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COUNTABLE AND UNCOUNTABLE COVERS

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If X is a topological space such that every cover consisting of uncountably many distinct open sets has a finite subcover, does it necessary follow that every cover consisting of countably many distinct open sets also has a finite subcover? In other words, in a non-countably compact space, can we always find a cover consisting of uncountably many distinct open sets without a finite subcover? The answer is negative in a trivial way. However, for T_1-spaces, the answer is affirmative.

Example. Let $X=\{1, 2, 3, \ldots\}$ and $A_n=\{1, 2, \ldots, n\}$ are the only proper open sets of X. Then the countable open cover $\{A_n : n=1, 2, \ldots\}$ has no finite subcover. On the other hand, there does not exist uncountably many distinct open sets in X.

Theorem. *Let X be a T_1-space. X is compact iff every cover consisting of uncountably many distinct open sets has a finite subcover.*

Proof. The necessity follows directly from the definition of compactness.

To see the sufficiency, suppose X is not compact. Then, by hypothesis, there is a countable open cover without a finite subcover, i.e. X is not countably compact. Since X is a T_1-space, we can obtain a countable discrete closed subset $H=\{x_1, x_2, \ldots\}$ of X. For each positive integer n, let U_n be an open set of X such that $U_n \cap H=\{x_n\}$. Let I be the collection of all infinite subsequences of the sequence $\{2, 4, 6, \ldots, 2n, \ldots\}$. Then the cardinality of I is uncountable. For each $i \in I$, $i=\{i(1), i(2), \ldots, i(n), \ldots\}$, let

$$V_i = \bigcup_{k=1}^{\infty} U_{i(k)}.$$

Then $\{U_1, U_3, U_5, \ldots, U_{2n+1}, \ldots\} \cup \{X \setminus H\} \cup \{V_i : i \in I\}$ is a cover of X consisting of uncountably many distinct open sets without a finite subcover.

REFERENCE

[1] S. Willard, *General Topology*, 1970.

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