ we can find a $b_i \in q_{i+1}(b_{i+1})$ and a rule $p_i \in p_{i+1}(t_{i+1})^{c_\beta}$ such that $s_i \leftarrow b_i$ and $p_i : b_i \to t_i$. Hence, $q_{i+1}(b_{i+1})$ is a basis for the topology.

Conversely, let τ_{i+1} be a topology on $(s_i, p_{i+1}(t_{i+1})$ and $q_{i+1}(b_{i+1})$ a basis. If $\mathcal{N}_{i+1}(s_i)$ is a subclass of $q_{i+1}(b_{i+1})$ and $r_{i+1}(a_{i+1})$ its $< c_\beta$ – subclass, then *lcc* over this class is in $p_{i+1}(t_{i+1})^{c_\beta}$. Provided k_i is the covertex of this *lcc*, there exists a unique $p_i : s_i \to k_i$. From the definition of a basis we have the existence of an object $b_i \in q_{i+1}(b_{i+1})$ and a rule $p'_i \in p_{i+1}(t_{i+1})^{c_\beta}$ such that $s_i \leftarrow_{p''_i} b_i$ and $p'_i : b_i \to k_i$. Hence we conclude that $\mathcal{N}_{i+1}(s_i)$ is a filter basis. Certainly, there exists one such topology on $(s_{i+1}, p_{i+1}(t_{i+1}))$.∎

One can also define the notion of a cobasis for topology, but we omit to do it.

4. Separation axioms.

In the introduction we have said that in formation of a topological structure in $\mathcal{U}$ there are conditions which regulate this formation and that among them there are separation axioms. The choice of the conditions which govern the formation of a topological structure ought to be such to ensure certain good properties of it. We choose separation conditions so to ensure the unique convergence of its filters and distinguishability of its objects. Thus, our choice of these conditions is the usual one. By their specifying we obtain various kinds or various realizations of the structure. We devote this section to formulation of separation conditions and characterization of structures embodied by them. By the order we have

Definition 9. A topology τ_{i+1} is a T_1*-topology*, and the space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ a T_1-space if the following axiom holds

$$(T_1) \qquad \neg_{i+1}(s_i') \vdash_{i+1} \tau_{i+1}(s_i) \Rightarrow s_i = s_i'.$$

Certainly, a discrete topology on $(s_{i+1}, p_{i+1}(t_{i+1}))$ is always a T_1-topology. We have also an obvious.

Proposition 9. *If Card* $(p_{i+1}(t_{i+1})) < c_\beta$, *then the discrete topology is the only* T_1*-topology possible on* $(s_{i+1}, p_{i+1}(t_{i+1}))$.∎

The next proposition gives a characterization of T_1-topological spaces.

Proposition 10. *The following statements are pairwise equivalent*:

1) $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ is a T_1*-space.*

2) *Given two object* $s_i, s_i' \in s_{i+1}$ *such that* $s_i \neq s_i'$, *there is an object* $t_i \in \tau_{i+1}(s_i)$ *such that* $\neg s_i' \prec t_i$.

3) *The only presequent of the objects of* $\tau_{i+1}(s_i)$ *is* s_i.

4) *The objects of* s_{i+1} *regarded as contained in* $p_{i+1}(t_{i+1})$ *are closed.*

