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SOLUTION OF SOME PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS BY MEANS OF THE ALGEBRAIC OPERATIONAL CALCULUS OF DISTRIBUTIONS*

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Summary

In the present paper are given the solutions of some parabolic integro-differential equations by using the algebraic operational Calculus of functions and Distributions of several variables as described in [1], [2], [3], [4], [5].

1. Preliminaries

Notations (cf. [2], [3], [4]).

$$\mathbf{N}=\{0, 1, 2, 3, \ldots\}; \quad \mathbf{Z}=\{\cdots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

$\mathbf{R}^2_+=[0, \infty[^2$; $(-\Gamma_0)=]-\infty, 0]^2$; $(-\Gamma_\lambda)=$ translated of $(-\Gamma_0)$ (cf. [1], § 2, no 1); $(\mathcal{D}_{-\Gamma_\lambda})=$ locally convex space of infinitely differentiable functions of support into $(-\Gamma_\lambda)$; $(\mathcal{D}_{-\Gamma})=\bigcup_{j\in\mathbf{Z}}(\mathcal{D}_{-\Gamma_j})=$ locally convex space of infinitely differentiable functions in $\mathbf{R}^2$, of support limited on the right.

$(\mathcal{D}_{-\Gamma})$ is the strict inductive limit of the family $(\mathcal{D}_{-\Gamma_j})_{j\in\mathbf{Z}}$

$(\mathcal{D}'_{+\Gamma})=$ topological dual space of $(\mathcal{D}_{-\Gamma})$ (cf. [2], § 2, nos 2, 3).

$(\mathcal{D}'_{\mathbf{R}^2_+})=$ convolution algebra of distribution of support into $\mathbf{R}^2_+=[0, \infty[^2$.

$(\mathcal{D}'_{\mathbf{R}^2})$ is an algebra without zeros divisors.

$\mathfrak{K}_{\mathcal{D}'_{\mathbf{R}^2_+}}=$ field of fractions of the algebra $(\mathcal{D}'_{\mathbf{R}^2_+})$.

$T_+=[0, \infty[_t$; $X_+=[0, \infty[_x$;

$(\mathcal{D}'_{T_+})$ (resp. $\mathcal{D}'_{X_+}$) $=$ convolution algebra of Distributions of support in T_+ (resp. X_+).

$(\mathfrak{K}_{\mathcal{D}'_{T_+}})$ (resp. $(\mathfrak{K}_{\mathcal{D}'_{X_+}})$) $=$ field of fractions of the convolution algebra $(\mathcal{D}'_{T_+})$ (resp. $(\mathcal{D}'_{X_+})$).

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2. Parabolic Integro-differential equations

In this paper we consider the parabolic integro-differential equations

$$\lambda \frac{\partial u}{\partial t} + \mu \frac{\partial}{\partial x} \int_0^x \frac{1}{\sqrt{\pi \eta}} u(t, x-\eta)\, d\eta = f(t, x) \tag{1}$$

$$\lambda \int_0^x \frac{\partial u(t, x-\eta)}{\partial t} \frac{\partial}{\partial \eta}\left(\frac{1}{\sqrt{\pi \eta}}\right) d\eta + \mu \frac{\partial u}{\partial x} = f(t, x) \tag{2}$$

$$\lambda \frac{\partial^2 u}{\partial x\, \partial t} + \mu \int_0^x \frac{1}{\sqrt{\pi (x-\eta)}} \frac{\partial^2 u(t, \eta)}{\partial \eta^2}\, d\eta = f(t, x) \tag{3}$$

and determine by means of the algebraic operational calculus of distributions the solutions of these equations satisfying respectively the following boundary values conditions, for $(t, x) \in \mathbf{R}_+^2$:

$$\lim_{t \to 0} u(t, x) = B(x); \tag{1'}$$

$$\lim_{x \to 0} u(t, x) = B(x); \quad \lim_{x \to 0} u(t, x) = A(t); \tag{2'}$$

$$\lim_{x \to 0} u(t, x) = A(t); \quad \lim_{t \to 0} u(t, x) = B(x); \quad \lim_{x \to 0} \frac{\partial u(t, x)}{\partial x} = c(t) \tag{3'}$$

By transfer of (1), (2), (3) into $(\mathcal{D}'_{\mathbf{R}_+^2})$ we obtain the equations:

$$\lambda \left\{\frac{\partial u}{\partial t}\right\} + \mu \left\{\frac{\partial}{\partial x} \int_0^x \frac{1}{\sqrt{\pi \eta}} u(t, x-\eta)\, d\eta\right\} = \{f(t, x)\}; \tag{4}$$

$$\lambda \left\{\int_0^x \frac{\partial u(t, x-\eta)}{\partial x} \frac{\partial}{\partial n} \frac{1}{\sqrt{\pi \eta}}\, d\eta\right\} + \mu \left\{\frac{\partial u}{\partial x}\right\} = \{f(t, x)\}; \tag{5}$$

$$\lambda \left\{\frac{\partial^2 u}{\partial t\, \partial x}\right\} + \mu \left\{\int_0^x \frac{1}{\sqrt{\pi \eta}} \frac{\partial^2 u(t, \eta)}{\partial \eta^2}\, d\eta\right\} = \{f(t, x)\}. \tag{6}$$

A) Consider, to begin, the equation (4);

We have (cf. [2], Chap. III, § 1, no 10, formula (48))

$$\left\{\frac{\partial u}{\partial t}\right\} = \frac{\partial \{u\}}{\partial t} - \delta(t) \otimes \{B(x)\} = \{\delta'(t) \otimes \delta(x)\} \overset{tx}{*} \{u\} - \delta(t) \otimes \{B(x)\} \tag{7}$$

and

$$\left\{\frac{\partial}{\partial x} \int_0^x \frac{u(t, x-\eta)}{\sqrt{\pi \eta}}\, d\eta\right\} = \frac{\partial}{\partial x} \left\{\int_0^x \frac{u(t, x-\eta)}{\sqrt{\pi \eta}}\, d\eta\right\}$$

