PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 15 (29), 1973, pp. 21—24

BALANCED LAWS ON *GD*-GROUPOIDS

Branka P. Alimpić

(Communicated June 21, 1972)

In this paper we examine a family of *GD*-groupoids [3] connected by a balanced law $w_1 = w_2$ [2]. In [1] we considered the analogous problem for quasigroups.

Let $w_1 = w_2$ be a balanced law, and $[w_1] = \{x_1, \ldots, x_{n+1}\}$ the set of variables appearing in w_1. Since $w_1 = w_2$ is a balanced law, we have $[w_1] = [w_2]$. We denote this set by W. Let $\Phi(w_1) = \{A_1, \ldots, A_n\}$ be the set of operation letters appearing in w_1, $\Phi(w_2) = \{B_1, \ldots, B_n\}$ the set of operation letters appearing in w_2, and $\Phi = \Phi(w_1) \cup \Phi(w_2)$.

Let $A_1 = \inf_{w_1} \Phi(w_1)$, $B_1 = \inf_{w_2} \Phi(w_2)$, i.e. let the law $w_1 = w_2$ be of the form $A_1(u_1, v_1) = B_1(u_1, v_2)$.

Further, let X_i, $i = 1, \ldots, n+1$, Q_i, P_i, $i = 1, \ldots, n$ be nonempty sets.

We define the *GD*-groupoids A_i, B_i, $i = 1, \ldots, n$, connected by $w_1 = w_2$ as follows:

1. $x_i \in X_i$, $i = 1, \ldots, n+1$.
2. If $A_k(u, v)$ is a subterm of the term w_1, $u \in U$, $v \in V$, A_k is a function from $U \times V$ into Q_k, i.e. $A_k : U \times V \to Q_k$.
3. If $B_k(u, v)$ is a subterm of the term w_2, $u \in U$, $v \in V$, B_k is a function from $U \times V$ into P_k, i.e. $B_k : U \times V \to P_k$.
4. $Q_1 = P_1 = S$.

Let $a_i \in X_i$, $i = 1, \ldots, n+1$ be fixed elements.

We consider a balanced law such that $\Phi = K\approx$ (see [1]).

Let $x, y \in W$ be such that $A_1 \overset{(x,\, y)}{\longleftrightarrow} B_1$. Then, by substitution of all variables x_i, except x and y by elements $a_i \in X_i$, we get

$$A_1(\alpha x, \beta y) = B_1^0(\gamma x, \delta y), \tag{1}$$

where α, β, γ, δ are certain surjections, B_1^0 denotes either B_1 or B_1^*

$\left(B_1^*(x, y) \overset{\text{def}}{=} B_1(y, x)\right)$.

Let the following condition hold:

$$\begin{cases} \text{Functions: (i) } L_1^{A_1} \text{ and } L_2^{A_1}, \text{ or (ii) } L_1^{B_1} \text{ and } L_2^{B_1}, \text{ or} \\ \text{(iii) } L_1^{A_1} \text{ and } L_2^{B_1^0}, \text{ or (iv) } L_1^{B_1^0} \text{ and } L_2^{A_1} \text{ are bijective.} \end{cases} \tag{2}$$

We prove the following assertion.

Theorem 1. *Let $w_1 = w_2$ be a balanced law and $\Phi = K\approx$. If the condition (2) holds, there exists a loop $\circ$, defined on the set S, which is homotopic image of all GD-groupoids connected by law $w_1 = w_2$. The loop $\circ$ satisfies the law $w_1(\circ) = w_2(\circ)$ obtained from $w_1 = w_2$ by substitution of all function letters of Φ by $\circ$. For every GD-groupoid $A \in \Phi$, the homotopy is of the form*

$$\sigma_A A(u, v) = \sigma_A L_1^A u \circ \sigma_A L_2^A v.$$

If Φ contains more than two function letters, the loop $\circ$ is a group.

Proof. Let us define the operation $\circ$ of the set S as follows:

In the case (i), (iii), (iv):

(3) $$A_1(u, v) = L_1^{A_1} u \circ L_2^{A_1} v.$$

In the case (ii):

(4) $$B_1(u, v) = L_1^{B_1} u \circ L_2^{B_1} v.$$

We prove that $\circ$ is well defined. In the case (i) and (ii), that is obviously.

Let $x, y \in S$. In the case (iii), there exists exactly one u such that $x = L_1^{A_1} u$.

The function $L_2^{A_1}$ is surjective, hence there exists at least one v such that $L_2^{A_1} v = y$.

We prove that the product $x \circ y$ does not depend on the choice of v, i.e. we prove:

$$L_2^{A_1} v' = L_2^{A_1} v'' \text{ implies, for every } u,\ A_1(u, v') = A_1(u, v'').$$

From (1) we have

(5) $$L_1^{A_1} \alpha = L_1^{B_1^0} \gamma$$

and

(6) $$L_2^{A_1} \beta = L_2^{B_1^0} \delta.$$

Since β is surjective, there exists s' such that $v' = \beta s'$ and s'' such that $v'' = \beta s''$. Hence we get

$$L_2^{A_1} \beta s' = L_2^{A_1} \beta s'',$$

and, using (6),

$$L_2^{B_1^0} \delta s' = L_2^{B_1^0} \delta s''.$$

Since $L_2^{B_1^0}$ is bijective, we have $\delta s' = \delta s''$.