Proof. Let s_i' be an object of s_{i+1} such that $s_i' \prec t_i$ for all $t_i \in \tau_{i+1}(s_i)$, then certainly $\neg_{i+1}(s_i') \vdash_{i+1} \tau_{i+1}(s_i)$. Hence because of 1) we have $s_i' = s_1$. Thus, for the case of disjoint objects s_i and s_i' there must exist an object $t_i \in \tau_{i+1}(s_i)$ such that $\neg s_i' \prec t_i$. From (T_1) we have that s_i is the presequent of the objects of $\tau_{i+1}(s_i)$. By 2) we have that s_i' is not their presequent for all $s_i' \neq s_i$. In such a way we have shown 1) $\Rightarrow$ 2) and 2) $\Rightarrow$ 3). Now we show 3) $\Rightarrow$ 4). If $s_i' \neq s_i$ then s_i is not the presequent of the objects of $\tau_{i+1}(s_i')$, hence $\tau_{i+1}(s_i') \wedge \neg_{i+1}(s_i) = \mathfrak{o}_{i+1}$. Thus, the only object adherent to s_i is just s_i. Hence, s_i is closed in $p_{i+1}(t_{i+1})$. Certainly, in a more precise writing ought to stay $I_{i+1}(s_i)$ instead of s_i. Finally we show 4) $\Rightarrow$ 1). From $\neg_{i+1}(s_i) \vdash_{i+1} \tau_{i+1}(s_i')$ $\Rightarrow \tau_{i+1}(s_i') \wedge \neg_{i+1}(s_i) \neq \mathfrak{o}_{i+1}$ follows that s_i' is adherent to s_i and because of 4) that $s_i' = s_i$.∎

Definition 10. A topology τ_{i+1} is a T_2*-topology*, and the space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ a T_2-space if the following axiom holds

$$(T_2) \qquad s_i \neq s_i' \Rightarrow \tau_{i+1}(s_i) \wedge \tau_{i+1}(s_i') = \mathfrak{o}_{i+1}.$$

The above condition means injectivity of the rule τ_{i+1}. Thus, a topological space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ is a T_2-space if the rule τ_{i+1} is injective. Certainly, this condition guarantees uniqueness of the limit, of course, if the limit exists. Thus, in a T_2-space every filter, except $\mathfrak{o}_{i+1}$ which in any topology converges to all objects of s_{i+1}, converges to at most one object of s_{i+1}. The converse statement, if every filter in a space converges to at most one object then this space is a T_2-space, is also valid.

From the expression $\rightharpoondown_{i+1}(s_i) \vdash_{i+1} \tau_{i+1}(s_i')$ we have $\rightharpoondown_{i+1}(s_i) \vdash_{i+1} \tau_{i+1}(s_i) \wedge \tau_{i+1}(s_i')$. Hence we have further that $\tau_{i+1}(s_i) \wedge \tau_{i+1}(s_i') \neq \mathfrak{o}_{i+1}$. If the topology is T_2-topology, then $s_i = s_i'$. In that way we have proved the following

Proposition 11. *Each T_2-topology is T_1-topology.*∎

Certainly, the converse statement is not true. The discrete topology on any space is always a T_2-topology. A characterization of T_2-spaces is given by the following

Proposition 12. *The following statements are pairwise equivalent:*

1) *$(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ is a T_2-space.*

2) *Any two distinct objects of s_{i+1} possess disjoint neighborhoods.*

3) *If $s_i \neq s_i'$, then there exists a neighborhood $t_i \in \tau_{i+1}(s_i)$ such that s_i' is not contained in $\mathbf{C}_{i+1}(t_i)$.*

Proof. The proof is obvious. From disjointness of filters $\tau_{i+1}(s_i)$ and $\tau_{i+1}(s_i')$ follows disjointness of their objects. Then for a $t_i \in \tau_{+1}(s_i)$ we have $\tau_{i+1}(s_i') \wedge \overline{t}_i = \mathfrak{o}_{i+1}$ and hence that s_i' is not in $\mathbf{C}_{i+1}(t_i)$. The converse is obvious.∎

So far we have not taken into account that besides open filters there are also clo ed filters in a topological space. For a filter we shall say to be closed if it possesses a basis consisting of closed objects. If $\tau_{i+1}(s_i)$ is a neighborhoods filter in a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ and $\mathbf{C}_{i+1}$ a clo ure operator on $p_{i+1}(t_{i+1})$, then $\mathbf{C}_{i+1}(\tau_{i+1}(s_i))$ is a filter whose basis consists of closures of basis objects of $\tau_{i+1}(s_i)$. The following axiom will concern these filters.