Then (4) may be written

$$\lambda \frac{\partial \{u\}}{\partial t} + \frac{\partial}{\partial x}\left\{ \int_0^x \frac{u(t, x-\eta)}{\sqrt{\pi \eta}} d\eta \right\} = \lambda\, \delta(t) \otimes \{B(x) + \{f\} \tag{8}$$

which is equivalent in $(\mathcal{D}'_{\mathbf{R}^2_+})$ to the convolution equation

$$\left\{\lambda\, \delta'(t) \otimes \delta(x) + \mu\, \delta(t) \otimes \delta'(x) \overset{x}{*} \frac{1}{\sqrt{\pi x}}\right\} \overset{tx}{*} \{u\} = \{f\} + \lambda\, \delta(t) \otimes \{B(x)\}. \tag{9}$$

and in $(\mathfrak{K}\mathcal{D}'_{\mathbf{R}^2_+})$ to the parabolic operational equation:

$$(\lambda p \otimes l_x + \mu\, l_t \otimes \sqrt{q})\, u = f + l_t \circ B \tag{10}$$

where u (resp. f) is the element of $\mathfrak{K}\mathcal{D}'_{\mathbf{R}^2_+}$ which corresponds to the element $\{u\}$ (resp. $\{f\}$) of $(\mathcal{D}'_{\mathbf{R}^2_+})$ and B the element of $(\mathfrak{K}_{X_+})$ which corresponds to the element $\{B\}$ of $(\mathcal{D}'_{X_+})$. If $\{F_0(t, x)\}$ is the fundamental solution of the parabolic convolution equa ion

$$\left\{\lambda\, \delta(t) \otimes \delta(x) + \mu \delta(t) \otimes \delta'(x) \overset{x}{*} \left\{\frac{1}{\sqrt{\pi x}}\right\}\right\} * \{F_0\} = \delta(t) \otimes \delta(x), \tag{11}$$

then, the operator $F_0 \in (\mathfrak{K}\mathcal{D}'_{\mathbf{R}^2_+})$ which corresponds to $\{F_0\} \in (\mathcal{D}'_{\mathbf{R}^2_+})$ is a solution in $(\mathcal{D}'_{X_+})$ of the equation:

$$(\lambda p + \mu \sqrt{q})\, F_0 = 1 = l_t \otimes l_x. \tag{12}$$

where l_t (resp. l_x) is the operator of $(\mathfrak{K}\mathcal{D}'_{\mathbf{T}_+})$ (resp. $(\mathfrak{K}\mathcal{D}'_{\mathbf{X}_+})$) corresponding to $\delta(t) \in (\mathcal{D}'_{\mathbf{T}_+})$ (resp. $\delta(x) \in (\mathcal{D}'_{\mathbf{X}_+})$. But F_0 is given by (cf. [5], formula (1) and [4] formula (14)):

$$(\mathfrak{K}\mathcal{D}'_{\mathbf{R}^2_+}) \ni F_0 = \frac{1}{\lambda p + \mu \sqrt{q}} = \left\{ \frac{\mu t}{2\lambda^2 x \sqrt{\pi x}} \exp\left(\frac{-\mu^2 t^2}{4\lambda^2 x}\right)\right\} = \{F_0(t, x)\} \in (\mathcal{D}'_{\mathbf{R}^2_+}).$$

Therefore, the solution of (8) is given by

$$\{u\} = \{f\} \overset{tx}{*} \{F_0\} + \lambda \{B(x)\} \overset{x}{*} \{F_0\} \text{ as element of } (\mathcal{D}_{\mathbf{R}^2_+}) \tag{13}$$

with $\{F_0\}$ solution of the equation:

$$\lambda \frac{\partial \{F_0\}}{\partial t} + \mu \left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial \{F_0\}}{\partial x} = \delta(t) \otimes \delta(x). \tag{14}$$

equivalent to the equation:

$$\lambda \frac{\partial \{F_0\}}{\partial t} + \mu \frac{\partial}{\partial x} \left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \{F_0\} = \delta(t) \otimes \delta(x). \tag{15}$$

whence for $x>0$, $t>0$ the equation:

(16) $$\lambda \frac{\partial F_0}{\partial t} + \mu \frac{1}{\sqrt{\pi x}} \overset{x}{*} \frac{\partial F_0}{\partial x} = 0$$

whence for $(t, x) \in \mathbf{R}^2_+$, the solution of (1), given by

(17) $$u(t, x) = \int_0^t \int_0^x f(t-\xi, x-\eta) F_0(\xi, \eta)\, d\xi\, d\eta + \lambda \int_0^x B(x-\eta) F_0(t, \eta)\, d\eta$$

Let us show that $\lim\limits_{\substack{t\to 0\\ t>0}} u(t, x) = B(x)$.

From (14) we get

$$\lim_{t\to 0} u(t, x) = \lambda \int_0^x B(x-\eta) \lim_{t\to 0} F_0(t, \eta)\, d\eta.$$

Therefore, we must determine $\lim\limits_{t\to 0} F_0(t, x)$.

The function $F_0(t, x)$ admits a singularity for $x=t=0$, and we shall determine $\lim\limits_{t\to 0} F_0$ in this point, the only where $F_0 \neq 0$ for $t\to 0$. To do this, we proceed as follows:

Let $(\alpha_j(x))_{\gamma \in \mathbf{N}}$ be a sequence of functions of the space $(\mathcal{D}_{X_+})$ (of infinitely differentiable functions of support in X_+), which converges to $\delta(x)$ in $(\mathcal{D}'_{X_+})$. (For example, we can take $\alpha_j>0$, of supports which converge to $0 \in \mathbf{R}$ and such that $\int_{\mathbf{R}} \alpha_j(x)\, dx = 1$, $\forall j \in \mathbf{N}$).

Under these conditions, if we consider in $(\mathcal{D}'_{X_+})$ the convolution

$$\{F_0(t, x)\} \overset{x}{*} \{\alpha_j(x)\} = \left\{\frac{1}{2\lambda\sqrt{\pi}} \int_0^x \frac{\alpha_j(x-\eta)}{\eta^{3/2}} \left(\frac{\mu t}{\lambda}\right) \exp\left(\frac{-\mu^2 t^2}{4\lambda^2 \eta}\right) d\eta\right\}$$

then we obtain

$$\lim_{\substack{t\to 0\\ x\to 0}} \{F_0(t, x)\} \overset{x}{*} \{\alpha_j(x)\} = \left\{\lim_{\substack{t\to 0\\ x\to 0}} \frac{2}{2\lambda\sqrt{\pi}} \int_0^x \frac{\alpha_j(x-\eta)}{\eta^{3/2}} \left(\frac{\mu t}{\lambda}\right) \exp\left(\frac{-\mu^2 t^2}{4\lambda^2 \eta}\right) d\eta\right\}.$$

But (cf. [6], chap. XXIX, No 544, p. 308):

