PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 15 (29), 1973), pp. 43—46

A THEOREM ON SPACES 2^X WITH THE COMPACT-OPEN TOPOLOGY

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(Communicated January 19, 1973)

1. Introduction

Let 2^X denote the set of all closed subsets of a topological space X (including the empty set $\varnothing$). If $\mathcal{K}$ is the family of all compact subset of X, then (see [1], p. 141) a basis for the *compact-open* topology on 2^X is the collection $\{\langle X-C\rangle \mid C\in\mathcal{K}\}$ where, for any $C\in\mathcal{K}$,

$$\langle X-C\rangle=\{F\in 2^X \mid F\subset X-C\}.$$

Lemma 1. *Let X be a topological space, Y a subspace of X, and let 2^X and 2^Y have compact-open topologies. Then, the function $f_Y: 2^X\to 2^Y$, defined by $f_Y(F)=F\cap Y$ for every $F\in 2^X$, is continuous.*

Proof. Let $\langle Y-C\rangle=\{A\in 2^Y \mid A\subset Y-C\}$ be an open basic subset in 2^Y, where C is a compact subset of the space Y (and, thus, $C\in\mathcal{K}$). Now, the conclusion of the lemma follows from the equalities:

$$f_Y^{-1}(\langle Y-C\rangle)=\{F\in 2^X \mid f_Y(F)\in\langle Y-C\rangle\}=\{F\in 2^X \mid F\cap Y\subset Y-C\}=$$
$$=\{F\in 2^X \mid F\cap C=\varnothing\}=\{F\in 2^X \mid F\subset X-C\}=\langle X-C\rangle.$$

In the set $\mathcal{K}$ let us introduce a binary relation $\leqslant$ as follows. If C_1 and C_2 by any two elements in $\mathcal{K}$, we define $C_1\leqslant C_2 \Leftrightarrow C_1\subset C_2$. Clearly, $\mathcal{K}$ is a directed set (if $C_1, C_2\in\mathcal{K}$ then $C_1\cup C_2\in\mathcal{K}$ and $C_1, C_2\subset C_1\cup C_2$).

Now, for each pair $C_1, C_2\in\mathcal{K}$ such that $C_1\leqslant C_2$, the function

$$\pi_{C_1}^{C_2}: 2^{C_2}\to 2^{C_1}$$

defined by $\pi_{C_1}^{C_2}(F)=F\cap C_1$ for every $F\in 2^{C_2}$, is continuous by Lemma 1.

Moreover, it is easy to see that, π_C^C is the identity on 2^C for every $C\in\mathcal{K}$, and that, for each three elements C_1, C_2, C_3 in the set $\mathcal{K}$ such that $C_1\leqslant C_2\leqslant C_3$, we have

$$\pi_{C_1}^{C_2}\circ\pi_{C_2}^{C_3}=\pi_{C_1}^{C_3}.$$

Consequently, $\{2^C, \pi_{C_1}^{C_2}, \mathcal{K}\}$ is an inverse system of maps and spaces

Recall that a topological space X is a *k-space* iff the following condition holds:

$$A\subset X \text{ is closed iff } A\cap C \text{ is closed in } C \text{ for each } C\in\mathcal{K}.$$

The purpose of this note is to prove the following theorem:

Theorem. *If X is a k-space, then the space 2^X with the compact-open topology is homeomorphic to the inverse limit of the inverse sistem $\{2^C, \pi_{C_1}^{C_2}, \mathcal{K}\}$ of the spaces 2^C with the compact-open topology i.e.*

$$2^X \approx \varprojlim \{2^C, \pi_{C_1}^{C_2}, \mathcal{K}\}.$$

2. Proof of the Theorem.

We will state first some results which will be used in the proof.

