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MULTIPLE SERIES RELATIONS INVOLVING HERMITE POLYNOMIALS

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1. Introduction

In the theory of Hermite polynomials $H_n(z)$ defined by

$$\sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!}=\exp(2zt-t^2), \tag{1.1}$$

the bilinear generating function

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{t^n}{n!}=(1-4t^2)^{\frac{-1}{2}}\exp\left\{\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right\} \tag{1.2}$$

is well known as Mehler's formula [3, p. 198]. Carlitz ([1], [2]) has given a number of extensions of (1,2). Recently, in an attempt to unify the various extensions of (1.2) given by Carlitz, Srivastava and Singhal [5] gave the following elegant formula.

$$\sum_{m,n,p=0}^{\infty} H_{n+p+r}(x)H_{p+m+s}(y)H_{m+n}(z)\frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!} \tag{1.3}$$

$$=\Delta^{-\frac{1}{2}(1+r+s)}(1-4u^2)^{\frac{r}{2}}(1-4v^2)^{\frac{s}{2}}\cdot$$

$$\cdot\exp\left\{\Sigma x^2-\frac{1}{\Delta}(\Sigma x^2-4\Sigma u^2x^2-4\Sigma wxy+8\Sigma uvxy)\right\}\cdot$$

$$\cdot\sum_{k=0}^{\min(r,s)}2^{2k}k!\binom{r}{k}\binom{s}{k}\left(\frac{w-2uv}{((1-4u^2)(1-4v^2))^{\frac{1}{2}}}\right)^k\cdot$$

$$\cdot H_{r-k}\left(\frac{(x-2vz)(1-4u^2)-2(y-2uz)(w-2uv)}{(\Delta(1-4u)^2))^{1/2}}\right)\cdot$$

$$\cdot H_{s-k}\left(\frac{(y-2uz)(1-4v^2)-2(x-2vz)(w-2uv)}{(\Delta(1-4v^2))^{1/2}}\right),$$

where $$\Delta = 1 - 4u^2 - 4v^2 - 4w^2 + 16uvw,$$

and Σx^2, $\Sigma u^2 x^2$, Σwxy, $\Sigma uvxy$ are symmetric functions in the indicated variables.

In a subsequent paper [6] they have also given three other formulae which are slight variations of (1.3).

The object of this paper is to give a new formula which includes (1.3) together with several other new results discussed in Section 3. The formula that we prove here is

$$(1.4)\quad \sum_{m,n,p,r,s=0}^{\infty} H_{2m+p+r+k}(x)\, H_{2n+p+s+l}(y)\, H_{r+s}(z)\, \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}\frac{t^r}{r!}\frac{h^s}{s!}$$
$$= U^{-\frac{1}{2}(l+k+1)} (1+4v-4h^2)^{\frac{k}{2}} (1+4u-4t^2)^{\frac{l}{2}} \exp\left(x^2+y^2+z^2-\frac{V}{U}\right)\cdot$$
$$\cdot \sum_{j=0}^{\min(1,k)} 2^{2j} j! \binom{k}{j}\binom{1}{j} \left(\frac{w-2th}{((1+4v-4h^2)(1+4u-4t^2))^{1/2}}\right)^j \cdot$$
$$\cdot H_{k-j}\left(\frac{x(1+4v-4h^2)-2(wy+tz-2thy-2whz+4vtz)}{(U(1+4v-4h^2))^{1/2}}\right)\cdot$$
$$\cdot H_{l-j}\left(\frac{y(1+4u-4t^2)-2(wx+hz-2thx-2wtz+4uhz)}{(U(1+4u-4t^2))^{1/2}}\right),$$

where, for convenience,

$$U = 1+4u+4v-4w^2-4t^2-4h^2+16uv-16vt^2-16uh^2+16wth,$$
$$V = x^2(1+4v-4h^2)+y^2(1+4u-4t^2)+z^2(1+4u+4v-4w^2+16uv)$$
$$(1.5)\quad -4xyw-4yzh-4zxt+8xyth+8yzwt+8xzwh$$
$$-16xzvt-16yzuh.$$

We also give here the following variation of (1.4).