Definition 11. A topology τ_{i+1} is a *T_3-topology*, and the space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ a T_3-space if the following axiom holds

$$(T_3) \qquad \mathbf{C}_{i+1}(\tau_{i+1}(s_i)) = \tau_{i+1}(s_i)$$

for each $s_i \in s_{i+1}$.

A space which is both a T_3-space and a T_1-space we shall call a *regular space.*

The condition (T_3) above means that the closure operator $\mathbf{C}_{i+1}$ acts fixed on the neighborhoods filters in con idered space. Thus, in a T_3-space all neighborhoods filters are closed. In the next proposition we give a characterization of these spaces.

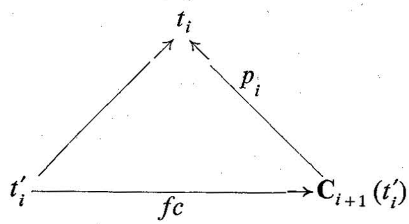
P r o p o s i t i o n 13. *The following statements are pairwise equivalent*:

1) $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ *is a* T_3*-space*.

2) $(\forall\, t_i \in \tau_{i+1}(s_i))\, (\exists\, t_i' \in \tau_{i+1}(s_i))\, (\exists!\, p_i \in \tau_{i+1}(s_i))\, (p_i : \mathbf{C}_{i+1}(t_i') \rightarrow t_i)$.

3) $(\forall\, t_i \in p_{i+1}(t_{i+1}))\, (\forall\, s_i \in s_{i+1})\, (\mathbf{C}_{i+1}(t_i) = t_i\ \&\ \neg\, s_i \prec t_i) \Rightarrow (\exists\, a_i, b_i \in p_{i+1}) (t_{i+1})^{c\beta})\, (\exists!\, p_i \in p_{i+1})\, (s_i \prec a_i\ \&\ p_i : t_i \rightarrow b_i\ \&\ a_i \wedge b_i = o_i^s)$.

P r o o f. 1) $\Leftrightarrow$ 2) follows from definitions. Namely, from the definitions of $\mathbf{C}_{i+1}$ and T_3-spaces we have the commutative diagram



Conversely, a subclass of $\tau_{i+1}(s_i)$, the class of objects of which consists of the objects $\mathbf{C}_{i+1}(t_i')$ is a basis of $\tau_{i+1}(s_i)$.

2) $\Rightarrow$ 3). Certainly, $\mathscr{C}_{i+1}(t_i)$ is open object such that $s_i \prec \mathscr{C}_{i+1}(t_i)$. Hence, $\mathscr{C}_{i+1}(t_i) \in \tau_i(s_i)$ and by 2) there exist an object $a_i \in \tau_{i+1}(s_i)$ and a unique rule $\mathbf{C}_{i+1}(a_i) \rightarrow \mathscr{C}_{i+1}(t_i)$ and hence further a unique rule $t_i \rightarrow \mathscr{C}_{i+1}\mathbf{C}_{i+1}(a_i)$. If we define $b_i = \mathscr{C}_{i+1}\mathbf{C}_{i+1}(a_i)$, then obviously $a_i \wedge b_i = o_i^s$.

3) $\Rightarrow$ 2). For an arbitrary object $a_i \in p_{i+1}(t_{i+1})$, $a_i \in \tau_{i+1}(s_i) \Rightarrow \mathbf{O}_{i+1}(a_i) \in \tau_{i+1}(s_i)$. Clearly, $\mathscr{C}_{i+1}\mathbf{O}_{i+1}(a_i)$ is closed object not containing s_i and hence there exist open objects b_i and c_i and a unique rule p_i such that $s_i \prec b_i$, $p_i : \mathscr{C}_{i+1}\mathbf{O}_{i+1}(a_i) \rightarrow c_i$ and $b_i \wedge c_i = o_i^s$. Since $s_i \prec b_i$ then $b_i \in \tau_{i+1}(s_i)$. We have further the unique rules $\mathbf{C}_{i+1}(b_i) \rightarrow \mathscr{C}_{i+1}(c_i)$ and $\mathscr{C}_{i+1}(c_i) \rightarrow a_i$ and hence the result.∎

