PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 15 (29), 1973, pp. 47—54

ON INVERSE LIMIT OF FUNCTION SPACES

M. M. Drešević

(Communicated February 9, 1973)

1. Introduction

In this note (following the results recently obtained by M. Marjanović for exp functor and exponentially complete spaces) we consider the covariant Map_X functor in the category of Hausdorff topological spaces and continuous mappings, and introduce the notion of functionally complete spaces. The obtained results, concerning this functor and this notion, are formally similar to those in [4] and [5] concerning exp functor and exponentially complete spaces.

I thank Professor M. Marjanović for several conversations on this paper.

2. Preliminaries

Let $\mathcal{H}=(O, M)$ be the category whose objects O are all Hausdorff topological spaces and morphisms M all continuous mappings of these spaces and let X be a fixed object of the category $\mathcal{H}$. We suppose that all spaces and all mappings that we consider in this paper belong to $\mathcal{H}$. For any $Y \in O$, let $\mathrm{Map}_X(Y)$ denote the set of all continuous functions from X into Y taken with the compact-open topology. We shall also use sometimes the notation Y^X for $\mathrm{Map}_X(Y)$. Since $Y \in O$, we also have $\mathrm{Map}_X(Y) \in O$ (see [3], p. 151 Prop. 1.1). If $f \in M$ and $f: Y \rightarrow Z$, let $\mathrm{Map}_X(f): \mathrm{Map}_X(Y) \rightarrow \mathrm{Map}_X(Z)$ be a mapping defined by $[\mathrm{Map}_X(f)](g)=f \circ g$ for each $g \in \mathrm{Map}_X(Y)$. Then, the mapping $\mathrm{Map}_X(f)$ is continuous (see [3], p. 165 Ex. 4C). If $i \in M$ is an identity, then $\mathrm{Map}_X(i)$ is also an identity and it is easy to see that if f_1 and f_2 are in M and $f_1 \circ f_2$ is defined, then $\mathrm{Map}_X(f_1 \circ f_2)=\mathrm{Map}_X(f_1) \circ \mathrm{Map}_X(f_2)$. So $\mathrm{Map}_X: \mathcal{H} \rightarrow \mathcal{H}$ is a covariant functor and the image of $\mathcal{H}$ under Map_X, denoted by $\mathrm{Map}_X(\mathcal{H})$, is a subcategory of $\mathcal{H}$.

The equivalences in $\mathcal{H}$ are homeomorphisms and for two spaces Y and Z in $\mathcal{H}$, $Y \approx Z$ will mean that Y and Z are homeomorphic.

Let $\{Y, \pi, A\}$ be an inverse system over the directed set A. Since Map_X is covariant, $\{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi), A\}$ will also be an inverse system.

3. Commutation of functor Map_X with inverse limit

Now, we will prove the following statement (see also [1], p. 57. Prop. 3.6):

3.1. $\mathrm{Map}_X(\varprojlim\{Y, \pi\}) \approx \varprojlim\{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi)\}$.

Proof. Put $Y_\infty = \varprojlim\{Y, \pi\}$ and consider the mapping

$$H: \mathrm{Map}_X(\varprojlim\{Y, \pi\}) \to \varprojlim\{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi)\}$$

defined by

$$H(f) = \{p_\alpha \circ f\},$$

for every $f \in \mathrm{Map}_X(Y_\infty)$, where p_α denotes the restriction to the subset Y_∞ of the natural projection of $\Pi\{Y_\alpha \mid \alpha \in A\}$ onto Y_α.

This mapping is well defined. Indeed, if $\alpha, \beta \in A$ and $\alpha \leqslant \beta$, then

$$[\mathrm{Map}_X(\pi_\alpha^\beta)](p_\beta \circ f) = p_\alpha \circ f$$

since, for each $x \in X$, and thus $f(x) \in Y_\infty$, we get

$$([\mathrm{Map}_X(\pi_\alpha^\beta)](p_\beta \circ f))(x) = (\pi_\alpha^\beta \circ p_\beta \circ f)(x) = \pi_\alpha^\beta(p_\beta[f(x)])$$
$$= p_\alpha[f(x)] = (p_\alpha \circ f)(x).$$