$$(1.6)\quad \sum_{m,n,p,r,s=0}^{\infty} H_{2m+p+r+k}(x)\, H_{2n+p+s}(y)\, H_{r+s+l}(z)\, \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}\frac{t^r}{r!}\frac{h^s}{s!}$$
$$= U^{-\frac{1}{2}(1+l+k)} (1+4v-4h^2)^{\frac{k}{2}} (1+4u+4v-4w^2+16uv)^{\frac{l}{2}}\cdot$$
$$\cdot \exp\left(x^2+y^2+z^2-\frac{V}{U}\right) \sum_{j=0}^{\min(1,k)} 2^{2j} j! \binom{k}{j}\binom{l}{j}\cdot$$
$$\cdot\left(\frac{t-2wh+4vt}{((1+4v-4h^2)(1+4u+4v-4w^2+16uv))^{1/2}}\right)^j\cdot$$
$$\cdot H_{k-j}\left(\frac{x(1+4v-4h^2)-2(wy+tz-2yth-2zwh+4zvt)}{(U(1+4v-4h^2))^{1/2}}\right)\cdot$$
$$\cdot H_{l-j}\left(\frac{z(1+4u+4v-4w^2+16uv)-2(hy+xt-2ywt-2xwh+4xvt+4yuh)}{(U(1+4u+4v-4w^2+16uv))^{1/2}}\right),$$

where U and V are given by (1.5)

We shall use the following results in proving the formula (1.4).

(1.7) $$D_x^r H_n(x) = 2^r r! \binom{n}{r} H_{n-r}(x),$$

where, $D_x = \dfrac{d}{dx}$.

(1.8) $$\exp(t D_x) f(x) = f(x+t).$$

(1.9) $$H_n(ax) = \left(-\frac{1}{a}\right)^n \exp(a^2x^2) D_x^n \exp(-a^2x^2),$$

which follows at once from Rodrigues' formula for the Hermite polynomials.

(1.10) $$\exp(t D_x^2)\{\exp(-x^2)\} = (1+4t)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{1+4t}\right),$$

which is Glaisher's operational formula.

(1.11) $$\exp(t D_x D_y) \exp(-a^2x^2 - b^2y^2)$$
$$= (1-4a^2b^2t^2)^{-\frac{1}{2}} \exp\left\{-a^2x^2 - \frac{(by-2a^2bxt)^2}{1-4a^2b^2t^2}\right\},$$

which was derived earlier by Singhal [4].

2. Proof of formula (1.4)

Denoting the left-hand side of (1.4) by Ω, if we make use of (1.9) and (1.1), we get

$$\Omega = \sum_{r,s=0}^{\infty} H_{r+s}(z) \frac{t^r}{r!} \frac{h^t}{s!} \sum_{m,n,p=0}^{\infty} H_{2m+p+r+k}(x) H_{2n+p+s+l}(y) \frac{u^m}{m!} \frac{v^n}{n!} \frac{w^p}{p!}$$
$$= \sum_{r,s=0}^{\infty} H_{r+s}(z) \frac{t^r}{r!} \frac{h^s}{s!} \sum_{m,n,p=0}^{\infty} (-)^{r+s+k+l} \exp(x^2+y^2) \cdot$$
$$\cdot D_x^{2m+p+r+k} D_y^{2n+p+s+l} \exp(-x^2-y^2) \frac{u^m}{m!} \frac{v^n}{n!} \frac{w^p}{p!}$$
$$= \exp(x^2+y^2)(-D_x)^k (-D_y)^l \sum_{j=0}^{\infty} H_j(z) \frac{(-tD_x - hD_y)^j}{j!} \cdot$$
$$\cdot \exp(uD_x^2 + vD_y^2 + wD_xD_y) \exp(-x^2-y^2)$$
$$= \exp(x^2+y^2)(-D_x)^k(-D_y)^l \cdot \exp\{-2tzD_x - 2hzD_y$$
$$+ (w-2th) D_xD_y + (u-t^2) D_x^2 + (v-h^2) D_y^2\} \exp(-x^2-y^2)$$
$$= \exp(x^2+y^2)(-D_x)^k(-D_y)^1 \exp\{-2hzD_y - 2tzD_x + (w-2th) D_xD_y\} \cdot$$
$$\cdot (1+4u-4t^2)^{-\frac{1}{2}} (1+4v-4h^2)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{1+4u-4t^2} - \frac{y^2}{1+4v-4h^2}\right),$$

by virtue of (1.10).