From the statement 3) of the above proposition we have that for every closed object t_i and every s_i of s_{i+1} which is not in this closed object, the objects a_i and b_i such that $t_i \rightarrow a_i$ unique and $s_i \prec b_i$ are disjoint. Thus, the axiom (T_3) means separation of closed objects of $p_{i+1}(t_{i+1})$ and objects of s_{i+1}.

So far formulated axioms are independent. However, there is the following connection among them.

P r o p o s i t i o n 14. (T_3) & $(T_1) \Rightarrow (T_2)$, i.e. *each regular space is* T_2*-space*.

P r o o f. Let $\tau_{i+1}(s_i) \wedge \tau_{i+1}(s_i') \neq 0_{i+1}$, then for all $t_i \in \tau_{i+1}(s_i)$ we have $s_i' \prec t_i$ and hence $\neg_{i+1}(s_i') \vdash_{i+1} \tau_{i+i}(s_i)$. Since $\tau_{i+i}(s_i)$ is open we have $\tau_{i+1}(s_i') \vdash_{i+1} \tau_{i+1}(s_i)$. Symmetrically we obtain $\tau_{i+1}(s_i) \vdash_{i+1} \tau_{i+1}(s_i')$. Thus $\tau_{i+1}(s_i) = \tau_{i+1}(s_i')$. If (T_1) holds, then $s_i = s_i'$. Hence (T_2) holds.∎

In the remainder of this section we formulate axioms by which we shall separate objects of $p_{i+1}(t_{i+1})$.

D e f i n i t i o n 12. A topology τ_{i+1} is a T_4*-topology*, and the space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ a T_4-space if the following axiom holds

(T_4) $\quad (\forall t_i, t_i' \in p_{i+1}\, t_{i+1}))\, (\mathbf{C}_{i+1}(t_i) = t_i\ \&\ \mathbf{C}_{i+1}(t_i') = t_i'\ \&\ t_i \wedge t_i' = o_i^s) \Rightarrow$

$(\exists\, a_i \in \overline{t_i})\, (\exists\, a_i' \in \overline{t_i'})\, (a_i \wedge a_i' = o_i^s)$.

A space which is both a T_4-space and a T_1-space we shall call a *normal space.*

This axiom is not a consequence of the axioms (T_1)-(T_3). However, the following assertion holds.

Proposition 15. *Every normal space is regular.*

Proof. It is enough to consider an object s_i instead of t_i. Because of (T_1), s_i is closed and the result follows immediately from (T_4).▮

In the next proposition we give a characterization of T_4-spaces.

Proposition 16. *The following statements are equivalent:*

1) $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ *is a T_4-space.*

2) $(\forall\, t_i \in p_{i+1}(t_{+1}); \mathbf{C}_{i+1}(t_1) = t_i)\,(\forall\, a_i \in p_{i+1}(t_{i+1});\ a_i \in \overline{t_i})\,(\exists\, a'_i \in p_{i+1}(t_{i+1});$
$a'_i \in \overline{t_i})\,(\exists!\, p_i \in p_{i+1})(p_i : \mathbf{C}_{i+1}(a'_i) \to a_i).$

Proof. Since $a_i \in \overline{t_i}$, then there exists a rule $t_i \to a_i$ and from the definition of $\mathbf{O}_{i+1}$ a unique rule $t_i \to \mathbf{O}_{i+1}(a_i)$. Certainly, t_i and $\mathscr{C}_{i+1}\mathbf{O}_{i+1}(a_i)$ are disjoint closed objects. From (T_4) there exist objects $a'_i \in \overline{t_i}$ and $b_i \in \overline{\mathscr{C}_{i+1}\mathbf{O}_{i+1}(a_i)}$ such that $a'_i \wedge b_i = o^s_i$. Hence we have the existence of a unique rule $\mathbf{C}_{i+1}(a'_i) \to \mathscr{C}_{i+1}\mathbf{O}_{i+1}(b_i)$. On the other side we have unique rules $\mathscr{C}_{i+1}\mathbf{O}_{i+1}(b_i) \to \mathbf{O}_{i+1}(a_i) \to a_i$. Thus there is a unique rule $\mathbf{C}_{i+1}(a'_i) \to a_i$.

