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META-ANALYTIC FUNCTIONS OF EQUAL MODULUS

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In [1], M. B. Balk characterized polyanalytic functions of constant modulus (see definition 1). Later Krajkiewicz and Bosch [4] gave necessary and sufficient conditions for two polyanalytic functions to have equal modulus or equal amplitude on a region. The following theorem [5, p. 35] played an important role in the derivation of those conditions: A holomorphic function $F(z, w)$ of two variables which is rational in z for each fixed w and which is rational in w for each fixed z, is a rational function of z and w.

Recently [8] M. F. Zuev extended Balk's result to meta-analytic functions, a class of functions containing polyanalytic functions (see definition 2). In this paper we give necessary and sufficient conditions for two meta-analytic functions to have equal modulus or equal amplitude on a region. In the derivation we encounter a holomorphic function $F(z, w)$ of two variables with the following properties: For each fixed z, $F(z, w)$ is a rational function of a particular set of holomorphic functions of w; for each fixed w, $F(z, w)$ is a rational function of a particular set of holomorphic functions z. By employing a technique Hurwitz used in proving a theorem of Weierstrass [3], [7], we are able to show that $F(z, w)$ must be a rational function of a particular set of functions which are holomorphic either in z or in w. This result allows us to obtain the desired conclusions.

Preliminaries: In this section we collect the relevant definitions and state some representation theorems.

Definition 1. Let G be a region of the complex plane Γ. The function $f: G \to \Gamma$ is said to be polyanalytic or n-analytic on the region G if $f \in C^n(G)$ and

$$\frac{\partial^n f}{\partial \bar{z}^n} = 0 \text{ on } G. \text{ Here } \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\, \frac{\partial f}{\partial y}\right)$$

is the complex form of the Cauchy-Riemann equations.

It is easy to show, by repeated antidifferention, that f is n-analytic on G if and only if f has a respresentation of the form $f(z) = \sum \bar{z}^n f_k(z)$, for all $z \in G$, where $k = 0, 1, \ldots, n\text{-}1$, each $f_k(z)$ is holomorphic on G, and $\bar{z}$ denotes the complex conjugate of z.

Definition 2. Let G be region of the complex plane Γ. A function $f: G \to \Gamma$ is said to be meta-analytic of order n on the region G if $f \in C^n(G)$ and $\sum a_k \frac{\partial^k f}{\partial \bar{z}^k} \equiv 0$ on G, where $k = 0, 1, \ldots, n$ and a_k is a complex constant for each k [8].

In [2], Fempl obtained a representation for meta-analytic functions when the roots of $\sum a_k \xi^k = 0$ are distinct. By modifying his argument one obtains the following general respresentation which we will take as the definition of meta-analytic functions:

Definition 3. Let G be a region of the complex plane Γ. The function $f: G \to \Gamma$ is meta-analytic on G if and only if f has a representation of the form

$$f(z) = \sum \left(\sum \bar{z}^i f_{ij}(z)\right) \exp\left(r_j \bar{z}\right),$$

for all $z \in G$, where $i = 0, 1, \ldots, m_j$; $j = 1, 2, \ldots, m$; m_j is the multiplicity of the root r_j of $\sum a_k \xi^k = 0$, and each $f_{ij}(z)$ is a holomorphic function of z on G.

We will find it advantageous to reverse the order of summation in the representation of f. We do this now and make the following standing assumptions: Henceforth f and g shall denote functions which are meta-analytic on a common region G with representations

$$f(z) = \sum \bar{z}^i \left(\sum f_{ij}(z) \exp\left(r_j \bar{z}\right)\right),$$

$$g(z) = \sum \bar{z}^h \left(\sum g_{hk}(z) \exp\left(s_k \bar{z}\right)\right),$$

$$\text{where } i = 0, 1, \ldots, n;\ j = 1, 2, \ldots, m;$$

$h = 0, 1, \ldots, q$; $k = 1, 2, \ldots p$ and each $f_{ij}(z)$ and each $g_{hk}(z)$ is holomorphic on G. Also G^* shall denote the region $G^* = \{z \in \Gamma : \bar{z} \in G\}$. Finally we assume $f \not\equiv 0$ and $g \not\equiv 0$ on G.

THE MAIN RESULTS.