(18) $$\left\{\lim_{\substack{t\to 0\\ x\to 0}} \frac{1}{\sqrt{\pi}} \int_0^x \frac{\alpha_j(x-\eta)}{2\lambda^2 \eta \sqrt{\eta}} (\mu t) \exp\left(\frac{-\mu^2 t^2}{4\lambda^2 \eta}\right) d\eta\right\} = \{\alpha_j(0)\} \frac{1}{\lambda}.$$

On the other hand, according to the continuity of the convolution in $(\mathcal{D}'_{\mathbf{R}^2_+})$, (18) gives us:

$$\lim_{j\to\infty} \Big(\lim_{\substack{t\to 0\\ x\to 0}} \{F_0\} \overset{x}{*} \{\alpha_j(x)\} = \lim_{\substack{t\to 0\\ x\to 0}} \{F_0(t, x)\} \overset{x}{*} \delta(x) = \lim_{j\to\infty} \frac{1}{\lambda} \{\alpha_j(0)\} = \frac{1}{\lambda} \delta(x),$$

whence

(18′) $$\lim_{t\to 0} \{F_0(t, x)\} = \frac{1}{\lambda} \delta(x)$$

Therefore,

$$\lim_{t\to 0} u(t,x) = \lambda \int_0^x B(x-\eta)\frac{1}{\lambda}\delta(\eta)\,d\eta = B(x),$$

i.e. $u(t,x)$ given by (17) satisfies the condition (1′). From (17) and (18)′ we obtain:

$$\frac{\partial u}{\partial t} = \frac{1}{\lambda} f(t,x) + \int_0^t\int_0^x f(\xi,\eta)\frac{\partial F_0(t-\xi, x-\eta)}{\partial t}\,\partial\xi\, d\eta +$$

$$+\lambda\int_0^x B(x-\eta)\frac{\partial F_0(t,\eta)}{\partial t}\,d\eta$$

and

$$\frac{\partial u}{\partial x} = \int_0^t\int_0^x f(\xi,\eta)\frac{\partial F_0(t-\xi,x-\eta)}{\partial x}\,d\xi\,d\eta + \lambda\int_0^x B(\eta)\frac{\partial F_0(t,x-\eta)}{\partial x}\,d\eta$$

since $F_0(t,0)=0$ for $t>0$. Then:

$$\frac{1}{\sqrt{\pi x}} \overset{x}{*} \frac{\partial u}{\partial x} = \frac{1}{\sqrt{\pi x}} \overset{x}{*} f \overset{xy}{*} \frac{\partial F_0}{\partial x} + \lambda\frac{1}{\sqrt{\pi x}} \overset{x}{*} B \overset{x}{*} \frac{\partial F_0}{\partial x}.$$

Whence:

$$\lambda\frac{\partial u}{\partial t} + \mu\frac{\partial}{\partial x}\int_0^x \frac{1}{\sqrt{\pi\eta}}\,u(t-\eta)\,d\eta =$$

$$= f(t,\xi) + \int_0^t\int_0^x f(\xi,\eta)\,\lambda\frac{\partial F_0(t-\xi,x-\eta)}{\partial t}\,d\xi\,d\eta +$$

$$\lambda^2\int_0^x B(\eta)\frac{\partial F_0(t,x-\eta)}{\partial t}\,\partial\eta +$$

$$+\int_0^t\int_0^x f(\xi,\eta)\left(\mu\frac{1}{\sqrt{\pi x}} \overset{x}{*} \frac{\partial F_0}{\partial x}\right)d\xi\,d\eta +$$

$$+\lambda\mu\int_0^x B(\eta)\left(\frac{1}{\sqrt{\pi x}}\right) \overset{x}{*} \frac{\partial F_0(t,x-\eta)}{\partial x}\,d\eta =$$

$$= f(t,\xi) + \int_0^t\int_0^x f(\xi,\eta)\left[\lambda\frac{\partial F_0(t-\xi,x-\eta)}{\partial t} + \mu\frac{1}{\sqrt{\pi y}} *\right.$$

$$* \frac{\partial F_0(t-\xi, x-\eta)}{\partial x}\Big] d\xi\, d\eta$$

$$+\lambda \int_0^x B(\eta)\left[\lambda \frac{\partial F_0(t, x-\eta)}{\partial t} + \mu \frac{1}{\sqrt{\pi x}} \overset{x}{*} \frac{\partial F_0(t, x-\eta)}{\partial x}\right] d\eta$$

But (cf. (16)):

$$\lambda \frac{\partial F_0}{\partial t} + \mu \frac{1}{\sqrt{\pi x}} * \frac{\partial F_0}{\partial x} = 0.$$

Therefore, $u(t, x)$ given by (17) satisfies the problem (1), (1′).

Remark: For supplementary properties of the function $F_0(t, x)$ c.f. [3] § 2, no 3.

B) Consider the parabolic int. diff. equation (2) with the boundary values conditions (2′).

The int. diff. eq. (2) is equivalent to

$$\lambda \frac{\partial}{\partial x} \int_0^x \frac{1}{\sqrt{\pi (x-\eta)}} \frac{\partial u(t, \eta)}{\partial t} d\eta + \mu \frac{\partial u}{\partial x} = f(t, x) \tag{19}$$

By transfer of (19) into $(\mathcal{D}'_{R^2_+})$ we have

$$\lambda \left\{\frac{\partial}{\partial x} \int_0^x \frac{1}{\sqrt{\pi (x-\eta)}} \frac{\partial u(t, \eta)}{\partial t} d\eta\right\} + \mu \left\{\frac{\partial u}{\partial x}\right\} = \{f\}. \tag{20}$$

But

$$\left\{\frac{\partial u}{\partial x}\right\} = \frac{\partial \{u\}}{\partial x} - \delta(x) \otimes A(t), \tag{21}$$

by virtue of (2′). Likewise

$$\left\{\frac{\partial}{\partial x} \int_0^x \frac{1}{\sqrt{\pi (x-\eta)}} \frac{\partial u(t, \eta)}{\partial t} d\eta\right\} = \lambda \left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial^2 \{u\}}{\partial t\, \partial x} \tag{22}$$

and

$$\left\{\frac{\partial u}{\partial t}\right\} = \frac{\partial \{u\}}{\partial t} - \delta(t) \otimes B(x) \tag{23}$$

by virtue of (2′).