Let us prove the converse. If t_i and t'_i are two arbitrary disjoint closed objects in $p_{i+1}(t_{i+1})$, then there is a rule $t_i \to \mathscr{C}_{i+1}(t'_i)$. Hence $\mathscr{C}_{i+1}(t'_i) \in \overline{t_i}$ and there exists an $a_i \in \overline{t_i}$ such that $\mathbf{C}_{i+1}(a_i) \to \mathscr{C}_{i+1}(t'_i)$ is a unique rule in $p_{i+1}(t_{i+1})$. Hence we have then $t'_i \to \mathscr{C}_{i+1}\mathbf{C}_{i+1}(a_i)$, i.e. $\mathscr{C}_{i+1}\mathbf{C}_{i+1}(a_i) \in \overline{t'_i}$. If we define $a'_i = \mathscr{C}_{i+1}\mathbf{C}_{i+1}(a_1)$, then $a_i \wedge a'_i = o^s_i$.▮

The last axiom in this section will sharpen the axiom (T_4). Its formulation is as follows

Definition 13. A topology τ_{i+1} is a *T_5-topology,* and the space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ a T_5-space if the following axiom holds

(T_5) $$(\forall\, t_i, t'_i \in p_{i+1}(t_{i+1}))(\mathbf{C}_{i+1}(t_i) = t_i \text{ or } \mathbf{C}_{i+1}(t'_i) = t'_i \ \&\ t_i \wedge t'_i = o^s_i) \Rightarrow (\exists\, a_i \in \overline{t_i})(\exists\, a'_i \in \overline{t'_i})(a_i \wedge a'_i = o^s_i).$$

A space which is both a T_5-space and a T_1-space we shall call a *completely normal space.*

The following proposition is obvious from the definition.

Proposition 17. *Each T_5-space is a T_4-space.*▮

Hence we have then that every completely normal space is normal.

5. Cardinality axioms.

In this section we shall deal with the second group of axioms which govern formation of topological structures in $\mathscr{U}$. They are cardinality axioms. By means of these axioms we shall go a step further in making precise the relationships betwen objects of s_{i+1} and certain subclasses of $p_{i+1}(t_{i+1})$ in a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$. By the way we shall define a new concept and a cardinal concerned with it.

In the Section 3 we have stipulated that the ranks of all neighborhoods filters are $\geqslant c_\beta$, where c_β is a fixed cardinal. Certainly, the ranks of all filters of $f_{i+1}(p_{i+1})$ are bounded above by Card $(p_{i+1}(t_{i+1}))$. Therefore, for every topology on $(s_{i+1}, p_{i+1}(t_{i+1}))$ there exist a least cardinal number α such thas $Rk(\tau_{i+1}(s_i)) \leqslant \alpha$ for all $s_i \in s_{i+1}$. This cardinal number we shall call the rank of the space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ with respect to the topology τ_{i+1}.

Definition 14. By the *rank* of a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ with respect to a topology τ_{i+1} we mean the sup of all neighborhoods filters $\tau_{i+1}(s_i)$, $s_i \in s_{i+1}$, i.e.

$$Rk(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) = Rk(\tau_{i+1}) = \sup\{Rk(\tau_{i+1}(s_i)) \mid s_i \in s_{i+1}\}.$$

If $Rk(\tau_{i+1}) \leqslant c_\beta$ for a space with the topology τ_{i+1}, then we shall say for it to be of the *first category*. The rank of spaces with the discrete and the trivial topologies is equal to 1.