Theorem 1. *Let f and g be two functions meta-analytic on a region G. Then $|f| \equiv |g|$ on G if and only if there exists a not identically zero polynomial $P\left(z, \bar{z}, \exp\left(\bar{r}_j z\right), \exp\left(s_k \bar{z}\right)\right)$ in the $q+m+2$ variables z, $\bar{z}$, $\exp\left(\bar{r}_j z\right)$, $\exp\left(s_k \bar{z}\right)$ such that $Pf \equiv \bar{P} g$ on G.*

Proof. If $Pf \equiv \bar{P} g$ on G, then obviously $|f| \equiv |g|$, on G. Now suppose $|f(z)| = |g(z)|$, for all $z \in G$. Then

$$\sum \bar{z}^i \left(\sum f_{ij}(z) \exp\left(r_j \bar{z}\right)\right) \sum z^i \left(\sum \bar{f}_{ij}(\bar{z}) \exp\left(\bar{r}_j z\right)\right)$$

$$= \sum \bar{z}^h \left(\sum g_{hk}(z) \exp\left(s_k \bar{z}\right)\right) \sum z^h \left(\sum \bar{g}_{hk}(\bar{z}) \exp\left(\bar{s}_k z\right)\right),$$

for all $z \in G$. By $\bar{f}_{ij}(\bar{z})$ we mean $\overline{f_{ij}(z)}$, for $z \in G$. Thus $\bar{f}_{ij}(w)$ is holomorphic in w on G^*. Now if we replace z by w in the above equality, we obtain two functions holomorphic in z and w on $G \times G^*$ which coincide when $w = \bar{z}$. Therefore by Vladimirov's theorem [6, p. 41], they coincide throughout $G \times G^*$; that is

$$\sum w^i \left(\sum f_{ij}(z) \exp(r_j w)\right) \sum z^i \left(\sum \bar{f}_{ij}(w) \exp(\bar{r}_j z)\right)$$
$$= \sum w^h \left(\sum g_{hk}(z) \exp(s_k w)\right) \sum z^h \left(\sum \bar{g}_{hk}(w) \exp(\bar{s}_k z)\right)$$

for all $(z, w) \in G \times G^*$. Since we are assuming $f \neq 0$ and $g \neq 0$. we can find a subdomain $D \times D^*$ of $G \times G^*$ on which each of the four double sums appearing in the above equality never vanish. On $D \times D^*$ we define the function

$$h(z, w) = \frac{\sum w^i \left(\sum f_{ij}(z) \exp(r_j w)\right)}{\sum w^h \left(\sum g_{hk}(z) \exp(s_k w)\right)} = \frac{\sum z^h \left(\sum \bar{g}_{hk}(w) \exp(\bar{s}_k z)\right)}{\sum z^i \left(\sum \bar{f}_{ij}(w) \exp(\bar{r}_j z)\right)}.$$

Note that for each fixed $w \in D^*$, $h(z, w)$ is a rational function in the arguments z, $\exp(\bar{s}_k z)$, $\exp(\bar{r}_j z)$. For each fixed $z \in D$, $h(z, w)$ is a rational function in the arguments w, $\exp(r_j w)$, $\exp(s_k w)$. We claim that this implies $h(z, w)$ is a rational function in the arguments z, w, $\exp(\bar{s}_k z)$, $\exp(s_k w)$, $\exp(\bar{r}_j z)$, and $\exp(r_j w)$

To prove this assertion we make use of a technique employed by Hurwitz in [3]. We will only outline the argument and refer the reader to Hurwitz for a more detailed exposition for the case when $h(z, w)$ is a rational function of z and w only.

From the definition of $h(z, w)$ we conclude that

$$h(z, w) \sum w^h \left(\sum g_{hk}(z) \exp(s_k w)\right) - \sum w^i \left(\sum f_{ij}(z) \exp(r_j w)\right) = 0,$$

for all $(z, w) \in D \times D^*$. Now we select any $t = (n+1)m + (q+1)p$ points $w_1, w_2, \ldots, w_t$ in D^* and form t linear homogeneous equations by substituting these values of w into the above equation. If we treat the functions $f_{ij}(z)$, $g_{hk}(z)$ as unknowns and apply Cramer's rule, we conclude that the coefficient matrix C of the system vanishes indentically for $z \in D$ and $w_1, w_2, \ldots, w_t \in D^*$. Let r be the maximum rank of the matrix C where the rank is taken over all $z \in D$, $w_1, w_2, \ldots, w_t \in D^*$. Clearly $r < t$. By making use of the fact that the sets $\{w_i \exp(r_j w)\}$ and $\{w^h \exp(s_k w)\}$ are sets of linearly independent functions, we can deduce that $r \geqslant (n+1)m$, $(q+1)p$. Hence there exists a $r \times r$ submatrix C_r of C whose determinant is not zero for some $z \in D$ and some fixed $w_1, w_2, , \ldots, w_r \in D^*$. Moreover C_r is a submatrix of a $(r+1) \times (r+1)$ submatrix C_{r+1} of C whose determinant does vanish identically for $z \in D$ and $w, w_1, w_2, \ldots, w_r \in D^*$. We may assume C_r is obtained from C_{r+1} by deleting the first row and a suitable column. Since $(r+1) \geqslant (n+1)m$, $h(z, w)$ is a factor of the first row column entry of C_{r+1}. If we expand $\det C_{r+1}$ by the first row, we obtain an equation of the form

$$h(z, w)\ P(z, w, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(s_k w))$$
$$= Q(z, w, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(r_j w)),$$

