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TAUBERIAN CONDITIONS FOR ABSOLUTE CONVERGENCE

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1. *Definitions and Notations.* Let $\sum_{n=1}^{\infty} a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Throughout the paper we suppose

$$\lambda_n = \mu_1 + \mu_2 + \cdots + \mu_n, \quad \mu_n = \lambda_n - \lambda_{n-1},$$

such that

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty, \text{ as } n \to \infty.$$

Let the sequence-to-sequence transformation is defined by

(1.1)
$$t_n = \frac{1}{\lambda_{n+1}} \sum_{\nu=1}^{n} \mu_{\nu+1} s_\nu .$$

If $\{t_n\} \in BV$, we say that $\{s_n\}$ $\left(\text{or } \sum_{n=1}^{\infty} a_n\right)$ is absolutely summable $(R', \lambda_n, 1)$ or symbolically we write

$$\{s_n\} \in |R', \lambda_n, 1|.$$

In fact $|R', \lambda_n, 1|$ is equivalent to $|R, \lambda_n, 1|$. See Mohanty [7], footnote to the page 298.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without any loss of generality, we assume the Fourier series of $f(t)$ to be given by

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

Then the conjugate series of the Fourier series of $f(t)$ will be given by

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t).$$

Throughout the paper we use the following notations:

(1.2)
$$\Phi(t) = \frac{1}{2}(f(x+t) + f(x-t)).$$

(1.3)
$$\psi(t) = \frac{1}{2}(f(x+t) - f(x-t)).$$

(1.4) $$F_1(t)=t^{-1}\int_0^t F(u)\,du, \text{ for any function } F(u).$$

(1.5) $$P(t)=\Phi(t)-\Phi_1(t).$$

(1.6) $$S(t)=\psi(t)-\psi_1(t).$$

(1.7) $$\Delta g_n=g_n-g_{n+1}.$$

(1.8) $$L_n=\exp(n/(\log(n+1))^c) \qquad (c>1).$$

2. *Introduction.* Recently, the present author [1], proved the following criterion, for the absolute convergence of Fourier series at a point $t=x$, by using a theorem on absolute Riesz summability of Fourier series obtained by author himself.

Theorem P. If (i) $\Phi(t)\in BV(0,\pi)$ (ii) $\beta(t)\,g(k/t)\in BV(0,\pi)$ and $\{n^{1-a}A_n(x)\}\in BV$, for $0<a<1$, then $\sum_{n=1}^{\infty} A_n(x)\in|C,0|$, where

$$\beta(t)=t^{-1}\int_0^t d\Phi(u)$$

and $g(k/t)$ stands for any one of the following functions:

$$\left(\log\frac{k}{t}\right)^{1+c},\ \log\frac{k}{t}\left(\log_2\frac{k}{t}\right)^{1+c},\ \ldots,\ \log\frac{k}{t}\ \cdots\ \log_{p-1}\frac{k}{t}\left(\log_p\frac{k}{t}\right)^{1+c},$$

where $\log_p=\log\log_{p-1}$, $\log_1=\log$, $c>0$ and k is some suitable positive constant taken for the convenience in analysis.

Many authors like Mohanty [6], [8], Mazhar [5], Chandra [1], [2], [3] have given the criteria for the absolute convergence of Fourier series and allied series, at a point $t=x$, by using the absolute Riesz summability theorems corresponding to the conditions imposed upon the generating function of Fourier series and allied series. In what follows we give a direct proof of the following NEW CRITERIA for the absolute convergence of Fourier series and conjugate series, at a point $t=x$.

Theorem 1. *Let*

(2.1) $$d\geqslant 0<b<1<c \quad \text{and} \quad bc=1+d,$$

and let $t^{-b}P(t)\in BV(y,\pi)$ *for all* y *with* $0<y<\pi$, *and that, as* $y\to 0$

(2.2) $$P_1(y)=O(y^b).$$

Then

(2.3) $$\{(\log(n+1))^{c+d}A_n(x)\}\in|R',L_n,1|$$

is the necessary and sufficient condition for

(2.4) $$\sum_{n=1}^{\infty} A_n(x)(\log(n+1))^d\in|C,0|.$$

Theorem 2. *Let* (2.1) *holds and let* $t^{-b}S(t)\in BV(y,\pi)$ *for all* y *with* $0<y<\pi$, *and that*

$$S_1(y)=O(y^b),\ \text{as } y\to 0. \quad \text{Then} \tag{2.5}$$

$$\{(\log(n+1))^{c+d}B_n(x)\}\in|R', L_n, 1| \tag{2.6}$$

is the necessary and sufficient condition for

$$\sum_{n=1}^{\infty} B_n(x)(\log(n+1))^d \in |C, 0|. \tag{2.7}$$

3. For the proof of the theorems, we require the following lemmas. Lemma 1 and Lemma 2 are due to Das [4].

Lemma 1. *Let* λ_n *be any sequence and let* $\sum_{n=1}^{\infty} a_n\in|R', \lambda_n, 1|$. *Then* $\sum_{n=1}^{\infty} a_n\in|C, 0|$ *implies and is implied by* $\left\{\frac{\lambda_n a_n}{\lambda_{n+1}-\lambda_n}\right\}\in|R', \lambda_n, 1|$.

