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SOME MERCERIAN THEOREMS FOR REGULARLY VARYING SEQUENCES

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1. Introduction

In this paper all the matrices considered are *triangular* and *invertibles*. If $A=(a_{n,k})$ $(n=1, 2, \ldots, 1 \leq k \leq n)$ is such a matrix, we will denote by $A^{-1}=(a_{n,k}^{(-1)})$ its inverse matrix, i.e., $A^{-1}A=AA^{-1}=I$. We will use the following terminology: If R is any fixed class of sequences, we will say that a matrix $(a_{n,k})$ is $0(R)$ — *regular* if, for any $(r_n) \in R$, we have

$$(1.1) \qquad s_n=0(r_n) \Rightarrow \sum_{k=1}^{n} a_{n,k}s_k=0(r_n), \quad (n \to \infty);$$

R — *regular* if

$$(1.2) \qquad s_n \cong r_n \Rightarrow \sum_{k=1}^{n} a_{n,k}s_k \cong r_n \cdot \quad (n \to \infty);$$

$0(R)$ — *mercerian* if

$$(1.3) \qquad \sum_{k=1}^{n} a_{n,k}s_k=0(r_n) \Rightarrow s_n=0(r_n), \quad (n \to \infty);$$

and R — *mercerian* if

$$(1.4) \qquad \sum_{k=1}^{n} a_{n,k}\, s_k \cong r_n \Rightarrow s_n \cong r_n, \quad (n \to \infty).$$

If R contains only the sequence $r_n=1$ $(n=1, 2, \ldots)$, then the matrix which satisfies (1.1), (1.2), (1.3) and (1.4) will be called $0(1)$ — *regular*, *regular*, $0(1)$ — *mercerian* and *mercerian*, respectively. For this classical case, we have shown the following result ([1, th. 1]):

Theorem A. A matrix $(\delta_{n,k})$ which satisfies the condition

$$(1.5) \qquad \liminf_{n \to \infty} \left\{ |\delta_{n,n}| - \sum_{k=1}^{n-1} |\delta_{n,k}| \right\} > 0$$

is $0(1)$ — *mercerian*. If, in addition, the matrix $(\delta_{n,k})$ is *regular*, then it is *mercerian*.

Using theorem A we gave a simple proof of Mercer's classical theorem, and established some other mercerian theorems (see [1] to [5]). *The purpose of this paper is to extend these results by assuming that R contains a certain class of regularly varying sequences.* A sequence (r_n) is said to be *regularly varying* if

$$\lim_{n\to\infty}\left(\frac{r_{[tn]}}{r_n}\right)=h(t) \text{ exists for every } t>0 \tag{1.6}$$

(see [6]). It is well known (see, for example [6, p. 56]) that (1.6) implies the existence of a real number β such that $h(t)=t^{\beta}$. This number β is called the *order* of the regularly varying sequence. We shall denote by R_{β} the class of all slowly varying sequences of order β. In particular, *a regularly varying sequence of order* 0 *is called a slowly varying sequence.* If $(r_n)\in R_{\beta}$, then it is easy to see that $r_n=n^{\beta}L_n$, where $(L_n)\in R_0$. The essential properties of regularly varying sequences have been studied by J. Karamata ([6] and [7]), and the $0(R_0)$ — regularity and R_0 — regularity theorems by M. Vuilleumier ([8, th. 4.1]). Her results specialized to triangular matrices can be stated as follows:

Theorem B. **1°**. In order that a matrix $(a_{n,k})$ will be $0(R_0)$ — *regular*, it is *necessary* and *sufficient* that

$$\sum_{k=1}^{n}|a_{n,k}|k^{-\eta}=0(n^{-\eta}),\quad (n\to\infty), \tag{1.7}$$

for some $\eta>0$.

2°. A matrix is R_0 — regular if and only if (1.7) holds together with the condition $\sum\limits_{k=1}^{n}a_{n,k}\to 1,\quad (n\to\infty)$.