Now we apply formulae (1.11) and (1.9), and observe that

$$\Omega=\exp(x^2+x^2)\,U^{-\frac{1}{2}}(-D_x)^k(-D_y)^l\exp(-2tzD_x-2hzD_y).$$

$$\cdot\exp\left[-\frac{1}{U}\{x^2(1+4v-4h^2)+y^2(1+4u-4t^2)-4xy(w-2th)\}\right]$$

$$=\exp(x^2+y^2)\,U^{-\frac{1}{2}}\exp(-2tzD_x-2hzD_y)(-D_x)^k\left(\frac{1+4u-4t^2}{U}\right)^{\frac{1}{2}}$$

$$\cdot\exp\left[-\frac{1}{U}\{x^2(1+4v-4h^2)+y^2(1+4u-4t^2)-4xy(w-2th)\}\right]\cdot$$

$$\cdot H_1\left(\frac{y(1+4u-4t^2)-2x(w-2th)}{(U(1+4u-4t^2))^{1/2}}\right)$$

$$=\exp(x^2+y^2)\,U^{-\frac{1}{2}}\left(\frac{1+4u-4t^2}{U}\right)^{\frac{1}{2}}\exp(-2tzD_x-2hzD_y)\cdot$$

$$\cdot\sum_{j=0}^{k}(-)^k\binom{k}{j}D_x^{k-j}\exp\left[-\frac{1}{U}\{x^2(1+4v-4h^2)+y^2(1+4u-4t^2)\right.$$

$$\left.-4xy(w-2th)\}\right]\cdot D_x^jH_1\left(\frac{y(1+4u-4t^2)-2x(w-2th)}{(U(1+4u-4t^2))^{1/2}}\right)$$

$$=\exp(x^2+y^2)\,U^{-\frac{1}{2}(1+k+1)}(1+4u-4t^2)^{\frac{1}{2}}(1+4v-4h^2)^{\frac{k}{2}}\cdot$$

$$\cdot\exp(-2tzD_x-2hzD_y)\exp\left[-\frac{1}{U}\{x^2(1+4v-4h^2)+y^2(1+4u-4t^2)\right.$$

$$\left.-4xy(w-2th)\}\right]\cdot\sum_{j=0}^{\min(1,k)}2^{2j}j!\binom{k}{j}\binom{l}{j}\left(\frac{w-2th}{((1+4u-4t^2)(1+4v-4h^2))^{1/2}}\right)^j\cdot$$

$$\cdot H_{k-j}\left(\frac{x(1+4v-4h^2)-2y(w-2th)}{(U(1+4v-4h^2))^{1/2}}\right)H_{l-j}\left(\frac{y(1+4u-4t^2)-2x(w-2th)}{(U(1+4u-4t^2))^{1/2}}\right),$$

which, in view of (1.8), leads us to the desired result (1.4).

We omit the details involved in the proof of (1.6) as it would run parallel to that of (1.4).

3. Particular cases of (1.4)

(i) Evidently (1.4) provides an extension of Srivastava and Singhal's formula (1.3) to which it would reduce when $u=v=0$.

Some other particular cases of (1.4) are worthy of note which are believed to be new and are given below.

(ii) If in (1.4) we set $t=h=0$, we are led to the the following result.

$$\sum_{m,n,p=0}^{\infty} H_{2m+p+k}(x)\, H_{2n+p+1}(y) \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!} \tag{3.1}$$

$$= P^{-\frac{1}{2}(1+k+1)} (1+4v)^{\frac{k}{2}} (1+4u)^{\frac{1}{2}} \cdot$$

$$\cdot \exp\left\{x^2+y^2-\frac{1}{P}\left(x^2(1+4v)+y^2(1+4u)-4xyw\right)\right\}\cdot$$

$$\cdot \sum_{j=0}^{\min(1,k)} 2^{2j} j! \binom{k}{j}\binom{1}{j}\left(\frac{w}{((1+4v)(1+4u))^{1/2}}\right)^j \cdot$$

$$\cdot H_{k-j}\left(\frac{x(1+4v)-2wy}{(P(1+4v))^{1/2}}\right) H_{1-j}\left(\frac{y(1+4u)-2xw}{(P(1+4u))^{1/2}}\right),$$

where $P=1+4u+4v-4w^2+16uv$.