Since α is surjective, there exists t such that $\alpha t = u$. Therefore we can put

$$B_1^0(\gamma t, \delta s') = B_1^0(\gamma t, \delta s''),$$

and, by using (1), we have

$$A_1(\alpha t, \beta s') = A_1(\alpha t, \beta s''),$$

or

$$A_1(u, v') = A_1(u, v'').$$

In the case (iv), the proof is analogous.

It is easy to prove that $\circ$ is a loop.

Since $\Phi = K_{\approx}$, we have from (3) and (4) (see [1]): For every $A \in K_{\approx}$,

$$\sigma_A A(u, v) = \sigma_A L_1^A u \circ \sigma_A L_2^A v,$$

and, by induction on number of variables in W, we prove

$$w_1(\circ) = w_2(\circ).$$

If Φ contains more than two function letters, there exist variables $x, y, z \in W$, such that, substituting all variables x_i, except x, y, z, by fixed $a_i \in X_i$, we get

$$A_1^0(\alpha_1 A_k^0(\alpha_2 x, \alpha_3 y), \alpha_4 z) = B_1^0(\beta_1 x, \beta_2 B_j^0(\beta_3 y, \beta_4 z)). \tag{7}$$

There are two possibilities, either the law $w_1 = w_2$ is of the first kind, or the loop $\circ$ is commutative. In both cases, substituting in $w_1(\circ) = w_2(\circ)$ all variables, except x, y, z from (7), by the identity element of the loop $\circ$, we get

$$(x \circ y) \circ z = x \circ (y \circ z).$$

Corollary. Let $w_1 = w_2$ be a balanced law and $\Phi = K_{\sim} = K_{\approx}^1 \cup K_{\approx}^2$. If the condition (2) holds, there exist loops $\circ$ and $*$, defined on the set S, so that $\circ$ is homotopic image of all *GD*-goupoids of the set $K_{\approx}^1$, and $*$ is homotopic image of all *GD*-groupoids of the set $K_{\approx}^2$. The loops $\circ$ and $*$ are connected by the law $u \circ v = v * u$, and by the law obtained from $w_1 = w_2$ by substitution of all functions letters of the set $K_{\approx}^1$ by $\circ$, and those of the set $K_{\approx}^2$ by $*$.

Let $w_1 = w_2$ be any balanced law. For every class $K_{\sim}^j$, $j = 1, \dots, s$, we have $K_{\sim}^j = K_1^j \cup K_2^j$, where $K_1^j \subset \Phi(w_1)$, and $K_2^j \subset \Phi(w_2)$. If A_j and B_j are function letters defined by $A_j = \inf_{w_1} K_1^j$, $B_j = \inf_{w_2} K_2^j$, there exist two variables $x_j, y_j \in W$ so that we have

$$\sigma_{A_j} A_j(\alpha_1^j x_j, \alpha_2^j y_j) = \sigma_{B_j} B_1^0(\beta_1^j x_j, \beta_2^j y_j).$$

Let the following condition hold:

$$\left\{\begin{array}{lll} \text{Functions:} & \text{(i)} & \sigma_{A_j} L_1^{A_j} \text{ and } \sigma_{A_j} L_2^{A_j}, \text{ or} \\ & \text{(ii)} & \sigma_{B_j} L_1^{B_j} \text{ and } \sigma_{B_j} L_2^{B_j}, \text{ or} \\ & \text{(iii)} & \sigma_{A_j} L_1^{A_j} \text{ and } \sigma_{B_j} L_2^{B_j^0}, \text{ or} \\ & \text{(iv)} & \sigma_{B_j} L_1^{B_j^0} \text{ and } \sigma_{A_j} L_2^{A_j} \\ \text{are bijective } (j = 1, \dots, s). & & \end{array}\right. \tag{8}$$

By induction on numbers of classes $K_{\sim}^j$, $j = 1, \dots, s$, we can prove:

Theorem 2. *Let $w_1 = w_2$ be any balanced law. If the condition (8) holds, there exist loops $\circ_i$ $(i = 1, \dots, t)$ defined on the set S, which are homotopic images of all GD-groupoids of classes $K_{\approx}^i$, respectively. Loops $\circ_i$ are connected by the law obtained from $w_1 = w_2$ by substitution of all function letters of a class $K_{\approx}^i$ by $\circ_i$, $i = 1, \dots, t$. For every $A \in K_{\approx}^i$, homotopy is of the form*

$$\sigma_A A(u, v) = \sigma_A L_1^A u \circ_i \sigma_A L_2^A v.$$

If the class $K_{\sim}$, containing $K_{\approx}^i$, has more than two function letters, the loop $\circ_i$ is a group.

REFERENCES

[1] Alimpić B., *Uravnoteženi zakoni na kvazigrupama*, Matematički vesnik 9 (24), Sv. 3 (1972), 249—255.

[2] Белоусов В. Д., *Уравновешенные тождества в квазигрупах*, Матем. сб. 70 (112), (966), 55—97.

[3] Milić S., *On GD-groupoids with applications to n-ary quasigroups*, Publ. Inst. Math. tome 13 (27), (1972).