Consider also the mapping

$$G: \varprojlim\{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi)\} \to \mathrm{Map}_X(\varprojlim\{Y, \pi\})$$

where, for an arbitrary point $f = \{f_\alpha\} \in \varprojlim\{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi)\}$ (hence $f_\alpha \in \mathrm{Map}_X(Y_\alpha)$), the mapping $G(f): X \to Y_\infty$ is defined by

$$[G(f)](x) = \{f_\alpha(x)\}$$

for every $x \in X$. To justify this definition, first of all, we observe that $[G(f)](x)$ is indeed a point of Y_∞ for each $x \in X$. If $\alpha \leqslant \beta$ we have

$$\pi_\alpha^\beta[f_\beta(x)] = (\pi_\alpha^\beta \circ f_\beta)(x) = ([\mathrm{Map}_X(\pi_\alpha^\beta)](f_\beta))(x) = f_\alpha(x).$$

Moreover, $G(f)$ is a continuous mapping i.e. $G(f) \in Y_\infty^X$ since, for each $x \in X$, we get

$$[p_\alpha \circ G(f)](x) = p_\alpha(\{f_\alpha(x)\}) = f_\alpha(x)$$

and, hence, $p_\alpha \circ G(f) = f_\alpha$.

We now show that G and H are bijections inverse to each other.

(a) $(G \circ H)(f) = f$ for each $f \in Y_\infty^X$, because if $x \in X$, we have

$$[(G \circ H)(f)](x) = (G[H(f)])(x) = \{(p_\alpha \circ f)(x)\} = \{p_\alpha[f(x)]\} = f(x).$$

(b) $(H \circ G)(f) = f$ for each $f = \{f_\alpha\} \in \varprojlim\{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi)\}$, since

$$(H \circ G)(f) = H[G(f)] = \{p_\alpha \circ G(f)\} = \{f_\alpha\} = f.$$

It remains to prove that the mappings G and H are continuous.

(c) *The continuity of H*. Let (C, U_α) be an open sub-basic set of the space Y_α^X, where C is a compact set in X and U_α is an open set in Y_α, and let q_α denote the restriction to the subset $\varprojlim \{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi)\}$ of the natural projection of $\Pi\{Y_\alpha^X \mid \alpha \in A\}$ onto Y_α^X. It is sufficient to show the continuity of the composition $q_\alpha \circ H$.

$$f \in (q_\alpha \circ H)^{-1}[(C, U_\alpha)] \Leftrightarrow q_\alpha[H(f)] \in (C, U_\alpha)$$

$$\Leftrightarrow q_\alpha(\{p_\alpha \circ f\}) \in (C, U_\alpha) \Leftrightarrow p_\alpha \circ f \in (C, U_\alpha) \Leftrightarrow p_\alpha[f(C)] \subset U_\alpha$$

$$\Leftrightarrow f(C) \subset p_\alpha^{-1}(U_\alpha) \Leftrightarrow f \in (C, p_\alpha^{-1}(U_\alpha))$$

and consequently

$$(q_\alpha \circ H)^{-1}[(C, U_\alpha)] = (C, p_\alpha^{-1}(U_\alpha)).$$

This completes the proof since $p_\alpha^{-1}(U_\alpha)$ is an open basic set in Y_∞.

(d) *The continuity of G*. Since X is a T_2-space, $S = (C, p_\alpha^{-1}(U_\alpha))$ is an open sub-basic set in Y_∞^X ([3], p. 153 Prop. 1.3). If $f = \{f_\alpha\} \in \varprojlim \{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi)\}$ we get

$$f \in G^{-1}(S) \Leftrightarrow [G(f)](C) \subset p_\alpha^{-1}(U_\alpha) \Leftrightarrow [G(f)](x) \in p_\alpha^{-1}(U_\alpha), \ \forall x \in C$$

$$\Leftrightarrow \{f_\alpha(x)\} \in p_\alpha^{-1}(U_\alpha), \ \forall x \in C \Leftrightarrow f_\alpha(x) \in U_\alpha, \ \forall x \in C \Leftrightarrow f_\alpha(C) \subset U_\alpha$$

$$\Leftrightarrow f_\alpha \in (C, U_\alpha) \Leftrightarrow f \in q_\alpha^{-1}(C, U_\alpha).$$

Hence, $G^{-1}(S) = q_\alpha^{-1}(C, U_\alpha)$ is an open basic set in $\varprojlim \{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi)\}$ and thus, G is a continuous mapping.