Let S denote the (Sierpiński) space having two points, 0 and 1, for which the open sets are $\emptyset$, S and $\{0\}$. Consider the set S^X of all continuous functions from X into S with the compact-open topology as defined for spaces of continuous functions (see [2], p. 121). In [1] (Lemma 3.2) the following fact is proved:

(a) *If S^X and 2^X have the respective compact-open topologies, then the function $\Gamma : S^X \to 2^X$, defined by $\Gamma(f) = f^{-1}(1)$ for every $f \in S^X$, is a homeomorphism i.e.*

$$S^X \approx 2^X.$$

On the other hand, we have the following theorem (see [2], p. 123, Th. 5):

(b) *If X is a k-space, then for every topological space Y the space Y^X with the compact-open topology is homeomorphic to the inverse limit of the inverse sistem $\{Y^C, \rho_{C_1}^{C_2}, \mathcal{K}\}$ of the spaces Y^C with the compact-open topology i.e.*

$$Y^X \approx \varprojlim \{Y^C, \rho_{C_1}^{C_2}, \mathcal{K}\}$$

where, for each pair $C_1, C_2 \in \mathcal{K}$ such that $C_1 \leqslant C_2$, the function

$$\rho_{C_1}^{C_2} : Y^{C_2} \to Y^{C_1}$$

defined by $\rho_{C_1}^{C_2}(f) = f \mid C_1$ for every $f \in Y^{C_2}$, is continuous.

Hence, by (a) and (b), to prove the assertion of the theorem, it suffices to show that

$$\varprojlim \{S^C, \rho_{C_1}^{C_2}, \mathcal{K}\} \approx \varprojlim \{2^C, \pi_{C_1}^{C_2}, \mathcal{K}\}.$$

For this purpose, consider the mapping

$$\Phi = \{I, \Gamma_C\} : \{S^C, \rho_{C_1}^{C_2}, \mathcal{K}\} \to \{2^C, \pi_{C_1}^{C_2}, \mathcal{K}\}$$

of these inverse sistems where, by (a), for each $C \in \mathcal{K}$, $\Gamma_C : S^C \to 2^C$ is a homeomorphism ($\Gamma_C(f) = f^{-1}(1)$ for $f \in S^C$) and I is the identity on $\mathcal{K}$. Therefore, using a well known theorem (see [2], p. 90, Th. 4], it remains to prove that the following diagram

$$\begin{array}{ccc} S^{C_1} & \xleftarrow{\ \rho_{C_1}^{C_2}\ } & S^{C_2} \\ \Gamma_{C_1} \downarrow & & \downarrow \Gamma_{C_2} \\ 2^{C_1} & \xleftarrow[\ \pi_{C_1}^{C_2}\ ]{} & 2^{C_2} \end{array}$$

commutes, that is to say,

$$\Gamma_{C_1} \circ \rho_{C_1}^{C_2} = \pi_{C_1}^{C_2} \circ \Gamma_{C_2}.$$

Indeed, for $f \in S^{C_2}$ we have

$$(\Gamma_{C_1} \circ \rho_{C_1}^{C_2})(f) = \Gamma_{C_1}(f \mid C_1) = (f \mid C_1)^{-1}(1) = f^{-1}(1) \cap C_1 =$$
$$= \pi_{C_1}^{C_2}(f^{-1}(1)) = (\pi_{C_1}^{C_2} \circ \Gamma_{C_2})(f)$$

which completes the proof.

3. Remark.

Let us observe that our Theorem can be proved also as follows.

Lemma 2. *If $F_0 \in 2^X$ and $\Phi \subset 2^X$ then*

$$F_0 \in \overline{\Phi} \Leftrightarrow f_C(F_0) \in \overline{f_C(\Phi)} \text{ for each } C \in \mathcal{K}.$$

Proof. By Lemma 1, the implication $\Rightarrow$ follows immediately from the continuity of functions $f_C: 2^X \to 2^C$.