Besides the just defined cardinal number we can define one more number. It is concerned with the bases of a space. Among all the bases of a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ there are those of minimal cardinality. We call their cardinality the basis degree of $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ or simply of the topology τ_{i+1}.

Definition 15. By the *basis degree* of a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ we mean the minimal cardinality of all its bases, i.e.

$Bd(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) = Bd(\tau_{i+1}) = \min\{\text{Card}(q_{i+1}(b_{i+1}) \mid q_{i+1}(b_{i+1})$ a basis of $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))\}$.

If $Bd(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) \leqslant c_\beta$, then we shall say that the space is of the *second category*. Any basis $q_{i+1}(b_{i+1}))$ such that

$$\text{Card}(q_{i+1}(b_{i+1})) = Bd(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$$

is a minimal basis.

Let $q_{i+1}(b_{i+1})$ be a minimal basis of $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ and $\mathcal{N}_{i+1}(s_i)$ a subclass of it being a basis of $\tau_{i+1}(s_i)$ for an $s_i \in s_{i+1}$. Then

$$Rk(\tau_{i+1}(s_i)) \leqslant \text{Card}(\mathcal{N}_{i+1}(s_i)) \leqslant Bd(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})).$$

Hence we have.

Proposition 18. $Rk(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) \leqslant Bd(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$.∎

The basis degree of spaces with the trivial topology is always 1, while those with the discrete topology is equal to the cardinality of s_{i+1}.

In the rest of this section we define a new concept and a cardinal number concerned with it.

Definition 16. For an object t_i of a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ we shall say to be *densely* near by an other object t_i' of the space if $\eta_{i+1}(t_i) : t_i \to \mathbf{C}_{i+1}(t_i)$ is reducible through t_i'.

Certainly, if t_i is densely near by the object 1_i of the space, then $\mathbf{C}_{i+1}(t_i) \approx 1_i$. The smallest possible cardinality of such objects in a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ we shall call the separability degree of the space or of τ_{i+1} for short.

Definition 17. By the *separability degree* of a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ we mean the minimal cardinality of objects being densely near by the object 1_1 of the space. In symbols

$$Sd(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) = Sd(\tau_{i+1}) = \min\{\text{Card}(t_i) \mid t_i \text{ densely near by the object } 1_i \text{ of } (s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))\}.$$

If $Sd(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) \leqslant c_\beta$, we shall say that the space and topology are *separable*.

The separability degree of the trivial topology is always 1, while of the discrete topology it is equal to the cardinality of s_{i+1}.

If we regard a minimal basis of a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ and take into account the definition of bases we can prove the following.

P r o p o s i t i o n 19. $Bd(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1})) \geqslant Sd(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$.∎

Since the proof of this proposition is obvious we omit it.

6. Compactness axioms.

In this section we shall formulate the third group of axioms which govern formation of topological structures in $\mathcal{U}$. A space formed on $(s_{i+1}, p_{i+1}(t_{i+1}))$ according to these axioms will distinguish itself by a "stronger" connection between objects of s_{i+1} and filters of $p_{i+1}(t_{i+1})$). Namely, the axioms will ensure that a sufficient number of filters have their d-limit objects in s_{i+1}, i.e. that the space possesses a sufficient number of convergent filters. We formulate here a type of these axioms and in a separate paper we shall consider some another types of them. For their formulations we shall have to involve certain new concepts.

For the present formulation of compactness axioms we need the concept of an adherent object to a filter. Its definition is analogous to the definition of an adherent object to an object of a space.

D e f i n i t i o n 18. For an object s_i of s_{i+1} in a space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ we shall say to be *adherent* to a filter $\mathfrak{f}_{i+1}$ of $p_{i+1}(t_{i+1})$ if the condition $\mathfrak{f}_{i+1} \wedge \tau_{i+1}(s_i) \neq \mathfrak{o}_{i+1}$ holds.