Let $\{Z, \rho, B\}$ be another inverse system in $\mathscr{H}$ over the directed set B. If $\Phi: \{Y, \pi\} \to \{Z, \rho\}$ is mapping of these two systems, then it obviously defines a mapping $\mathrm{Map}_X(\Phi): \{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi)\} \to \{\mathrm{Map}_X(Z), \mathrm{Map}_X(\rho)\}$. So we have two induced mappings $\mathrm{Map}_X(\varprojlim \Phi)$ and $\varprojlim \mathrm{Map}_X(\Phi)$ and we will prove that they are the same up to the composition with homeomorphisms. More precisely we have the following proposition:

3.2. *Given a mapping of inverse systems*

$$\Phi = \{\varphi, \varphi_\beta\}: \{Y, \pi, A\} \to \{Z, \rho, B\}.$$

Then, there exist two homeomorphisms H and K such that diagram

$$\begin{array}{ccc} \mathrm{Map}_X(\varprojlim \{Y, \pi\}) & \xrightarrow{H} & \varprojlim \{\mathrm{Map}_X(Y), \mathrm{Map}_X(\pi)\} \\ {\scriptstyle \mathrm{Map}_X(\varprojlim \Phi)} \downarrow & & \downarrow {\scriptstyle \varprojlim \mathrm{Map}_X(\Phi)} \\ \mathrm{Map}_X(\varprojlim \{Z, \rho\}) & \xrightarrow[K]{} & \varprojlim \{\mathrm{Map}_X(Z), \mathrm{Map}_X(\rho)\} \end{array}$$

commutes.

Proof. Let H and K be homeomorphisms form 3.1 related to $\{Y, \pi\}$ and $\{Z, \rho\}$ respectively, and let P_β and Q_β denote the restrictions to the subsets $\varprojlim \{Z, \rho\}$ and $\varprojlim \{\mathrm{Map}_X(Z), \mathrm{Map}_X(\rho)\}$ of the natural projections of $\Pi\{Z_\beta \mid \beta \in B\}$

onto Z_β and of $\Pi\{Z_\beta^X \mid \beta \in B\}$ onto Z_β^X respectively. Now, for $f \in Y_\infty^X$, $\beta \in B$, we have

$$[Q_\beta \circ K \circ \mathrm{Map}_X (\underleftarrow{\lim}\, \Phi)] (f) = (Q_\beta \circ K) (\underleftarrow{\lim}\, \Phi \circ f)$$

$$= Q_\beta (\{P_\beta \circ \underleftarrow{\lim}\, \Phi \circ f\}) = P_\beta \circ \underleftarrow{\lim}\, \Phi \circ f.$$

On the other hand, according to the definition of a limit mapping,

$$[Q_\beta \circ \underleftarrow{\lim}\, \mathrm{Map}_X (\Phi) \circ H] (f) = [Q_\beta \circ \underleftarrow{\lim}\, \mathrm{Map}_X (\Phi)] (\{p_\alpha \circ f\})$$

$$= [\mathrm{Map}_X (\varphi_\beta)] (p_{\varphi(\beta)} \circ f) = \varphi_\beta \circ p_{\varphi(\beta)} \circ f.$$

But, for $x \in X$

$$[P_\beta \circ \underleftarrow{\lim}\, \Phi \circ f] (x) = P_\beta (\underleftarrow{\lim}\, \Phi [f(x)]) = \varphi_\beta (p_{\varphi(\beta)} [f(x)])$$

$$= (\varphi_\beta \circ p_{\varphi(\beta)} \circ f) (x).$$

Thus the commutativity of the diagram follows.

Hence, by 3.1 and 3.2, two functors Map_X and inverse limit commute.

4. Functionally complete spaces

Let us introduce the following definition: A topological space Y is *functionally complete relative to* X if $Y \approx \mathrm{Map}_X (Y)$. A trivial example of such a space is the singleton space.

For a $Y \in O$, put $Y = Y^{(0)}$, and for $n = 1, 2, \ldots$ let

$$Y^{(n)} = \mathrm{Map}_X (Y^{(n-1)}).$$

Let $a \in X$ be a fixed point. Consider the mapping

$$p_a : Y^{(1)} \to Y^{(0)}$$

defined by

$$p_a (f) = f(a)$$

for every $f \in Y^{(1)}$, which is continuous and onto (see [3], p. 165 Prop. 4.5).