(iii) On the other hand, the choice $w=0$ would transform (1.4) into

$$\sum_{m,n,r,s=0}^{\infty} H_{2m+r+k}(x)\, H_{2n+s+1}(y)\, H_{r+s}(z) \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^r}{r!}\frac{t^s}{s!} \tag{3.2}$$

$$= Q^{-\frac{1}{2}(1+k+1)} (1+4v-4t^2)^{\frac{k}{2}} (1+4u-4w^2)^{\frac{1}{2}} \exp\left(x^2+y^2+z^2-\frac{R}{Q}\right)\cdot$$

$$\cdot \sum_{j=0}^{\min(1,k)} 2^{2j} j! \binom{k}{j}\binom{l}{j}\left(\frac{-2wt}{((1+4v-4t^2)(1+4u-4w^2))^{1/2}}\right)^j \cdot$$

$$\cdot H_{k-j}\left(\frac{x(1+4v-4t^2)-2(wz-2ywt-4zvw)}{(Q(1+4v-4t^2))^{1/2}}\right)\cdot$$

$$\cdot H_{l-j}\left(\frac{y(1+4u-4w^2)-2(tz-2xwt+4utz)}{(Q(1+4u-4w^2))^{1/2}}\right),$$

where
$$Q=1+4u+4v-4w^2-4t^2+16uv-16vw^2-16ut^2,$$

and
$$R=x^2(1+4v-4t^2)+y^2(1+4u-4w^2)+z^2(1+4u+4v+16uv)$$
$$-4yzt-4xzw+8xywt-16xzvw-16yzut\,.$$

(iv) On setting $t=0$ and making a slight change of variables, we get

$$\sum_{m,n,r,s=0}^{\infty} H_{2m+r+k}(x)\, H_{2n+r+s+l}(y)\, H_s(z) \frac{u^m}{m!}\frac{v^n}{n!}\frac{w^r}{r!}\frac{t^s}{s!} \tag{3.3}$$

$$= S^{-\frac{1}{2}(1+1+k)} (1+4v-4t^2)^{\frac{k}{2}} (1+4u)^{\frac{l}{2}} \exp\left(x^2+y^2+z^2-\frac{T}{S}\right)\cdot$$

$$\cdot \sum_{j=0}^{\min(1,k)} 2^{2j} j! \binom{k}{j}\binom{l}{j}\left(\frac{w}{((1+4u)(1+4v-4t^2))^{1/2}}\right)^j \cdot$$

$$\cdot H_{k-j}\left(\frac{x(1+4v-4t^2)-2(wy-2twz)}{(S(1+4v-4t^2))^{1/2}}\right)\cdot$$

$$\cdot H_{l-j}\left(\frac{y(1+4u)-2(wx+tz+4utz)}{(S(1+4u))^{1/2}}\right),$$

where $S=1+4u+4v-4w^2-4t^2+16uv-16ut^2$,

$$T=x^2(1+4v-4t^2)+y^2(1+4u)+z^2(1+4u+4v-4w^2+16uv)$$
$$-4xyw-4yzt+8xzwt-16yzut.$$

(v) For $u=t=0$, (1.4) would assume the form:

(3.4) $$\sum_{m,n,p=0}^{\infty} H_{n+k}(x)\,H_{2m+n+p+l}(y)\,H_p(z)\frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}$$

$$=X^{-\frac{1}{2}(1+l+k)}(1+4u-4w^2)^{\frac{k}{2}}\exp\left(x^2+y^2+z^2-\frac{Y}{X}\right)\cdot$$

$$\cdot\sum_{j=0}^{\min(1,k)} 2^{2j}j!\binom{k}{j}\binom{l}{j}\left(\frac{v}{(1+4u-4w^2)^{1/2}}\right)^j\cdot$$

$$\cdot H_{k-j}\left(\frac{x(1+4u-4w^2)-2(vy-2zvw)}{(X(1+4u-4w^2))^{1/2}}\right)H_{1-j}\left(\frac{y-2(vx+zw)}{X^{1/2}}\right),$$

where $X=1+4u-4v^2-4w^2$,

$$Y=x^2(1+4u-4w^2)+y^2+z^2(1+4u-4v^2)-4xyv-4yzw+8xzvw.$$

Further on taking $v=w=0$, (3.4) would correspond to the elegant result (6.6) in [2].

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