From theorem B we deduce a necessary and sufficient condition for a matrix to be $0(R_{\beta})$ — *regular*, which is:

Theorem C. A matrix $(\gamma_{n,k})$ is $0(R_{\beta})$ — *regular* if and only if there exists $\alpha<\beta$ such that

$$\sum_{k=1}^{n}|\gamma_{n,k}|\left(\frac{k}{n}\right)^{\alpha}=0(1),\quad (n\to\infty), \tag{1.8}$$

and is R_{β} — *regular* if and only if (1.8) holds together with the condition

$$\sum_{k=1}^{n}\gamma_{n,k}\left(\frac{k}{n}\right)^{\beta}\to 1,\quad (n\to\infty). \tag{1.9}$$

Then, we will show (theorem 1), that a matrix $(a_{n,k})$ is $0(R_{\beta})$ — *mercerian* if and only if

$$\sum_{k=1}^{n}|a_{n,k}^{(-1)}|\left(\frac{k}{n}\right)^{\alpha}=0(1),\quad (n\to\infty),$$

for some $\alpha<\beta$, and that a matrix $(a_{n,k})$ is R_{β} — *mercerian* if and only if it is $0(R_{\beta})$ — mercerian and if

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}a_{n,k}^{(-1)}\left(\frac{k}{n}\right)^{\beta}\right)=1.$$

On the other hand, we shall show in theorem 2 that a $0\,(R_\beta)$ – *mercerian matrix* $(a_{n,k})$ is R_β – *mercerian if*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} a_{n,k}\left(\frac{k}{n}\right)^{\beta}\right)=1. \tag{1.10}$$

Then, using theorem 1, we shall *extend our theorem A to regularly varying sequences*, by showing that a matrix $(a_{n,k})$ is $0\,(R_\beta)$ – *mercerian if*

$$\liminf_{n\to\infty}\left\{|a_{n,n}|-\sum_{k=1}^{n-1}|a_{n,k}|\left(\frac{k}{n}\right)^{\alpha}\right\}>0,$$

for some $\alpha<\beta$ (theorem 3); if in addition the relation (1.10) holds, then in view of theorem 2, the matrix $(a_{n,k})$ is already R_β – mercerian. Finally, as a consequence of these results, we shall show that *Mercer's classical theorem can be extended to slowly varying sequences* (*theorem 4*).

2. Results and Proofs

Theorem 1. A matrix $(a_{n,k})$ is $0\,(R_\beta)$ – *mercerian* if and only if there exists $\alpha<\beta$ such that

$$\sum_{k=1}^{n}|a_{n,k}^{(-1)}|\left(\frac{k}{n}\right)^{\alpha}=0\,(1),\quad (n\to\infty); \tag{2.1}$$

and is R_β – *mercerian* if and only if (1.1) holds together with the additional condition

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} a_{n,k}^{(-1)}\left(\frac{k}{n}\right)^{\beta}\right)=1. \tag{2.2}$$

Proof of theorem 1. Set $t_n=\sum_{k=1}^{n} a_{n,k}\,s_k,\quad (n=1,2,\ldots)$.

As $(a_{n,k})$ is an invertible matrix,

$$s_n=\sum_{k=1}^{n} a_{n,k}^{(-1)}\,t_k,\quad (n=1,2,\ldots).$$

In other words, it is evident that the relations (1.3) and (1.4) for $R=R_\beta$ means that

$$t_n=0\,(r_n)\ \Rightarrow\ \sum_{k=1}^{n} a_{n,k}^{(-1)}\,t_k=0\,(r_n), \tag{2.3}$$

respectively that

$$t_n\cong r_n\ \Rightarrow\ \sum_{k=1}^{n} a_{n,k}^{(-1)}\,t_k\cong r_n,\quad (n\to\infty), \tag{2.4}$$

for every $r_n\in R_\beta$. Setting $\gamma_{n,k}=a_{n,k}^{(-1)}$ $(n=1,2,\ldots,\ 1\le k\le n)$, we see that the conclusions of theorem 1 follow from theorem C. In fact, according to theorem C, the property (2.3) is equivalent to (1.8), and (2.3) together with (2.4) is equivalent to the relations (1.8) and (1.9).