Denote $p_a : Y^{(1)} \to Y^{(0)}$ by $p_a^{(0)}$, and for $n = 1, 2, \ldots$ let

$$p_a^{(n)} = \mathrm{Map}_X (p_a^{(n-1)}) : Y^{(n+1)} \to Y^{(n)},$$

then all mappings $p_a^{(n)}$, $n = 0, 1, \ldots$, are continuous and onto. So we get an inverse system of maps and spaces

$$Y^{(0)} \xleftarrow{p_a^{(0)}} Y^{(1)} \xleftarrow{p_a^{(1)}} Y^{(2)} \leftarrow \cdots \leftarrow Y^{(n)} \xleftarrow{p_a^{(n)}} Y^{(n+1)} \leftarrow \cdots$$

Let

$$Y^{(\omega)} = \underleftarrow{\lim}\, \{Y^{(n)}, p_a^{(n)}\},$$

then $Y^{(\omega)}$ is an object in $\mathscr{H}$, and applying 3.1 to $\underleftarrow{\lim}\, \{Y^{(n)}, p_a^{(n)}\}$ we immediately get

4.1. *The space $Y^{(\omega)}$ is functionally complete relative to X.*

Proof.

$$\mathrm{Map}_X(Y^{(\omega)}) \approx \varprojlim \{\mathrm{Map}_X(Y^{(n)}),\ \mathrm{Map}_X(p_a^{(n)})\}$$

$$= \varprojlim \{Y^{(n+1)},\ p_a^{(n+1)}\} \approx Y^{(\omega)},$$

where the last relation follows from Th. 3.15 in [2], p. 219.

Consider now a mapping $f\colon Y \to Z$ belonging to $\mathcal{H}$. Put $f = f^{(0)}$, and for $n = 1, 2, \ldots$ let

$$f^{(n)} = \mathrm{Map}_X(f^{(n-1)})\colon Y^{(n)} \to Z^{(n)}.$$

So we get the following diagram

$$\begin{array}{ccccccccc} Y^{(0)} & \xleftarrow{p_a^{(0)}} & Y^{(1)} & \leftarrow \cdots \leftarrow & Y^{(n)} & \xleftarrow{p_a^{(n)}} & Y^{(n+1)} & \leftarrow \cdots \\ {\scriptstyle f^{(0)}}\downarrow & & \downarrow{\scriptstyle f^{(1)}} & & {\scriptstyle f^{(n)}}\downarrow & & \downarrow{\scriptstyle f^{(n+1)}} & \\ Z^{(0)} & \xleftarrow[p_a^{(0)}]{} & Z^{(1)} & \leftarrow \cdots \leftarrow & Z^{(n)} & \xleftarrow[p_a^{(n)}]{} & Z^{(n+1)} & \leftarrow \cdots \end{array}$$

which we will call the *induced* diagram.

4.2. *All rectangles of the induced diagram commute.*

Proof. It is known that the first rectangle commutes. (see [3], p. 165 Ex. 4C). Since Map_X is a covariant functor, the commutativity of all other rectangles immediately follows.

Let $\{f^{(n)}\}\colon \{Y^{(n)}, p_a^{(n)}\} \to \{Z^{(n)}, p_a^{(n)}\}$ be the mapping of these inverse systems and let $f^{(\omega)} = \varprojlim \{f^{(n)}\}$, then $f^{(\omega)}$ is continuous ([2], p. 218 Th. 313). We will prove that $f^{(\omega)}$ is also *functionally complete relative to* X in the sense that $f^{(\omega)}$ is equal to $\mathrm{Map}_X(f^{(\omega)})$ up to the composition with homeomorphisms.