Lemma 2. *For any* $\{\lambda_n\}$, $\{b_n\}\in|R', \lambda_n, 1|$ *and* $\{d_n\}\in BV$ *implies* $\{b_n d_n\}\in|R', \lambda_n, 1|$.

Lemma 3. *Let* $\lambda_n=L_n$, *that is* $\exp(n/(\log(n+1))^c)$. *Then* $\left\{\frac{L_n a_n}{L_{n+1}-L_n}\right\}\in|R', L_n, 1|$ *is equivalent to* $\{(\log(n+1))^c a_n\}\in|R', L_n, 1|$.

This follows by combining Lemmas 1 and 2.

Lemma 4. *Let* (2.1) *holds. Then, uniformly in* $0<t\leqslant\pi$,

$$\sum=\sum_{n=1}^{\infty}\Delta\left(\frac{1}{L_n}\right)\left|\sum_{m=1}^{n} L_m(\log(m+1))^d g(m,t)\right|=O(t^{-b}),$$

where $g(n,t)$ *stands for either* $\frac{\cos nt}{n}$ *or* $\frac{\sin nt}{n}$.

Proof. Let $T=$ the integral part of $\exp(t^{-b/(1+d)})$. Then, we write

$$\sum=\sum_{n=1}^{T}+\sum_{T+1}^{\infty}=\sum\nolimits_1+\sum\nolimits_2, \text{ say.}$$

Now, by using $g(m,t)=O(m^{-1})$ and changing the order of summation, we have

$$\begin{aligned}\sum\nolimits_1&=O\left\{\sum_{m=1}^{T}(\log(m+1))^d m^{-1}\right\}\\&=O\{(\log T)^{d+1}\}\\&=O(t^{-b}),\end{aligned}$$

uniformly in $0<t\leqslant\pi$. And since $\left\{\frac{L_n(\log(n+1))^d}{n}\right\}$ is monotonic increasing with $n\geqslant e^{2c}\Gamma(q+4)$, where q is the integral part of c, we have, by Abel's lemma

$$\sum\nolimits_2 = O\left\{\sum_{n=T}^{\infty} \frac{|\Delta L_n|}{nL_{n+1}} (\log(n+1))^d \max_{1\leqslant m'\leqslant n} \left|\sum_{m=m'}^{n} mg(m,t)\right|\right\}$$

$$= O\left\{t^{-1}\sum_{n=T}^{\infty} n^{-1}(\log(n+1))^{d-c}\right\}$$

$$= O\{t^{-1}(\log T)^{1+d-c}\} = O(t^{-b}),$$

uniformly in $0<t\leqslant\pi$.

Combining $\sum_1$ and $\sum_2$, we follow the proof of the lemma.

4. Proof of Theorem 1. We have

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} \Phi(t)\left(\cos nt - \frac{\sin nt}{nt}\right)dt + \frac{2}{\pi}\int_0^{\pi}\Phi(t) - \frac{\sin nt}{nt}\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} t\,\Phi(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt$$

$$- \frac{2}{\pi}\int_0^{\pi}\Phi(t)\,dt\int_t^{\pi}\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu}\right)du$$

$$= \frac{2}{\pi}\int_0^{\pi} t\,\Phi(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt$$

$$- \frac{2}{\pi}\int_0^{\pi} u\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu}\right)du\cdot\frac{1}{u}\int_0^{u}\Phi(t)\,dt$$

(by changing the order of integration)

$$\frac{2}{\pi}\int_0^{\pi}(\Phi(t)-\Phi_1(t))\,t\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt \quad \text{(by (1.4))}$$

$$= \frac{2}{\pi}\int_0^{\pi} t\,P(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt \qquad \text{(by 1.5))}$$

$$= \frac{2}{\pi}\int_0^{\frac{1}{n}} t\,P(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt$$

$$+ \frac{2}{\pi}\int_{\frac{1}{n}}^{\pi} t\,P(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt$$

$$= I_1 + I_2, \text{ say.}$$

Now, integrating by parts, we have

$$I_1 = \frac{2}{\pi}\left[tP_1(t) t \frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)\right]_0^{\frac{1}{n}}$$

$$-\frac{2}{\pi}\int_0^{\frac{1}{n}} tP_1(t)\frac{\partial}{\partial t}\left\{t\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)\right\} dt$$

$$= O(n^{-1-b}) + \frac{2}{\pi}\int_0^{\frac{1}{n}} P_1(t)\left(nt \sin nt + \cos nt - \frac{\sin nt}{nt}\right) dt$$