According to (21), (22) and (23), the equation (20) is transformed in the convolution equation:

$$\left\{\lambda \delta'(t) \otimes \frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{\pi x}}\right\} + \mu \delta(t) \otimes \delta'(x)\right\} \overset{tx}{*} \{u\} = \tag{24}$$

$$= \{f\} + \mu \{A(t)\} \otimes \delta(x) + \lambda \delta(t) \otimes \frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \{B(x)\},$$

which is equivalent to the equation:

$$(25)\qquad \lambda\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial^2\{u\}}{\partial t\,\partial x}+\mu\frac{\partial\{u\}}{\partial x}=\{f\}+\mu A(t)\otimes\delta(x)+$$
$$+\lambda\delta(t)\otimes\frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \{B(x)\}.$$

Let $\{G_0(t, x)\}$ be the fundamental solution, of (25) i.e.

$$(26)\qquad \lambda\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial^2\{G_0\}}{\partial t\,\partial x}+\mu\frac{\partial\{G_0\}}{\partial x}=\delta(t)\otimes\delta(x).$$

By transfer of (26) into the field $(\mathfrak{K}_{\mathcal{D}'_{\mathbf{R}^2_+}})$ we get the parabolic operational equation

$$(27)\qquad (\lambda p\sqrt{q}+\mu q)\,G_0=1,$$

whence for G_0, the operational relation (cf. [3] § 2, no 5, formula (59), and [5] formula (2)):

$$(28)\qquad (\mathfrak{K}_{\mathcal{D}'_{\mathbf{R}^2_+}})\ni\frac{1}{\sqrt{q}\,(\lambda p+\mu\sqrt{q})}=G_0\Leftrightarrow\left\{\frac{1}{\lambda}\,\frac{1}{\sqrt{\pi x}}\exp\left(\frac{-\mu^2t^2}{4\lambda^2x}\right)\right\}=$$
$$=\{G_0(t, x)\}\in(\mathcal{D}'_{\mathbf{R}^2_+}).$$

From (27) we get

$$G_0(p, q)=\frac{1}{\sqrt{q}\,(\lambda p+\mu q)}\Leftrightarrow G_0(t, q)=\frac{1}{\lambda}\,\frac{1}{\sqrt{q}}\left(\frac{-\mu t}{\lambda}\sqrt{q}\right)$$

whence:

$$\frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \{G_0(t, x)\}=q\frac{1}{\sqrt{q}}\,\frac{1}{\lambda}\,\frac{1}{\sqrt{q}}\exp\left(\frac{-\mu t}{\lambda}\sqrt{q}\right)=\frac{1}{\lambda}\exp\left(\frac{-\mu t}{\lambda}\sqrt{q}\right).$$

But (cf. [5], formula (1) and [3], § 2, formula (45)) we have

$$(29)\qquad \frac{1}{\lambda}\exp\left(\frac{-\mu t}{\lambda}\sqrt{q}\right)=\frac{1}{\lambda^2}\left\{\frac{\mu t}{2x\sqrt{\pi x}}\exp\left(\frac{-\mu^2t^2}{4\lambda^2x}\right)\right\}=\{F_0(t, x)\},$$

whence

$$(30)\qquad \frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \{G_0(t, x)\}=\{F_0(t, x)\}.$$

Therefore, the solution of (25) is given by

$$(31)\qquad \{u(t, x)\}=\{f(t, x)\} \overset{tx}{*} \{G_0(t, x)\}+\mu\{A(t)\} \overset{t}{*} \{G_0(t, x)\}+$$
$$+\lambda\frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \{G_0(t, x)\} \overset{x}{*} \{B(x)\}$$

Let us show that $\{u(t, x)\}$ given by (31) satisfies the equation (25). Indeed, from (31) we obtain

$$(32)\qquad \frac{\partial\{u\}}{\partial x}=\{f\} \overset{tx}{*} \frac{\partial\{G_0\}}{\partial x}+\mu\{A(t)\} \overset{t}{*} \frac{\partial\{G_0\}}{\partial x}+\frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial\{G_0\}}{\partial x}$$

and (31), (32) give us

$$\lambda\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial^2\{u\}}{\partial t\,\partial x}+\mu\frac{\partial\{u\}}{\partial x}=\{f\} \overset{tx}{*} \left[\lambda\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial^2\{G_0\}}{\partial t\,\partial x}+\mu\frac{\partial\{G_0\}}{\partial x}\right]+$$
$$+\mu\{A(t)\} \overset{t}{*} \left[\lambda\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial^2\{G_0\}}{\partial t\,\partial x}+\mu\frac{\partial\{G_0\}}{\partial x}\right]$$
$$+\lambda\frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \{B(x)\} \overset{x}{*} \left[\lambda\left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial^2\{G_0\}}{\partial t\,\partial x}+\mu\frac{\partial\{G_0\}}{\partial x}\right].$$

whence, according to (26), we obtain the equation (25). On the other hand keeping in mind the relation (30), we can write the solution (31) as follows:

(33) $$\{u(t,x)\}=\{f\} \overset{tx}{*} \{G_0(t,x)\}+\mu\{A(t)\} \overset{t}{*} \{G_0(t,x)\}+$$
$$+\lambda B(x) \overset{x}{*} \{F_0(t,x)\}.$$

Therefore, the solution of (2) is given by

(34) $$u(t,x)=\int_0^t\int_0^x f(\xi,\eta)\,G_0(t-\xi,x-\eta)\,d\xi\,d\eta+$$
$$+\lambda\int_0^x B(\eta)\,F_0(t,x-\eta)\,d\eta+\mu\int_0^t A(\xi)\,G_0(t-\xi,x)\,d\xi.$$

Let us now prove that $u(t,x)$ given by (34) satisfies the problem (2), (2'). Indeed, we have

$$\lim_{t\to 0} u(t,x)=\lambda\int_0^x B(\eta)\lim_{t\to 0} F_0(t,x-\eta)\,d\eta$$

But $\lim_{t\to 0} F_0(t,x-\eta) \Leftrightarrow \lim_{t\to 0}\frac{1}{\lambda}\exp\left(\frac{-\mu t}{\lambda}\sqrt{q}\right) \Leftrightarrow \frac{l_x}{\lambda} \Leftrightarrow \frac{\delta_{(x-\eta)}}{\lambda}$

$$\left(\text{since } \exp\left(\frac{-\mu t}{\lambda}\sqrt{q}\right)=l_x-\left(\frac{\mu t}{\lambda}\right)\sqrt{q}+\left(\frac{\mu t}{\lambda}\right)^2\frac{1}{2!}q\ldots\right),$$

whence

$$\lim_{t\to 0} u(t,x)=B(x).$$

Likewise,

$$\lim_{x\to 0} u(t,x)=\mu\lim_{x\to 0}\int_0^t A(\xi)\,G_0(t-\xi,x)\,d\xi=$$
$$=\mu\int_0^t A(\xi)\lim_{x\to 0} G_0(t-\xi,x)\,d\xi.$$