4.3. *For any $f\colon Y \to Z$ in $\mathcal{H}$, the mapping $f^{(\omega)}\colon Y^{(\omega)} \to Z^{(\omega)}$ is functionally complete relative to X.*

Proof. Applying 3.2 to the mapping of the inverse systems $\{f^{(n)}\}$: $\{Y^{(n)}, p_a^{(n)}\} \to \{Z^{(n)}, p_a^{(n)}\}$ we have

$$\begin{array}{ccccc} \mathrm{Map}_X(Y^{(\omega)}) & \xrightarrow{H} & \varprojlim \{Y^{(n+1)},\ p_a^{(n+1)}\} & \xrightarrow{h} & Y^{(\omega)} \\ {\scriptstyle \mathrm{Map}_X(f^{(\omega)})}\downarrow & & \downarrow{\scriptstyle \varprojlim \{f^{(n+1)}\}} & & \downarrow{\scriptstyle f^{(\omega)}} \\ \mathrm{Map}_X(Z^{(\omega)}) & \xrightarrow[K]{} & \varprojlim \{Z^{(n+1)},\ p_a^{(n+1)}\} & \xrightarrow[k]{} & Z^{(\omega)} \end{array}$$

h and k being the obvious homeomorphisms. Since the rectangles are commutative we obtain

$$\mathrm{Map}_X(f^{(\omega)}) = (k \circ K)^{-1} \circ f^{(\omega)} \circ (h \circ H).$$

5. Imbedding theorem

Now we propose to determine a subcategory of $\mathcal{H}$ all of whose objects are functionally complete relative to X and to prove that for each $Y \in O$ there is a $Z \in O$ which contains Y and is functionally complete relative to X.

Consider the mapping

$$j_0 : Y^{(0)} \to Y^{(1)}$$

where, for each point $y \in Y^{(0)}$, $j_0(y) : X \to Y$ denotes the constant mapping in $Y^{(1)}$ which sends X into the single point y. The mapping j_0 is an imbedding ([3], p. 163 Prop. 4.1). For each $n = 1, 2, \ldots$ let

$$j_n = \mathrm{Map}_X(j_{n-1}) : Y^{(n)} \to Y^{(n+1)}.$$

The mappings j_n, $n = 0, 1, \ldots$, are continuous and so we get the following sequence of spaces and maps

$$Y^{(0)} \xrightarrow{j_0} Y^{(1)} \xrightarrow{j_1} Y^{(2)} \to \cdots \to Y^{(n)} \xrightarrow{j_n} Y^{(n+1)} \to \cdots.$$

5.1. *For each* $n = 0, 1, \ldots$, $j_n(Y^{(n)})$ *is a retract of* $Y^{(n+1)}$ *with the retraction* $j_n \circ p_a^{(n)} : Y^{(n+1)} \to j_n(Y^{(n)})$.

Proof. This assertion is valid for $n = 0$ ([3], p. 165 Prop. 4.6).

Let $n > 0$. If $f \in j_n(Y^{(n)})$, then there is $g \in Y^{(n)}$ such that

$$f = j_n(g) = [\mathrm{Map}_X(j_{n-1})](g) = j_{n-1} \circ g.$$

Obviously, $p_a^{(n)} \circ j_n$ is the identity mapping on $Y^{(n)}$, for each $n = 0, 1, \ldots$, and so we have

$$(j_n \circ p_a^{(n)})(f) = [\mathrm{Map}_X(j_{n-1} \circ p_a^{(n-1)})](f) = j_{n-1} \circ p_a^{(n-1)} \circ f =$$

$$= j_{n-1} \circ p_a^{(n-1)} \circ j_{n-1} \circ g = j_{n-1} \circ 1_{Y^{(n-1)}} \circ g = j_{n-1} \circ g = f.$$

Hence, $j_n \circ p_a^{(n)}$ is a retraction of $Y^{(n+1)}$ onto $j_n(Y^{(n)})$.

For each $n = 0, 1, \ldots$ let

$$j_{0,n} = j_n \circ j_{n-1} \circ \cdots \circ j_0 : Y^{(0)} \to Y^{(n+1)}.$$

Consider the mapping

$$j_{0,\omega} : Y \to \varprojlim \{Y^{(n+1)}, p_a^{(n+1)}\} = Y^{\infty}$$

defined by

$$j_{0,\omega}(y) = \{j_{0,n}(y)\}_{n=0,1,\ldots}$$

for every $y \in Y$. This mapping is well defined. Indeed, for $n = 1, \ldots$, we get

$$p_a^{(n)}[j_{0,n}(y)] = p_a^{(n)}[(j_n \circ j_{0,n-1})(y)] = (p_a^{(n)} \circ j_n)(j_{0,n-1}(y)) =$$