Theorem 2. An $0(R_\beta)-$*mercerian* matrix $(a_{n,k})$ is $R_\beta-$*mercerian* if

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} a_{n,k}\left(\frac{k}{n}\right)^{\beta}\right)=1. \tag{2.5}$$

Proof of theorem 2. Since $(a_{n,k})$ is $0(R_\beta)-$mercerian, we know by the first part of theorem 1, that there exists $\alpha<\beta$ such that

$$\sum_{k=1}^{n}\left|a_{n,k}^{(-1)}\right|\left(\frac{k}{n}\right)^{\alpha}=0(1),\quad (n\to\infty). \tag{2.6}$$

Hence, by the second part of theorem 1, we have only to show that (2.5) and (2.6) imply

$$\sum_{k=1}^{n} a_{n,k}^{(-1)}\left(\frac{k}{n}\right)^{\beta}\to 1,\quad (n\to\infty). \tag{2.7}$$

Let

$$c_n \overset{\text{det}}{=} \sum_{k=1}^{n} a_{n,k}\left(\frac{k}{n}\right)^{\beta},\quad (n=1, 2,\dots); \tag{2.8}$$

we have

$$1=\sum_{k=1}^{n} a_{n,k}^{(-1)}\left(\frac{k}{n}\right)^{\beta}c_k,\quad (n=1, 2,\dots)$$

and

$$\sum_{k=1}^{n} a_{n,k}^{(1-)}\left(\frac{k}{n}\right)^{\beta}-1=\sum_{k=1}^{n} a_{n,k}^{(-1)}\left(\frac{k}{n}\right)^{\beta}(1-c_k). \tag{2.9}$$

Writing

$$A_{n,k}=a_{n,k}^{(-1)}\left(\frac{k}{n}\right)^{\beta},\quad (n=1, 2,\dots,\ 1\le k\le n),$$

it follows from (2.6) that

$$\sum_{k=1}^{n}|A_{n,k}|\le\sum_{k=1}^{n}\left|a_{n,k}^{(-1)}\right|\left(\frac{k}{n}\right)^{\beta}\le\sum_{k=1}^{n}\left|a_{n,k}^{(-1)}\right|\left(\frac{k}{n}\right)^{\alpha}=0(1),\ (n\to\infty). \tag{2.10}$$

Also, it follows from (2.6) that

$$|A_{n,k}|=\left|a_{n,k}^{(-1)}\right|\left(\frac{k}{n}\right)^{\beta}=\left(\frac{k}{n}\right)^{\beta-\alpha}\left|a_{n,k}^{(-1)}\right|\left(\frac{k}{n}\right)^{\alpha}\le \tag{2.11}$$

$$\le\left(\frac{k}{n}\right)^{\beta-\alpha}\sum_{k=1}^{n}\left|a_{n,k}^{(-1)}\right|\left(\frac{k}{n}\right)^{\alpha}\to 0,\quad (n\to\infty)$$

for $k=1, 2, 3,\dots$. Since by (2.5) and (2.8), $c_n\to 1$ $(n\to\infty)$, it follows from Toeplitz-Schur's theorem according to (2.10) and (2.11) that

$$\sum_{k=1}^{n} a_{n,k}^{(-1)}\left(\frac{k}{n}\right)^{\beta}(1-c_k)\to 0,\quad (n\to\infty),$$

which proves, in view of (2.9), our statement (2.7).

Theorem 3. A matrix $(a_{n,k})$ is $0(R_\beta)-$mercerian if there exists $a<\beta$ such that

$$\liminf_{n\to\infty}\left\{|a_{n,n}|-\sum_{k=1}^{n-1}|a_{n,k}|\left(\frac{k}{n}\right)^{\alpha}\right\}>0. \tag{2.12}$$

Proof of theorem 3. According to theorem A, a matrix $(\delta_{n,k})=(a_{n,k}k^{\alpha}n^{-\alpha})$ which satisfies condition (1.5) is $0(1)$-mercerian, i.e., (2.12) implies

$$\sum_{k=1}^{n}|\delta_{n,k}^{(-1)}|=0(1), \quad (n\to\infty).$$

Now as

$$(\delta_{n,k}^{(-1)})=(a_{n,k}^{(-1)}k^{\alpha}n^{-\alpha}),$$

we have

$$\sum_{k=1}^{n}|a_{n,k}^{(-1)}|\left(\frac{k}{n}\right)^{\alpha}=0(1), \quad (n\to\infty),$$

for some $\alpha<\beta$, which implies our result according to theorem 1.

Using theorems 2 and 3, we will now generalize Mercer's classical theorem for slowly varying sequences.