Conversely, assume that $F_0 \notin \overline{\Phi} \Leftrightarrow F_0 \in \operatorname{int}(2^X - \Phi)$. Thus, there exists a compact set C_0 in X such that

$$F_0 \in \langle X - C_0 \rangle \text{ and } \langle X - C_0 \rangle \subset 2^X - \Phi.$$

Since the set $\{A \in 2^{C_0} \mid A \subset C_0 - C_0\} = \{\emptyset\}$ is an open basic neighborhood of $f_{C_0}(F_0) = \emptyset$ in 2^{C_0} which, because $\langle X - C_0 \rangle \cap \Phi = \emptyset$, does not meet $f_{C_0}(\Phi)$, it follows that $f_{C_0}(F_0) \notin \overline{f_{C_0}(\Phi)}$ and the conclusion of the lemma is valid.

Now, we shall show that, the function

$$f: 2^X \to \varprojlim \{2^C, \pi_{C_1}^{C_2}, \mathcal{K}\},$$

defined by $f(F) = \{f_C(F)\}$ for every $F \in 2^X$, is a homeomorphism.

First of all f is well defined, since, for $C_1 \subset C_2$, we have

$$\pi_{C_1}^{C_2}(f_{C_2}(F)) = \pi_{C_1}^{C_2}(F \cap C_2) = (F \cap C_2) \cap C_1 = F \cap C_1 = f_{C_1}(F).$$

(α) It is easy to see that f is a *one-to-one* function, since if

$$f(F_1) = f(F_2) \Rightarrow f_C(F_1) = f_C(F_2),\ \forall C \in \mathcal{K} \Rightarrow F_1 \cap C = F_2 \cap C,\ \forall C \in \mathcal{K}$$
$$\Rightarrow F_1 \cap \{x\} = F_2 \cap \{x\},\ \forall x \in X \Rightarrow F_1 = F_2.$$

(β) Let us prove that f is a function *onto* $\varprojlim \{2^C, \pi_{C_1}^{C_2}, \mathcal{K}\}$. Let $\{A_C\} \in \varprojlim \{2^C, \pi_{C_1}^{C_2}\ \mathcal{K}\}$ be arbitrarily given. We must find a point $F \in 2^X$, such that

$$f(F) = \{A_C\} \Leftrightarrow f_C(F) = A_C,\ \forall C \in \mathcal{K} \Leftrightarrow F \cap C = A_C,\ \forall C \in \mathcal{K}.$$

Put

$$F = \{x \in X \mid A_{\{x\}} = \{x\}\}$$

and let us prove that $F \cap C = A_C$ for any $C \in \mathcal{K}$. Indeed,

$$x \in A_C \Rightarrow x \in C \Rightarrow \{x\} \subset C \Rightarrow \pi_{\{x\}}^C(A_C) = A_{\{x\}} \Rightarrow A_C \cap \{x\} = A_{\{x\}} \Rightarrow$$
$$\{x\} = A_{\{x\}} \Rightarrow x \in F \Rightarrow x \in F \cap C$$

and, hence, $A_C \subset F \cap C$. Conversely, if

$$x \in F \cap C \Rightarrow A_{\{x\}} = \{x\} \wedge \{x\} \subset C \Rightarrow A_{\{x\}} = \{x\} \wedge A_C \cap \{x\} = A_{\{x\}} \Rightarrow$$
$$A_C \cap \{x\} = \{x\} \Rightarrow \{x\} \subset A_C \Rightarrow x \in A_C$$

and, hence, $F \cap C \subset A_C$.

Since X is a k-space and $F \cap C = A_C \in 2^C$ for each $C \in \mathcal{K}$, F is a closed set in X.

Therefore, the proof is completed by (α), (β), Lemma 2 and [3] (p. 167, Th. 3).

REFERENCES

[1] L. J. Billera, *Topologies for 2^X; set-valued functions and their graphs*, Trans. Amer. Math. Soc. 155 (1971), 137—147.

[2] Engelking R, *Outline of General Topology*, Amsterdam, (1968).

[3] Kuratowski K, *Topology* (Russian), Moscow. vol. II, (1969).