(by (2.2))

$$= O(n^{-1-b}) + O\left(\int_0^{\frac{1}{n}} | P_1(t) | \, dt\right)$$

$$= O(n^{-1-b}),$$

by (2.2). And, integrating by parts, we have

$$I_2 = \frac{2}{\pi^{1+b}} P(\pi) \int_{\frac{1}{n}}^{\pi} t^{1+b} \frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right) dt$$

$$-\frac{2}{\pi}\int_{\frac{1}{n}}^{\pi} d(t^{-b} P(t)) \int_{\frac{1}{n}}^{t} u^{1+b}\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu}\right) du.$$

And, for $\frac{1}{n} < t \leqslant \pi$ and $0 < b < 1$, we have

$$-\int_{\frac{1}{n}}^{t} u^{1+b}\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu}\right) du = -t^b \frac{\sin nt}{n} + n^{-1-b}\sin 1$$

$$+(1+b)\int_{\frac{1}{n}}^{t} u^{b-1}\frac{\sin nu}{n} du$$

$$= -t^b \frac{\sin nt}{n} + n^{-1-b}\sin 1$$

$$+ (1+b) n^{-b}\int_{\frac{1}{n}}^{t'} \sin nu \, du$$

$$\left(\frac{1}{n} < t' < t\right)$$

(by the second mean value theorem)

$$= -t^b \frac{\sin nt}{n} + O(n^{-1-b}).$$

Therefore, combining I_1 and I_2, we have

$$A_n(x) = O(n^{-1-b}) + O(n^{-1-b})$$

$$+ \frac{2}{\pi} \int_{\frac{1}{n}}^{\pi} d(t^{-b} P(t)) \left\{ -t^b \frac{\sin nt}{n} + O(n^{-1-b}) \right\}$$

(4.1) $$= O(n^{-1-b}) - \frac{2}{\pi} \int_{\frac{1}{n}}^{\pi} d(t^{-b} P(t))\, t^b \frac{\sin nt}{n},$$

since, for large n,

$$t^{-b}(P(t) \in BV\left(\frac{1}{n}, \pi\right).$$

In view of Lemma 3, (2.3) is equivalent to

(4.2) $$\left\{ \frac{L_n A_n(x) (\log (n+1))^d}{L_{n+1} - L_n} \right\} \in |R', L_n, 1|,$$

where L_n is defined by (1.8). Now to prove the theorem we first prove

(2.3) is necessary Since $\sum_{n=1}^{\infty} A_n(x) (\log (n+1))^d \in |C, 0|$ implies

$$\sum_{n=1}^{\infty} A_n(x) (\log (n+1))^d \in |R', L_n, 1|.$$

Therefore it follows, from Lemma 1, that (4.2) is necessary. And hence (2.3) is necessary.

Now, finally, we prove

(2.3) is sufficient. Since (2.3) is equivalent to (4.2), it is enough to show that (4.2) is sufficient. Now, from (4.2), it follows that $\{t_n\} \in BV$, where, by (1.1),

(4.3) $$t_n = \frac{1}{L_{n+1}} \sum_{m=1}^{n} L_m A_m(x) (\log (m+1))^d.$$

And, from (4.3), we have

$$A_n(x) (\log (n+1))^d = -\Delta t_{n-1} - \frac{\Delta L_n}{L_n} t_n.$$

Therefore

$$\sum_{n=1}^{\infty} |A_n|(x)| (\log (n+1))^d \leqslant \sum_{n=1}^{\infty} |\Delta t_{n-1}| + \sum_{n=1}^{\infty} \frac{|\Delta L_n|}{L_n} | t_n|$$

$$= O(1) \qquad \text{(by (4.2))}$$

$$+ \sum_{n=1}^{\infty} \Delta\left(\frac{1}{L_n}\right) \left| \sum_{m=1}^{\infty} L_m A_m(x) (\log (m+1))^d \right|. \qquad \text{(by (4.3))}$$