$$= [\mathrm{Map}_X(p_a^{(n-1)} \circ j_{n-1})](j_{0,n-1}(y)) = p_a^{(n-1)} \circ j_{n-1} \circ j_{0,n-1}(y) =$$

$$= 1_{Y^{(n-1)}} \circ j_{0,n-1}(y) = j_{0,n-1}(y).$$

5.2. *If* f *is an arbitrary point in* $j_{0,\omega}(Y)$ *and* $\pi_n : \Pi\{Y^{(n)} \mid n = 1, 2, \ldots\} \to Y^{(n)}$ *denotes the natural projection, then*

$$[\pi_{n+1}(f)](x) \in j_{n-1}(Y^{(n-1)})$$

for each $x \in X$ *and* $n = 1, 2, \ldots$.

Proof. Since $f \in j_{0,\omega}(Y)$ there exists $y \in Y$ such that $j_{0,\omega}(y)=f$ and thus $j_{0,n-1}(y)=\pi_n(f)$ for each $n=1, 2, \dots$. Let $n=1$ and $x \in X$ be arbitrarily given. We must show that $[\pi_2(f)](x) \in j_0(Y)$, that is to say, $[\pi_2(f)](x): X \to Y$ is a constant mapping in $Y^{(1)}$. But, this is true because

$$[\pi_2(f)](x)=[j_{0,1}(y)](x)=[(j_1 \circ j_0)(y)](x)=$$
$$=[j_0 \circ j_0(y)](x)=j_0([j_0(y)](x))=j_0(y).$$

Suppose that $[\pi_{n+1}(f)](x) \in j_{n-1}(Y^{(n-1)})$. Since

$$\pi_{n+2}(f)=j_{0,n+1}(y)=(j_{n+1} \circ j_{0,n})(y)=[\mathrm{Map}_X(j_n)](j_{0,n}(y))=$$
$$=j_n \circ \pi_{n+1}(f)$$

we have

$$[\pi_{n+2}(f)](x)=j_n([\pi_{n+1}(f)](x)) \in j_n(Y^{(n)})$$

which completes the inductive proof.

5.3. *Each Hausdorff space Y can be imbedded in an functionally complete space.*

Proof. We will prove that Y can be imbedded into the functionally complete space $Y^{(\omega)}$. Since $h: Y^{\infty} \to Y^{(\omega)}$ is a homeomorphism (see 4.3) it is sufficient to show that $j_{0,\omega}$ is an imbedding of Y into Y^{∞}.

It is easy to verify that $j_{0,\omega}$ is one-to-one. Moreover, since all mappings $j_{0,n}$ are continuous, so is $j_{0,\omega}$ ([3], p. 40 Cor. 5.8). Therefore, it remains to prove that $j_{0,\omega}$ is an open mapping. For this purpose, let V denote an open set in Y. Then (a, V) is an open sub-basic set in $Y^{(1)}$. Let π_n^* denote the restriction of the natural projection π_n to the subset Y^{∞}. Then $(\pi_1^*)^{-1}((a, V))$ is an open basic set in Y^{∞} and, hence

$$P=j_{0,\omega}(Y) \cap (\pi_1^*)^{-1}((a, V))=j_{0,\omega}(Y) \cap Y^{\infty} \cap \pi_1^{-1}((a, V))$$

is an open set in $j_{0,\omega}(Y)$. Therefore, to prove that $j_{0,\omega}$ is an open mapping it suffices to show that $j_{0,\omega}(V)=P$.

First, let $f \in j_{0,\omega}(V)$. Then, there exists $y \in V$, such that $j_{0,\omega}(y)=f$ and, consequently, $j_0(y)=\pi_1(f)$. By this equality

$$[\pi_1(f)](a)=[j_0(y)](a)=y \in V$$

and, hence, $\pi_1(f) \in (a, V)$. This implies that $f \in \pi_1^{-1}((a, V))$ and, hence, clearly, $f \in P$.

Conversely, if $f \in P$ consider the point $y_0=[\pi_1(f)](a) \in V$. It suffices to show that $j_{0,\omega}(y_0)=f$ or equivalently,

$$j_{0,n-1}(y_0)=\pi_n(f) \qquad (n=1, 2, \dots)$$

Let us establish these equalities by induction.