Certainly, if s_i is adherent to $\mathfrak{f}_{i+1}$ then there exists a filter $\mathfrak{g}_{i+1}$ being different of $\mathfrak{o}_{i+1}$ and such that $\mathfrak{g}_{i+1} \vdash_{i+1} \tau_{i+1}(s_i)$ and $\mathfrak{g}_{i+1} \vdash_{i+1} \mathfrak{f}_{i+1}$. For instance $\mathfrak{g}_{i+1} = \mathfrak{f}_{i+1} \wedge \tau_{i+1}(s_i)$ is such a filter. Conversely, if there exists a filter $\mathfrak{g}_{i+1} \neq \mathfrak{o}_{i+1}$ which converges to s_i and such that $\mathfrak{g}_{i+1} \vdash_{i+1} \mathfrak{f}_{i+1}$ for a filter $\mathfrak{f}_{i+1} \in f_{i+1}(p_{i+1})$, then $\tau_{i+1}(s_i) \wedge \overline{a}_i \vdash_{i+1} \mathfrak{g}_{i+1}$ for each object a_i of $\mathfrak{f}_{i+1}$. Since $\mathfrak{g}_{i+1} \neq \mathfrak{o}_{i+1}$, s_i is adherent to every object of $\mathfrak{f}_{i+1}$ and thus to $\mathfrak{f}_{i+1}$ itself. Hence we have the following

P r o p o s i t i o n 20. *An object $s_i \in s_{i+1}$ is adherent to a filter $\mathfrak{f}_{i+1}$ iff there exists a filter converging to s_i and satisfying the conditions $\mathfrak{g}_{i+1} \neq \mathfrak{o}_{i+1}$ and $\mathfrak{g}_{i+1} \vdash_{i+1} \mathfrak{f}_{i+1}$*.∎

Now we can formulate desired axioms. These axioms will guarantee the existence of adherent objects of all or certain filters in a space. Let c_γ be a cardinal number $\geqslant c_\beta$.

D e f i n i t i o n 19. A space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ and its topology τ_{i+1} we shall say to be *completely compact* (*c_γ-compact*) if the following axiom holds

$$(C)\ (\forall \mathfrak{f}_{i+1} \in f_{i+1}(p_{i+1}); \mathfrak{f}_{i+1} \neq \mathfrak{o}_{i+1}\ (\&\ Rk(\mathfrak{f}_{i+1}) \leqslant c_\gamma))\ (\exists s_i \in s_{i+1})\ (\mathfrak{f}_{i+1} \wedge \tau_{i+1}(s_i) \neq \mathfrak{o}_{i+1}).$$

Thus, a space is completely compact if each its filter possesses at least one adherent object. Certainly, a space is completely compact iff it is c_γ-compact for all $c_\gamma \leqslant c_{p(t)}$. If $c'_\gamma \leqslant c_\gamma$ and a space is c_γ-compact then it is also c'_γ-compact. From $Rk(\mathfrak{f}_{i+1}) < c_\beta$ we have that $\mathfrak{f}_{i+1}$ is a principial filter and thus that it possesses an adherent object. Hence, the notion of c_γ-compactness is only important for cardinals $\geqslant c_\beta$. The complete compactnesss can be characterized by convergence of ultrafilters.

Proposition 21. *A space* $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ *is completely compact iff every one of its ultrafilters is convergent.*