By using (4.1), we have

$$\sum_{n=1}^{\infty} \Delta\left(\frac{1}{L_n}\right) \left| \sum_{m=1}^{n} L_m A_m (x) (\log (m+1)^d \right|$$

$$= O\left\{\sum_{n=1}^{\infty} \Delta\left(\frac{1}{L_n}\right) \sum_{m=1}^{n} L_m \frac{(\log (m+1))^d}{m^{1+b}}\right\}$$

$$+ O\left\{\int_{\frac{1}{n}}^{\pi} |d(t^{-b}P(t))| t^b \sum_{n=1}^{\infty} \Delta\left(\frac{1}{L_n}\right).\right.$$

$$\left. \cdot \left| \sum_{m=1}^{n} L_m (\log (m+1))^d \frac{\sin mt}{m} \right|\right\}$$

$$= O\left\{ \sum_{m=1}^{\infty} L_m \frac{(\log (m+1))^d}{m^{1+b}} \sum_{n=m}^{\infty} \Delta\left(\frac{1}{L_n}\right)\right\}$$

(by changing the order of summation)

$$+ O\left\{\int_{\frac{1}{n}}^{\pi} |d(t^{-b}P(t))|\right\} \qquad \text{(by Lemma 4)}$$

$$= O(1),$$

since, for large n,

$$t^{-b}P(t) \in BV\left(\frac{1}{n}, \pi\right)$$

and

$$\sum_{m=1}^{\infty} \frac{(\log (m+1))^d}{m^{1+b}} \quad \text{is finite.}$$

This terminates the proof of Theorem 1.

5. Proof of Theorem 2. We have

$$\frac{\pi}{2} B_n(x) = \int_0^\pi \psi(t) \sin nt \, dt$$

$$= \int_0^\pi \psi(t) \left(\sin nt + \frac{\cos nt}{nt} \right) dt$$

$$- \int_0^\pi \psi(t) \frac{\cos nt}{nt} dt$$

$$= - \int_0^\pi t \psi(t) \frac{\partial}{\partial t} \left(\frac{\cos nt}{nt} \right) dt$$

$$+ \int_0^\pi \psi(t) \, dt \int_t^{\left(\pi + \frac{\pi}{2n}\right)} \frac{\partial}{\partial u} \left(\frac{\cos nu}{nu} \right) du$$

$$= - \int_0^\pi t \psi(t) \frac{\partial}{\partial t} \left(\frac{\cos nt}{nt} \right) dt$$

$$+ \int_0^\pi \psi(t) \, dt \int_t^\pi \frac{\partial}{\partial u} (\cos nu/nu) \, du$$

$$+ \int_0^\pi \psi(t) \, dt \int_\pi^{\pi + \frac{\pi}{2n}} \frac{\partial}{\partial u} \left(\frac{\cos nu}{nu} \right) du$$

$$= - \int_0^\pi t \psi(t) \frac{\partial}{\partial t} \left(\frac{\cos nt}{nt} \right) dt$$

$$+ \int_0^\pi \frac{\partial}{\partial t} \left(\frac{\cos nt}{nt} \right) dt \int_0^t \psi(u) \, du$$

(by changing the order of integration)

$$-\frac{\cos n\pi}{n}\frac{1}{\pi}\int_0^\pi \psi(t)\,dt$$

$$=\int_0^\pi S(t)\left(\sin nt+\frac{\cos nt}{nt}\right)dt-\psi_1(\pi)\frac{\cos n\pi}{n}$$

(by (1.4) and (1.6))

$$=-\psi_1(\pi)\frac{\cos n\pi}{n}+\left(\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\pi}\right)\left(S(t)\left(\sin nt+\frac{\cos nt}{nt}\right)dt\right)$$

$$=-\psi_1(\pi)\frac{\cos n\pi}{n}+I_1+I_2,\text{ say.}$$

Now, proceeding as in Theorem 1 for I_1 and I_2 and using $t^{-b}S(t)\in BV(y,\pi)$ and (2.5), for $y=\frac{1}{n}$, where n is sufficiently large, we have

$$I_1=O(n^{-1-b}),$$

and

$$I_2=O(n^{-1-b})-S(\pi)\frac{\cos n\pi}{n}$$

$$-\int_{\frac{1}{n}}^{\pi} d(t^{-b}S(t))\,t^b\frac{\cos nt}{n}.$$

Therefore

$$B_n(x)=O(n^{-1-b})-\frac{2}{\pi}(\psi_1(\pi)+S(\pi))\frac{\cos n\pi}{n}$$

$$+\frac{2}{\pi}\int_{\frac{1}{n}}^{\pi} d(t^{-b}S(t))\,t^b\frac{\cos nt}{n}$$

$$=O(n^{-1-b})+\frac{2}{\pi}\int_{\frac{1}{n}}^{\pi} d(t^{-b}S(t))\,t^b\frac{\cos nt}{n},$$

by using (1.6) and the fact that $\psi(\pi)=0$.

Now, proceeding as in Theorem 1, it is easy to see that (2.6) is necessary and sufficient for (2.7).

This terminates the proof of Theorem 2.

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