Let $n=1$. Since $f \in j_{0,\omega}(Y)$ there exists $y \in Y$ such that $j_{0,\omega}(y)=f$. This implies $j_0(y)=\pi_1(f)$ and, hence, $\pi_1(f)$ is a constant mapping.

Therefore

$$[\pi_1(f)](x)=[\pi_1(f)](a)=y_0=[j_0(y_0)](x)$$

and, hence, $j_0(y_0)=\pi_1(f)$. Now suppose that $j_{0,n-1}(y_0)=\pi_n(f)$. According to this inductive hypothesis, we have

$$j_{0,n}(y_0)=(j_n\circ j_{0,n-1})(y_0)=j_n[\pi_n(f)]=j_n(p_a^{(n)}[\pi_{n+1}(f)])=$$
$$=[\mathrm{Map}_X(j_{n-1}\circ p_a^{(n-1)})](\pi_{n+1}(f))=j_{n-1}\circ p_a^{(n-1)}\circ\pi_{n+1}(f).$$

Hence, for each $x\in X$, by 5.1 and 5.2, we obtain

$$[j_{0,n}(y_0)](x)=(j_{n-1}\circ p_a^{(n-1)})([\pi_{n+1}(f)](x))=[\pi_{n+1}(f)](x)$$

and, thus, $j_{0,n}(y_0)=\pi_{n+1}(f)$. This completes the inductive proof.

Hence, $u=h\circ j_{0,\omega}:Y\to Y^{(\omega)}$ is an imbedding of Y into $Y^{(\omega)}$.

By this theorem $Y^{(\omega)}$ is non-empty if Y is non-empty and, hence, there exist non-trivial examples of the functionally complete spaces.

5.4. *$u(Y)$ is a retract of $Y^{(\omega)}$.*

Proof. To prove this, we shall show that $j_{0,\omega}(Y)$ is a retract of Y^∞ with the retraction $r=j_{0,\omega}\circ p_a^{(0)}\circ\pi_1^*:Y^\infty\to j_{0,\omega}(Y)$.

Let $f\in j_{0,\omega}(Y)$. Then there exists $y\in Y$ such that $j_{0,\omega}(y)=f$ and, thus, $j_0(y)=\pi_1^*(f)$. Consequently, we have

$$r(f)=j_{0,\omega}(p_a^{(0)}[\pi_1^*(f)])=j_{0,\omega}(p_a^{(0)}[j_0(y)])=j_{0,\omega}(y)=f.$$

Denote by $\mathrm{Map}_X^{(\omega)}$ the functor corresponding to each $Y\in O$ the space $Y^{(\omega)}$, and to each mapping $f:Y\to Z$ in $\mathcal{H}$ the mapping $f^{(\omega)}:Y^{(\omega)}\to Z^{(\omega)}$. Then we have

5.5. $\mathrm{Map}_X^{(\omega)}$ *is a covariant functor from $\mathcal{H}$ to itself.*

Proof. If $i:Y\to Y$ is the identity mapping, then all $i^{(n)}:Y^{(n)}\to Y^{(n)}$ are identity mappings and so is $i^{(\omega)}$. If $f:Y\to Z$ and $g:Z\to W$ are in $\mathcal{H}$, then $(g\circ f)^{(n)}=g^{(n)}\circ f^{(n)}$ which easily implies $(g\circ f)^{(\omega)}=g^{(\omega)}\circ f^{(\omega)}$.

Hence, $\mathrm{Map}_X^{(\omega)}(\mathcal{H})$ is a subcategory of the category $\mathcal{H}$ which has the desired property.

REFERENCES

[1] Bucur I. and Deleanu A., *Introduction to the Theory of Categories and Functors*, London, (1968).

[2] Eilenberg S. and Steenrod N., *Foundations of Algebraic Topology*, Princeton, (1952).

[3] Hu S. T., *Elements of General Topology*, Holden-Day, San Francisco (1964).

[4] Marjanović M. M., *Exponentially complete spaces I*, Glasnik Matematički 6 (26) (1971), 143—147.

[5] Marjanović M. M., *Exponentially complete spaces II*, Publ. Inst. Math Math. 13 (27) (1972), 77—79.