But

$$\mu\lim_{x\to 0} G_0(t-\xi,x)=\delta(t-\xi).$$

Indeed, for $x>0$, we have

$$\frac{\mu}{2\lambda\sqrt{\pi x}}\int_{-\infty}^{\infty} -\exp\left(-\frac{\mu^2 t^2}{4\lambda^2 x}\right) dt = \int_{-\infty}^{\infty} - e^{-u^2}\, du = 1$$

and for any $b>0$,

$$\frac{\mu}{2\lambda}\cdot\frac{1}{\sqrt{\pi x}}\int_{b}^{\infty}\exp\left(\frac{-\mu^2 t^2}{4\lambda^2 x}\right) dt \leq \frac{\lambda}{\mu}\sqrt{x}\int_{\frac{b^2\mu^2}{4\lambda^2 x}}^{\infty} e^{-u^2}\, du =$$

$$= \frac{\lambda}{\mu}\sqrt{x}\,\exp\left(\frac{-b^2\mu^2}{4\lambda^2 x}\right) \to 0 \text{ for } x\to 0.\quad x>0.$$

Therefore, the integral over $[b, \infty[$, $b>0$, has for limit 0, for $x\to 0$. The same is true for the interval $]-\infty, a]$, $a<0$. Then, (cf. [7], chap. 1, § 2, no 5) for each $x>0$ the function

$$g_x(t) = \frac{\mu}{2\lambda}\cdot\frac{1}{\sqrt{\pi x}}\exp\left(\frac{-\mu^2 t^2}{4\lambda^2 x}\right) \to \delta(t) \text{ when } x\to 0,\quad x>0$$

whence

$$\mu \lim_{x\to 0} G_0(t-\xi, x) = \delta(t-\xi)$$

and

$$\lim_{x\to 0} u(t,x) = \int_0^t A(\xi)\,\delta(t-\xi)\, d\xi = A(t).$$

On the other hand, we have $G_0(0,x) = \dfrac{1}{\lambda\sqrt{\pi x}}$, whence

$$\frac{\partial u}{\partial t} = \frac{1}{\lambda}\int_0^x f(t,\eta)\frac{1}{\sqrt{\pi(x-\eta)}}\, d\eta + \lambda\int_0^x B(\eta)\frac{\partial F_0(t, x-\eta)}{\partial t}\, d\eta +$$

$$+\mu\int_0^t A(\xi)\,\frac{\partial G_0(t-\xi, x)}{\partial t}\, d\xi + \mu A(t)\frac{1}{\lambda\sqrt{\pi x}} +$$

$$+\int_0^t\int_0^x f(\xi,\eta)\frac{\partial G_0(t-\xi, x-\eta)}{\partial t}\, d\xi\, d\eta$$

whence

$$\frac{\lambda}{\sqrt{\pi x}} \overset{x}{*} \frac{\partial u}{\partial t} + \mu u = \int_0^x f(t,\eta)\, d\eta +$$

$$+\lambda\int_0^x B(\eta)\left[\frac{\lambda}{\sqrt{\pi x}} \overset{x}{*} \frac{\partial F_0(t, x-\eta)}{\partial t} + \mu F_0(t, x-\eta)\right] d\eta +$$

$$+\mu\int_0^t A(\xi)\left[\frac{\lambda}{\sqrt{\pi x}}\overset{x}{*}\frac{\partial G_0(t-\xi,x)}{\partial t}+\mu G_0(t-\xi,x)\right]d\xi+$$

$$+\mu\int_0^t\int_0^x f(\xi,\eta)\left[\frac{\lambda}{\sqrt{\pi x}}\overset{x}{*}\frac{\partial G_0(t-\xi,x-\eta)}{\partial t}+\mu G(t-\xi,x-\eta)\right]d\xi\,d\eta+\mu A(t).$$

But (cf. [3], § 2, formula (57)):

$$G_0(t,x)=\frac{1}{\sqrt{\pi x}}\overset{x}{*}F_0(t,x)$$

and (loc. cit., no 5)

$$\lambda\frac{\partial G_0(t,x)}{\partial t}+\mu F_0(t,x)=0,$$

whence

$$\lambda\ \frac{1}{\sqrt{\pi x}}\overset{x}{*}\frac{\partial F_0}{\partial t}+\mu F_0=0 \tag{35}$$

Likewise

$$\lambda\frac{1}{\sqrt{\pi x}}\overset{x}{*}\frac{\partial G_0}{\partial t}+\mu G_0=$$

$$=\lambda\frac{1}{\sqrt{\pi x}}\overset{x}{*}\frac{1}{\sqrt{\pi x}}*F_0(t,x)+\mu\frac{1}{\sqrt{\pi x}}\overset{x}{*}F_0(t,x)=$$

$$=\frac{1}{\sqrt{\pi x}}\overset{x}{*}\left[\lambda\frac{1}{\sqrt{\pi x}}\overset{x}{*}\frac{\partial F_0}{\partial t}+\mu F_0\right]=0.$$

whence

$$\lambda\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\pi x}}\overset{x}{*}\frac{\partial u}{\partial t}\right)+\mu\frac{\partial u}{\partial x}=f(t,x).$$

Therefore, u given by (34) satisfies the problem (2), (2′).

