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A FIXED POINT THEOREM FOR NON-EXPANSIVE MAPPINGS ON COMPACT METRIC SPACES

Karol Baron and *Janusz Matkowski*

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A self-mapping F of a metric space (X, ρ) is said to be non-expansive provided that $\rho(F(x_1), F(x_2)) \leqslant \rho(x_1, x_2)$ holds for every $x_1, x_2 \in X$; F is said to be ρ-contractive provided that $\rho(F(x_1), F(x_2)) < \rho(x_1, x_2)$ holds for every $x_1, x_2 \in X$, $x_1 \neq x_2$.

We introduce the following

Definition. We say that a metric space (X, ρ) is S-space iff there exists an $x_0 \in X$ such that for every $t \in (0, 1)$ there is a ρ-contractive self-mapping f_t of (X, ρ) such that $\rho(f_t(x), x) \leqslant (1-t)\rho(x_0, x)$ holds for every $x \in X$.

The theorem presented below generalizes theorem 1 in [2].

Theorem. *Every non-expansive self-mapping of a compact S-space has a fixed point.*

In the proof of this theorem we shall use the following simple fact. If a sequence of continuous self-mappings of a compact metric space having a fixed point is uniformly convergent, then its limit has also a fixed point.

Proof of the theorem. Let (X, ρ) be a compact metric S-space and suppose that $F\colon X \to X$ is non-expansive. We shall show that F has a fixed point. To this end we take $t_n \in (0, 1)$, $n = 1, 2, \ldots$, such that $\lim_{n\to\infty} t_n = 1$, and put $F_n = f_{t_n} \circ F$, $n = 1, 2, \ldots$. It is clear that for every $n = 1, 2, \ldots$ F_n is ρ-contractive self-mapping of (X, ρ), thus in view of the Edelstein theorem (see [3]) for every $n = 1, 2, \ldots$ F_n has a fixed point. Moreover $\rho(F_n(x), F(x)) \leqslant (1-t_n)\rho(x_0, F(x))$ for all $x \in X$ which means that $\{F_n\}_{n=1}^{\infty}$ tends uniformly to F (since X is compact) and so F has a fixed point.

Note that uniqueness cannot be asserted (cf. [2], Remark 1).

Proposition. *Let (X, ρ) be a complete metric space, and suppose that there exists an $x_0 \in X$ such that for every $x \in X$ the set $I(x) = \{y \in X : \rho(x_0, x) = \rho(x_0, y) + \rho(y, x)\}$ is metrically convex (cf. [1], p. 41). If for every $x_1, x_2 \in X$, $x_1 \neq x_2$, $x_i \neq x_0$, $i = 1, 2$, and $y_i \in I(x_i)$, $i = 1, 2$, the condition*

$$\frac{\rho(x_0, y_1)}{\rho(x_0, x_1)} = \frac{\rho(x_0, y_2)}{\rho(x_0, x_2)}$$

implies $\rho(y_1, y_2) < \rho(x_1, x_2)$, then (X, ρ) is an S-space.

Proof. Let $t \in (0, 1)$. Since for every $x \in X$, $I(x)$ is metrically convex and closed subset of X, then by Menger's theorem ([1], theorem 14.1) for every $x \in X$ there exists an $y \in I(x)$ such that $\rho(x_0, y) = t\rho(x_0, x)$. We define the self-mapping f_t of X by the formula $f_t(x) = y$, where y is chosen to x from the quoted above Menger's theorem. We shall show that f_t is ρ-contractive. Indeed, let $x_1, x_2 \in X$, $x_1 \neq x_2$. Since $\rho(x_0, f_t(x_i)) = t\rho(x_0, x_i)$, $i = 1, 2$, then

$$\frac{\rho(x_0, f_t(x_i))}{\rho(x_0, x_i)} = t, \qquad i = 1, 2,$$

whenever $x_i \neq x_0$, $i = 1, 2$, and consequently $\rho(f_t(x_1), f_t(x_2)) < \rho(x_1, x_2)$. If however, one of the elements x_1 and x_2 is equal to x_0, say x_1, then

$$\rho(f_t(x_1), f_t(x_2)) = \rho(x_0, f_t(x_2)) = t\rho(x_0, x_2) < \rho(x_1, x_2),$$

that is to say f_t is ρ-contractive. Moreover, for every $x \in X$

$$\rho(f_t(x), x) = \rho(x_0, x) - \rho(x_0, f_t(x)) = (1-t)\rho(x_0, x).$$

This ends the proof.

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Silesian University
Katowice, Poland