Proof. Let the space be completely compact. Then each ultrafilter $\mathfrak{v}_{i+1}$ possesess an adherent object $s_i \in s_{i+1}$. Hence we have $\mathfrak{v}_{i+1} \wedge \tau_{i+1}(s_i) \neq \mathfrak{o}_{i+1}$ and from the property of ultrafilters, being atoms in $f_{i+1}(p_{i+1})$, we have $\mathfrak{v}_{i+1} \vdash_{i+1} \tau_{i+1}(s_i)$. Conversely, let every ultrafilter converges in the space. Then from $\mathfrak{f}_{i+1} \neq \mathfrak{o}_{i+1}$ we have the existence of a ultrafilter $\mathfrak{v}_{i+1}$ such that $\mathfrak{v}'_{i+1} \vdash_{i+1} \mathfrak{f}_{i+1}$. If $\mathfrak{v}'_{i+1} \vdash_{i+1} \tau_{i+1}(s_i)$, then according to the proposition 20. we have that s_i is a dherent to $\mathfrak{f}_{i+1}$.∎

In what follows we shall be concerned with properties of objects in a compact space. For that purpose we must define the concept of a compact object in a space.

Definition 20. For an object $t_i \in p_{i+1}(t_{i+1})$ we shall say to be completely compact (c_γ-compact) in $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$ if every filter $\mathfrak{f}_{i+1}$ of $p_{i+1}(t_{i+1})$ such that $\mathfrak{f}_{i+1} \neq \mathfrak{o}_{i+1}$ and $\mathfrak{f}_{i+1} \vdash_{i+1} t_i$ (and $Rk(\mathfrak{f}_{i+1}) \leqslant c_\gamma$) possesses an adherent object in it.

Proposition 22. *Every closed object of a completely compact (c_γ-compact) space is completely compact (c_γ-compact).*

Proof. Let t_i be a closed object in a completely compact (c_γ-compact) space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$. Every filter $\mathfrak{f}_{i+1}$ such that $\mathfrak{f}_{i+1} \neq \mathfrak{o}_{i+1}$ and $\mathfrak{f}_{i+1} \vdash_{i+1} \overline{t_i}$ (and $Rk(\mathfrak{f}_{i+1}) \leqslant c_\gamma$) possesess an adherent object $s_i \in s_{i+1}$. Thus $\tau_{i+1}(s_i) \wedge t_i \,{}_{i+1}\!\dashv$ $\dashv \tau_{i+1}(s_i) \wedge \mathfrak{f}_{i+1} \neq \mathfrak{o}_{i+1}$. Hence s_i is contained in $\mathbf{C}_{i+1}(t_i)$. Since t_i is closed then s_i is also contained in it.∎

The closedness of an object in a compact space is obviously a compactification of that object. The converse statement of the above one is also valid if the axiom (T_2) holds.

Proposition 23. *Every completely compact object of a T_2-space is closed.*

Proof. Let t_i be a completely compact object of a T_2-space $(s_{i+1}, \tau_{i+1}, p_{i+1}(t_{i+1}))$. Then for an s_i of $\mathbf{C}_{i+1}(t_i)$ we have $\tau_{i+1}(s_i) \wedge \overline{t_i} \neq \mathfrak{o}_{i+1}$, and then the existence of a ultafilter $\mathfrak{v}_{i+1}$ such that $\mathfrak{v}_{i+1} \vdash_{i+1} \tau_{i+1}(s_i) \wedge \overline{t_i}$. Since t_i is completely compact then $\mathfrak{v}_{i+1}$ also converges to an object s'_i of t_i. Because of (T_2), $s_i = s'_i$ and s_i is contained in t_i.∎

7. Conclusion.

In this paper we have dealt with formation of topological structure on a fixed level in $\mathcal{U}$ and with the conditions which govern this formation. By these conditions we have distinguished certain kinds of topological structures that we have characterized then. We have not entered into a deeper study of these structures since it has not been our main purpose. Because of that we have presented here only those results that we have regarded the most important ones. These results overlap, in the main, the well-known results given in the cited literature. However, they have now a wider sense and make a logical whole in the new approach to the problem. This approach is the most important for us because it enables us to attain the wanted aim emphasized in the Introduction.

Later, in a paper, we shall deal with a more general kind of organization of spatial wholes in $\mathcal{U}$. This organization will be more f.ee than the above one. We shall show that each topology is only a special case of such a general organization in $\mathcal{U}$.

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