C) Consider finally the int. diff. equation (3) which by transfer into $(\mathcal{D}'_{\mathbf{R}_+^2})$ is given by

$$\lambda\left\{\frac{\partial^2 u}{\partial t\,\partial x}\right\}+\mu\left\{\int_0^x\frac{1}{\sqrt{\pi(x-\eta)}}\frac{\partial^2 u(t,\eta)}{\partial\eta^2}d\eta\right\}=\{f\}. \tag{36}$$

But (cf. [2], chap. III, § 1, no 10, formula (54) and (3′)):

$$\frac{\partial^2\{u\}}{\partial t\,\partial x}=\left\{\frac{\partial^2 u}{\partial t\,\partial x}\right\}+\delta(t)\otimes\{\delta'(x)\overset{x}{*}\{B(x)\}+ \tag{37}$$

$$+\{\delta'(t)\overset{x}{*}\{A(t)\otimes\delta(x)-\{\delta(t)\otimes\delta(x)A(0)$$

and (loc. cit. formula (51)):

$$\frac{\partial^2\{u\}}{\partial x^2}=\left\{\frac{\partial^2 u}{\partial x^2}\right\}+\{A(t)\}\otimes\delta'(x)+\{c(t)\}\otimes\delta(x) \tag{38}$$

On the other hand, according to (38), we have:

$$\left\{\int_0^x \frac{1}{\sqrt{\pi(x-\eta)}} \frac{\partial^2 u(t,\eta)}{\partial \eta^2} d\eta\right\} = \left\{\frac{1}{\sqrt{\pi y}}\right\} \overset{x}{*} \left\{\frac{\partial^2 u}{\partial x^2}\right\} = \tag{39}$$

$$= \left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \left\{\frac{\partial^2 \{u\}}{\partial x^2}\right\} - \{A(t)\} \otimes \left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \{\delta'(x)\} - \left\{\frac{1}{\sqrt{\pi x}}\right\} \otimes \{c(t)\}.$$

Then (36) is transformed in the following equation of $(\mathcal{D}'_{R^2_+})$:

$$\lambda \frac{\partial^2 \{u\}}{\partial t\, \partial x} + \mu \left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial^2 \{u\}}{\partial x^2} = \{f\} + \lambda \{\delta(t) \otimes \delta'(x)\} \overset{x}{*} \{B(x)\} + \tag{40}$$

$$+ \lambda (\delta'(t) \overset{t}{*} \{A(t)\}) \otimes \delta(x) - \lambda (\delta(t) \otimes \delta(x)) A(0) +$$

$$+ \mu \{A(t)\} \otimes \left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \{\delta'(x)\} + u \{c(t)\} \otimes \left\{\frac{1}{\sqrt{\pi x}}\right\}.$$

Let $\{H_0(t,x)\}$ be the fundamental solution of (40), i.e. solution of the equation

$$\lambda \frac{\partial^2 \{H_0\}}{\partial t\, \partial x} + \mu \left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \frac{\partial^2 \{H_0\}}{\partial x^2} = \delta(t) \otimes \delta(x), \tag{41}$$

which is equivalent to the convolution equation of $(\mathcal{D}'_{R^2_+})$

$$\left[\lambda \delta'(t) \otimes \delta'(x) + \mu \left\{\frac{1}{\sqrt{\pi x}}\right\} \overset{x}{*} \delta''(x) \otimes \delta(t)\right] \overset{tx}{*} \{H_0\} = \delta(t) \otimes \delta(x) \tag{42}$$

By transfer of (42) into $(\mathfrak{R}_{\mathcal{D}'_{R^2_+}})$, we obtain the parabolic opeiational equation:

$$(\lambda p \otimes q + \mu q \sqrt{q})\, H_0 = 1, \tag{43}$$

whence (cf. [5], formula (3) and [3] § 2, no 6, formula (74)), the operational relation.

$$(\mathfrak{R}_{\mathcal{D}'_{R^2_+}}) \ni \frac{1}{\lambda p \otimes q + \mu q \sqrt{q}} = \left\{\frac{1}{\lambda} \operatorname{erf}_c \left(\frac{\mu t}{2 \lambda \sqrt{\pi x}}\right)\right\} = \{H_0\} \in (\mathcal{D}'_{R^2_+}). \tag{44}$$

Therefore, the solution of (40) is given by:

$$\{u(t,x)\} = \{f\} \overset{tx}{*} \{H_0\} + \lambda \{H_0\} \overset{x}{*} \frac{d\{B\}}{dx} + \lambda \{H_0\} \overset{t}{*} \frac{d\{A\}}{dt} - \tag{45}$$

$$- \lambda \{H_0\} A(0) + \mu \left(\{A(t)\} \otimes \delta'(x) \overset{x}{*} \left\{\frac{1}{\sqrt{\pi x}}\right\}\right) \overset{tx}{*} \{H_0\} +$$

$$+ \mu \{H_0\} \overset{tx}{*} \left(\left\{\frac{1}{\sqrt{\pi x}}\right\} \otimes \{c(t)\}\right)$$

which can be written as follows:

$$(46)\qquad \{u(t,x)\}=\{f\}\overset{tx}{*}\{H_0\}+\lambda\{H_0\}\overset{x}{*}\left\{\frac{dB}{dx}\right\}+\lambda\{H_0\}B(0)+$$
$$+\lambda\left\{\frac{\partial H}{\partial t}\right\}\overset{t}{*}\{A(t)\}+\lambda H_0(0,x)\otimes\{A(t)\}-\lambda\{H_0\}A(0)+$$
$$+\mu\left(\{A(t)\}\otimes\delta'(x)\overset{x}{*}\left\{\frac{1}{\sqrt{\pi x}}\right\}\right)\overset{tx}{*}\{H_0\}+$$
$$+\mu\{H_0\}\overset{tx}{*}\left(\left\{\frac{1}{\sqrt{\pi x}}\right\}X\{c(t)\}\right)$$

For $x>0$, $t>0$, we have $\delta'(x)=0$; hence:

$$(47)\qquad u(t,x)=\int_0^t\int_0^x f(\xi,\eta)H_0(t-\xi,x-\eta)\,d\xi\,d\eta+\lambda\int_0^x H_0(t,x-\eta)\frac{dB}{d\eta}d\eta+$$
$$+\lambda\int_0^t\frac{\partial H(\xi,x)}{\partial\xi}A(t-\xi)\,d\xi+A(t)+$$
$$+\mu\int_0^t c(\xi)\left(\int_0^x\frac{1}{\sqrt{\pi(x-\eta)}}H_0(t-\xi,\eta)\,d\eta\right)d\xi$$

On the other hand, we have
(cf. [3], § 2, no 5, tormula (64)):

$$(48)\qquad G_0(t,x)=\int_0^x\frac{1}{\sqrt{\pi(x-\eta)}}F_0(t,\eta)\,d\eta$$

and (loc. cit. no 7, formula (79)):

$$(49)\qquad \frac{\partial H_0}{\partial x}=F_0(t,x)$$

hence:

$$(50)\qquad \int_0^x\frac{1}{\sqrt{\pi(x-\eta)}}\frac{\partial H_0(t-\xi,\eta)}{\partial\eta}d\eta=\frac{1}{\lambda}\int_0^x\frac{1}{\sqrt{\pi(x-\eta)}}F_0(t-\xi,\eta)\,d\eta=$$
$$=G_0(t-\xi,x)$$