where P is a not identically zero polynomial in z, w, $\exp(\bar{r}_j z)$, $\exp(\bar{s}_k z)$, $\exp(s_k w)$ and Q is a not identically zero polynomial in z, w, $\exp(\bar{r}_j z)$, $\exp(\bar{s}_k z)$, $\exp(r_j w)$. We may assume that P and Q are relatively prime. Thus we obtain

$$P(z, \bar{z}, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(s_k \bar{z}))\, f(z)$$

$$= Q(z, \bar{z}, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(r_j \bar{z}))\, g(z),$$

for all $z \in D$. We will finish the proof by showing that

$$\overline{P(z, \bar{z}, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(s_k \bar{z}))}$$

$$= \lambda\, Q(z, \bar{z}, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(r_j \bar{z}))$$

for some constant λ with $|\lambda| = 1$.

From the above equation, we see that

$$|P(z, \bar{z}, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(s_k \bar{z}))|$$

$$= |Q(z, \bar{z}, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(r_j \bar{z}))|,$$

for all $z \in D$. Hence if we let $P^* = \bar{P}$, $Q^* = \bar{Q}$ we have that $P P^* = Q Q^*$ for all $z \in D$. Therefore, by Vladimirov's theorem we get

$$P(z, w, \exp(r_j \bar{z}), \exp(s_k \bar{z}), \exp(s_k w))\, P^*(z, w), \exp(r_j w, \exp(\bar{s}_k z), \exp(s_k w))$$

$$= Q(z, w, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(r_j w))\, Q^*(z, w, \exp(r_j w), \exp(s_k w), \exp(\bar{r}_j z))$$

for all $(z, w) \in D \times D^*$ and therefore for all $(z, w) \in \Gamma \times \Gamma$. We claim this identity remains an identity if we replace each argument $z, w, \exp(\bar{r}_j w)$, etc., by a distinct complex variable $z_1, z_2, \ldots, z_s$. To see this, we write $\bar{P} P^*$ and $\bar{Q} Q^*$ as polynomials which are linear in two sets of linearly independent functions of the form $\{\exp(\alpha z)\}$ and $\{\exp(\beta w)\}$. This can be done by noting that $[\exp(\bar{r}_j z)]^p = \exp(p \bar{r}_j z)$, for integral values of p. Then one can argue that the identity $P P^* = Q Q^*$ is preserved if we replace $\exp(p \bar{r}_j z)$ by z_2 $\exp(\bar{r}_j z)$ by z_3, etc. Finally we let $z_2 = z_3^p$ and so on. We therefore have an indentity of the form

$$P(z_1, \ldots, z_s)\, P^*(z_1, \ldots, z_s)$$

$$= Q(z_1, \ldots, z_s)\, Q^*(z_1, \ldots, z_s)$$

where P and Q are relatively prime. Clearly P^* and Q^* are also relatively prime. Hence $P^* R = Q$ for some polynomial R. Similarly $Q S = P^*$ for some polynomial S. Thus $P^* R S = P^*$ and $R = \lambda$, where λ is a complex constant. Then $P^* \lambda = Q$ implies that

$$\overline{P(z, \bar{z}, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(s_k \bar{z}))}$$

$$= \lambda^*\, Q(z, \bar{z}, \exp(\bar{r}_j z), \exp(\bar{s}_k z), \exp(r_j \bar{z})),$$

for all $z \in \Gamma$. None of the arguments $\exp(r_j \bar{z})$ appear in $\bar{P}$. Therefore they do not appear. in Q. Likewise the arguments $\exp(s_k \bar{z})$ do not appear in Q. Therefore they do not appear in $\bar{P}$. This proves the theorem.

Theorem 2. *Let f and g be two functions meta-analytic on a region G. Then $f(z)\overline{g(z)} \equiv \overline{f(z)}\, g(z)$ on G if and only if there exist two not identically zero real-valued polynomials P and Q in the variables $z, \bar{z}$,* $\exp(\bar{r}_j z)$, $\exp(r_j \bar{z})$, , $\exp(\bar{s}_k z)$, *and* $\exp(s_k \bar{z})$ *such that $Pf \equiv Qg$ on G.*

The proof of theorem 2 is similar to the proof of theorem 1. By using theorem 2 and reasoning as in [4] we obtain the following corollary which we also state without proof.

Corollary. *If f and g are two meta-analytic functions which never vanish on a region G, then*

$$\operatorname{amp} f(z) = \operatorname{amp} g(z) \qquad (\text{modulo } 2\pi),$$

for all $z \in G$, if and only if the following condition holds: There exist two not identically zero real-valued polynomials P and Q in the variables $z, \bar{z}$, $\exp(\bar{r}_j z)$, $\exp(r_j \bar{z})$, $\exp(\bar{s}_k z)$, *and* $\exp(s_k \bar{z})$ *with PQ positive for one point of G and such that $Pf = Qg$, for all $z \in G$.*

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