Theorem 4. Let α denote a real number, and let

$$c_1(s_n)=\frac{1}{n}\sum_{k=1}^{n}s_k, \quad (n=1, 2, 3, \dots).$$

Then,

$$s_n\cong L_n \Leftrightarrow \alpha s_n+(1-\alpha)c_1(s_n)\cong L_n, \quad (n\to\infty), \tag{2.13}$$

for every $(L_n)\in R_0$, if and only if $\alpha>0$.

Proof of theorem 4. For $\alpha=0$, the conclusion is obvious, because we know that $(c_1(s_n))$ is not a mercerian transformation. For $\alpha<0$, we can use the following lemma of the author ([2, p. 28]): Let

$$t_n=\sum_{k=1}^{n}\alpha_{n,k}s_k, \quad (n=1, 2, \dots),$$

be a regular summability method. If there exists a divergent sequence (s_n^*) such that

$$\sum_{k=1}^{n}\alpha_{n,k}s_k^*=0(s_n^*), \quad (n\to\infty), \tag{2.14}$$

then $(\alpha_{n,k})$ *sums at least one divergent sequence.* In fact, setting in the last lemma

$$t_n=\alpha s_n+(1-\alpha)c_1(s_n), \quad (n=1, 2, \dots),$$

i.e.,

$$\alpha_{n,k}=\begin{cases}\dfrac{1-\alpha}{n}, & 1\le k\le n-1\\[2mm] \alpha+\dfrac{1-\alpha}{n}, & k=n\end{cases} \tag{2.15}$$

and $s_n^*=n^{-1/\alpha}$ $(n=1, 2, \dots)$, we see that (2.14) is satisfied and consequently that (2.13) cannot hold for any $\alpha<0$.

For $\alpha>0$, we first note that if the matrix $(\alpha_{n,k})$ is defined by (2.15) then

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\alpha_{n,k}\right)=1$$

and

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}|\alpha_{n,k}|\left(\frac{k}{n}\right)^{-r}\right)=\alpha+\frac{1-\alpha}{1-r},$$

for every $0<r<1$. Consequently, $(\alpha_{n,k})$ *satisfies the conditions of theorem C for* $\beta=0$. It follows form the last theorem that

$$s_n\cong L_n \Rightarrow \sum_{k=1}^{n}\alpha_{n,k}s_k\cong L_n, \quad (n\to\infty),$$

for every $(L_n)\in R_0$, and thus according to *theorem* 2, it is enough to show that

$$t_n=0(L_n) \Rightarrow s_n=0(L_n), \quad (n\to\infty),$$

for every $(L_n)\in R_0$. But as

$$s_n=\frac{t_n+(\alpha-1)c_1(s_n)}{\alpha}, \quad (\alpha\neq 0),$$

it is sufficient to show that

(2.16) $$t_n=0(L_n) \Rightarrow c_1(s_n)=0(L_n), \quad (n\to\infty),$$

for every $(L_n)\in R_0$. Now since

$$s_1=c_1(s_1) \text{ and } s_n=nc_1(s_n)-(n-1)c_1(s_{n-1}), \quad n=2,3,\ldots),$$

we have

$$t_n=\sum_{k=1}^{n}b_{n,k}c_1(s_k), \quad (n=2,3,\ldots),$$

with

(2.17) $$b_{n,k}=\begin{cases}0, & k\neq n \text{ or } n-1\\ -\alpha(n-1), & k=n-1\\ \alpha n-\alpha+1, & k=n.\end{cases}$$

According to theorem 3, the relation (2.16) will be proved if we show that for every $\alpha>0$ there exists $\eta>0$ such that

(2.18) $$\liminf_{n\to\infty}\left(|b_{n,n}|-|b_{n,n-1}|\left(1-\frac{1}{n}\right)^{-\eta}\right)>0.$$

As by (2.17),

$$|b_{n,n}|-|b_{n,n-1}|\left(1-\frac{1}{n}\right)^{-\eta}=\alpha n-\alpha+1-\alpha(n-1)\left(1-\frac{1}{n}\right)^{-\eta}=$$
$$=1-\alpha\eta+0\left(\frac{1}{n}\right), \quad (n\to\infty),$$

we see that (2.18) holds, and thus theorem 4 is proved.

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