Moreover, the relation (loc. cit. formula (70))

$$\{H_0(t,x)\}=\left\{\frac{1}{\sqrt{\pi x}}\right\}\overset{x}{*}\{G_0(t,x)\}\ \text{give us}$$

$$\left\{\frac{1}{\sqrt{\pi x}}\right\}\overset{x}{*}\{H_0(t,x)\}=Y(x)\overset{x}{*}\{G_0(t,x)\}$$

hence

$$\int_0^x \frac{1}{\sqrt{\pi(x-\eta)}} H_0(t-\xi\,\eta)\,d\eta = \int_0^x G_0(t-\xi\,\eta)\,d\eta. \tag{51}$$

Therefore the solution (47) can be written as follows

$$u(t,x) = \int_0^t \int_0^x f(\xi,\eta) H_0(t-\xi, x-\eta)\,d\xi\,d\eta + \lambda \int_0^x H_0(t, x-\eta) \frac{dB}{d\eta}\,d\eta + \tag{52}$$

$$+\lambda \int_0^t \frac{\partial H_0(\xi,x)}{\partial \xi} A(t-\xi)\,d\xi + A(t) +$$

$$+\mu \int_0^t \int_0^x G_0(t-\xi, x-\eta)\,C(\xi)\,d\xi\,d\eta.$$

Let us show that $u(t,x)$ given by (52) satisfies the conditions (3′). We have

$$\lim_{t\to 0} u(t,x) = \lambda \lim_{t\to 0} \int_0^x H_0(t, x-\eta)\frac{dB}{d\eta}\,d\eta +$$

$$+\lambda \lim_{t\to 0} \int_0^t \frac{\partial H_0(\xi,x)}{\partial \xi} A(t-\xi)\,d\xi + A(0) +$$

$$+\mu \lim_{t\to 0} \int_0^t \int_0^x G_0(t-\xi, x-\eta)\,C(\xi)\,d\xi\,d\eta.$$

But

$$\lim_{t\to 0} H_0(t, x-\eta) = \frac{1}{\lambda} \Rightarrow \lambda \lim_{t\to 0} \int_0^x H_0(t, x-\eta)\frac{dB}{d\eta}\,d\eta = B(x) - B(0).$$

On the other hand, we have:

$$\frac{\partial H_0(\xi,x)}{\partial \xi} = -\frac{u}{\lambda} G_0(t,x), \text{ whence } \left|\frac{\partial H_0}{\partial \xi}\right| = -\frac{\partial H_0}{\partial \xi}$$

and

$$\left|\int_0^t \frac{\partial H_0(\xi,x)}{\partial \xi} A(t-\xi)\,d\xi\right| \leqslant \sup_{0\leqslant t\leqslant a} |A(t)| \int_0^t -\frac{\partial H_0(\xi,x)}{\partial \xi}\,d\xi =$$

$$= \sup_{0\leqslant t\leqslant a} |A(t)|\,[H_0(0,x) - H_0(t,x)].$$

Whence

$$\lim_{t\to 0}\left|\int_0^t \frac{\partial H_0(\xi,x)}{\partial \xi} A(t-\xi(d\xi\right| = \left|\lim_{t\to 0}\int_0^t \frac{\partial H_0(\xi,x)}{\partial \xi} A(t-\xi)\,d\xi\right| = 0, \text{ i.e.}$$

$$\lim_{t\to 0}\int_0^t \frac{\partial H_0(\xi,x)}{\partial \xi} A(t-\xi)\,d\xi = 0.$$

Moreover

$$\left|\int_0^t\int_0^x G_0(t-\xi, x-\eta)\, C(\xi)\, d\xi\, d\eta\right| \leqslant \int_0^t |C(\xi)| \left|\int_0^x G_0(t-\xi, \eta)\, d\eta\right| \leqslant$$

$$\leqslant \sup_{0\leqslant t\leqslant a} |C(t)| \cdot \sup_{0\leqslant t\leqslant a} G_0(t, x)\cdot tx = \sup_{0\leqslant t\leqslant a} |C(t)| \frac{\sqrt{x}}{\lambda\sqrt{\pi}}\cdot t.$$

Whence

$$\lim_{t\to 0}\int_0^t\int_0^x G_0(t-\xi, x-\eta)\, C(\xi)\, d\xi\, d\eta = 0.$$

Therefore $\lim_{t\to 0} u(t, x) = B(x)$, i.e. $u(t, x)$ given by (52) satisfies the first condition (3′).

In the same way, we obtain from (52):

$$\lim_{x\to 0} u(t, x) = \lambda \lim_{x\to 0}\int_0^t \frac{\partial H_0(\xi, x)}{\partial\xi} A(t-\xi)\, d\xi + A(t) +$$

$$+\mu \lim_{x\to 0}\int_0^t\int_0^x G_0(t-\xi, x-\eta)\, C(\xi)\, d\xi\, d\eta.$$

But

$$\left\{\int_0^t \frac{\partial H_0(\xi, x)}{\partial\xi} A(t-\xi)\, d\xi\right\} \in (\mathcal{D}'_+)_t \Rightarrow$$

$$\Rightarrow \left\{\int_0^t \frac{\partial H_0(\xi, x)}{\partial\xi} A(t-\xi)\, d\xi\right\} = \left\{\frac{\partial H_0}{\partial t}\right\} \overset{t}{*} \{A(t)\} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0}\left\{\int_0^t \frac{\partial H_0(\xi, x)}{\partial\xi} A(t-\xi)\, d\xi\right\} = \lim_{x\to 0}\left\{\frac{\partial H_0(t, x)}{\partial t}\right\} \overset{t}{*} \{A(t)\}$$

and

$$\lim_{x\to 0}\left\{\frac{\partial H_0(t, x)}{\partial t}\right\} = \lim_{x\to 0} < \frac{\partial\{H_0\}}{\partial t},\ \underset{\varphi\in(\mathcal{D}_-)}{\varphi(t)} > =$$

$$= < \lim_{x\to 0} H_0(t, x),\ -\varphi'(t) > = < 0,\ \varphi'(t) > =$$

$$= \{0\} \in (\mathcal{D}'_+)_t \Rightarrow \lim_{x\to 0}\int_0^t \frac{\partial H_0(\xi, x)}{\partial\xi} A(t-\xi)\, d\xi = 0;$$

Since

$$\Rightarrow \lim_{x\to 0}\left\{\int_0^t \frac{\partial H_0(\xi, x)}{\partial\xi} A(t-\xi)\, d\xi\right\} = \left\{\lim_{x\to 0}\int_0^t \frac{\partial H_0(\xi, x)}{\partial\xi} A(t-\xi)\, d\xi\right\}.$$

Likewise

$$\left\{\int_0^t\int_0^x G_0(t-\xi, x-\eta)\ C(\xi)\, d\xi\, d\eta\right\} = \{C(t)\} \overset{t}{*} \left\{\int_0^x G_0(t, \eta)\, d\eta\right\} \in (\mathcal{D}'_+)_t.$$

$$\Rightarrow \lim_{x\to 0}\{c(t)\} \overset{t}{*} \left\{\int_0^x G_0(t, \eta)\, d\eta\right\} = \{c(t)\} \overset{t}{*} \lim_{x\to 0}\left\{\int_0^x G_0(t, \eta)\, d\eta\right\}$$

and

$$\left|\int_0^x G_0(t,\eta)\,d\eta\right| \le \int_0^x |G_0(t,\eta)\,d\eta| \le \frac{1}{\lambda\sqrt{\pi x}}\cdot\int_0^x d\eta = \frac{1}{\lambda\sqrt{\pi}}\sqrt{x}.$$

$$\Rightarrow \lim_{x\to 0}\int_0^x G_0(t,\eta)\,d\eta = 0 \Rightarrow \lim_{x\to 0}\int_0^t\int_0^x G_0(t-\xi,\ x-\eta)\,C(\xi)\,d\xi\,d\eta = 0.$$

Therefore,

$$\lim_{x\to 0} u(t,x) = A(t)$$

and $u(t, x)$ satisfies also the second condition (3′).

At last, from (52) we obtain

$$\frac{\partial u}{\partial x} = \int_0^t f(\xi,x)\,H_0(t-\xi,0)\,d\xi + \int_0^t\int_0^x f(\xi,\eta)\,\frac{\partial H_0(t-\xi,\ x-\eta)}{\partial x}\,d\xi\,d\eta +$$

$$+\lambda\int^x \frac{\partial H_0(t,\ x-\eta)}{\partial x}\,\frac{dB}{d\eta}\,d\eta + \lambda\int_0^t \frac{\partial^2 H_0(\xi,x)}{\partial\xi\,\partial x}\,A(t-\xi)\,d\xi +$$

$$+\mu\int_0^t G_0(t-\xi,\ x)\,c(\xi)\,d\xi.$$

But $$H_0(t,0)=0 \Rightarrow \int_0^t f(\xi,x)\,H_0(t-\xi,0)\,d\xi = 0.$$

and

$$\left|\int_0^t\int_0^x f(\xi,\eta)\,\frac{\partial H_0(t-\xi,\ x-\eta)}{\partial x}\,d\xi\,d\eta\right| = \left|\int_0^t\int_0^x f(\xi,\ x-u)\,\frac{\partial H_0(t-\xi,u)}{\partial u}\,d\xi\,du\right|$$

$$\le \sup_{\substack{0\le t\le a\\ 0\le x\le b}}|f(t,x)|\int_0^t\left|\int_0^x \frac{\partial H_0(t-\xi,u)}{\partial u}\,du\right|d\xi.$$

$$\le \sup_{\substack{0\le t\le a\\ 0\le x\le b}}|f(t,x)|\int_0^t |H_0(\xi,x)-H_0(\xi,0)|\,d\xi,$$

whence

$$\lim_{x\to 0}|H_0(\xi,x)-H_0(\xi,0)| = 0 \Rightarrow$$

$$\lim_{x\to 0}\int_0^t\int_0^x f(\xi,\eta)\,\frac{\partial H_0(t-\xi,\ x-\eta)}{\partial x}\,d\xi\,d\eta = 0.$$

In the same way we find that

$$\lim_{x\to 0} \int_0^x \frac{\partial H_0(, x-\eta)}{\partial x} \frac{dB}{\partial \eta} d\eta = 0.$$

On the other hand

$$\left| \int_0^t \frac{\partial^2 H_0(\xi, x)}{\partial \xi \, \partial \xi} A(t-\xi)\, d\xi \right| = \left| \int_0^t \frac{\partial F_0(\xi, x)}{\partial \xi} A(t-\xi)\, d\xi \right| =$$

$$= \left| \int_0^t \left(\frac{1}{\xi} - \left(\frac{\mu}{\lambda}\right)^2 \frac{1}{x} \right) F_0(\xi, x) A(t-\xi)\, d\xi \right| \leq$$

$$= \sup_{0 \leq t \leq a} |A(t)| \left(\int_0^t \frac{F_0(\xi, x)}{\xi} d\xi + \left(\frac{\mu}{\lambda}\right)^2 \int_0^t \frac{F_0(\xi, x)}{x} d\xi \right),$$

and $\lim_{x\to 0} \frac{F_0(\xi,x)}{\xi} = \lim_{x\to 0} \frac{F_0(\xi,x)}{x} = 0 \Rightarrow \lim_{x\to 0} \int_0^t \frac{\partial^2 H_0(\xi, x)}{\partial \xi \, \partial x} A(t-\xi)\, d\xi = 0.$

Likewise,

$$\lim_{x\to 0} \left\{ \int_0^t G_0(t-\xi, x)\, C(\xi)\, d\xi \right\} = \lim_{x\to 0} \{G_0(t,x)\} \overset{t}{*} \{C(t)\}$$

$$= \frac{\delta(t)}{\mu} \overset{t}{*} \{C(t)\} = \{C(t)\} \Rightarrow \lim_{x\to 0} \mu \int_0^t G_0(t-\xi, x)\, c(\xi)\, d\xi = c(t)$$

Whence $\lim_{x\to 0} \frac{\partial u}{\partial x} = C(t) \qquad x \to 0$

and $u(t,x)$ satisfies also the third condition (3′).

R e m a r k. The function $F_0(t,x)$, $G_0(t,x)$, $H_0(t,x)$ satisfy for $x \geq 0$, $t \geq 0$, the parabolic partial differential equation (cf. [3], formulas (23), (54) and (72)):

$$\lambda^2 \frac{\partial^2 u}{\partial t^2} - \mu^2 \frac{\partial u}{\partial u} = 0.$$

For this reason we say that the equations (1), (2), (3) are parabolic integro-differential equations.

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