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# CERTAIN THEOREMS ON SELF-RECIPROCAL FUNCTIONS

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„The object of this paper is to prove certain theorems on self-reciprocal functions".

## 1. Introduction

Let

$$\psi(p)=p\int_0^\infty e^{-pt}f(t)\,dt,\quad R(p)>0, \tag{1.1}$$

then we say that $\psi(p)$ is operationally related to $f(t)$ and symbolically we write as $\psi(p)\doteqdot f(t)$ or $f(t)\doteqdot\psi(p)$.

Mainra, V.P., [3] has defined the kernel $\tilde{w}_{u,v}^{\lambda}(x)$ as

$$\tilde{w}_{u,v}^{\lambda}(x)=\sqrt{x}\int_0^\infty\int_0^\infty (y/t)\,J_u(xy/t)\,J_\lambda(y)\,J_v(t)\,dt\,dy, \tag{1.2}$$

$R(u, v, \lambda)\geqslant -\dfrac{1}{2}$ and proved that it is a Fourier kernel.

Two functions $f(x)$ and $g(x)$ are called $\tilde{w}_{u,v}^{\lambda}(x)$ transform of each other if they satisfy the integral equation

$$f(x)=\int_0^\infty \tilde{w}_{u,v}^{\lambda}(xy)\,g(y)\,dy. \tag{1.3}$$

If $g(x)=f(x)$ i.e. $f(x)=\int_0^\infty \tilde{w}_{u,v}^{\lambda}(xy)f(y)\,dy$, then $f(x)$ is said to be self-reciprocal in the $\tilde{w}_{u,v}^{\lambda}(x)$ transform and is denoted by $R_{u,v}^{\lambda}$.

**2.** Theorem 1 (a): *Let* (i) $f(x)\doteqdot g(p)$ (ii) $x^{2m-1}f(1/x)\doteqdot\psi(p)$

(iii) $x^{m-1}f(1/x)$ *be* $R_{u,v}^{\lambda}$, *then*

*$x^{-m}\psi(1/x)$ is the $\tilde{w}_{u,v}^{\lambda}(x)$ transform of $t^{m-1}g(t)$, provided that $x^{m-1}g(x)$, $x^{m-1}f(1/x)$ and $x^{2m-1}f(1/x)$ are bounded and absolutely integrable in* $(0, \infty)$, $R\left(m+u+\dfrac{3}{2}\right)>0$, $R(m+v+3/2)>0$, $R(m+\lambda+3/2)>0$.

Proof: Let

(2.1) $$\varphi(p) = x^m \tilde{w}_{u,v}^{\lambda}(x).$$

Then

(2.2) $$\varphi(ap) \doteqdot (x/a)^m \tilde{w}_{u,v}^{\lambda}(x/a).$$

Also

(2.3) $$g(p) \doteqdot f(x).$$

We notice that $p^m \varphi(p)$ is continuous in $(0, \infty)$.

From (2.2) and (2.3) applying Goldstein's theorem, we have

$$\int_0^\infty \varphi(ta) f(t) \frac{dt}{t} = \int_0^\infty g(t) \tilde{w}_{u,v}^{\lambda}(t/a)\, t^{m-1}\, dt$$

or

(2.4) $$\int_0^\infty \varphi(pt) f(t) \frac{dt}{t} = p^{-m} \int_0^\infty \tilde{w}_{u,v}^{\lambda}(t/p)\, g(t)\, t^{m-1}\, dt.$$

Interpreting we have

$$\int_0^\infty \tilde{w}_{u,v}^{\lambda}(x/t)\,(x/t)^m f(t) \frac{dt}{t} \doteqdot p^{-m} \int_0^\infty \tilde{w}_{u,v}^{\lambda}(t/p)\, g(t)\, t^{m-1}\, dt$$

or

(2.5) $$x^m \int_0^\infty \tilde{w}_{u,v}^{\lambda}(xt)\, t^{m-1} f(1/t)\, dt \doteqdot p^{-m} \int_0^\infty \tilde{w}_{u,v}^{\lambda}(t/p)\, t^{m-1} g(t)\, dt.$$

Since $t^{m-1} f(1/t)$ is $R_{u,v}^{\lambda}$ we have

$$x^{2m-1} f\left(\frac{1}{x}\right) \doteqdot p^{-m} \int_0^\infty \tilde{\omega}_{u,v}^{\lambda}(t/p)\, t^{m-1} g(t)\, dt.$$

Also $x^{2m-1} f\left(\frac{1}{x}\right) \doteqdot \psi(p)$, by Lerch's theorem we have

$$\int_0^\infty \tilde{w}_{u,v}^{\lambda}(tp)\, t^{m-1} g(t)\, dt = p^{-m} \psi(1/p).$$

Or $x^{-m} \psi(1/x)$ is the $\tilde{w}_{u,v}^{\lambda}(x)$ transform of $x^{m-1} g(x)$.

Thus the theorem is proved.

Now suppose that $x^{m-1} g(x)$ is $R_{u,v}^{\lambda}$, then from (2.5) we have

(2.6) $$x^m \int_0^\infty \tilde{w}_{u,v}^{\lambda}(xt)\, t^{m-1} f(1/t)\, dt \doteqdot p^{1-2m} g(1/p).$$

Suppose $p^{1-2m} g(1/p) \doteqdot h(x)$, then we have from (2.6)

$$\int_0^\infty \tilde{w}_{u,v}^{\lambda}(xt)\, t^{m-1} f(1/t) = x^{-m} h(x)$$

or $x^{-m} h(x)$ is the $\tilde{w}_{u,v}^{\lambda}(x)$ transforms of $x^{m-1} f(1/x)$.

Hence we can state the theorem as:

Theorem 1(b): *Let* (i) $f(x) \doteqdot g(p)$

(ii) $h(x) \doteqdot p^{1-2m} g(1/p)$

(iii) $x^{m-1} g(x)$ *be* $R_{u,v}^{\lambda}$, *then*

$x^{-m} h(x)$ *will be* $\tilde{w}_{u,v}^{\lambda}(x)$ *transform of* $x^{m-1} f(1/x)$, *provided that conditions of the theorem 1 (a) are satisfied.*

Theorem 2(a): *Let* (i) $f(x) \doteqdot g(p)$

(ii) $x^{2m+1} f(1/x) \doteqdot \psi(p)$

(iii) $x^{-m-1} f(x)$ *be* $R_{u,v}^{\lambda}$, *then*

$x^m \psi(x)$ *is the* $\tilde{w}_{u,v}^{\lambda}(x)$ *transform of* $x^{-m-1} g(1/x)$, *provided that* $x^{2m+1} f(1/x)$, $x^{-m-1} f(x)$, $x^{-m-1} g(1/x)$ *are bounded and absolutely integrable in* $(0, \infty)$.

Theorem 2(b): *Let* (i) $f(x) \doteqdot g(p)$,

(ii) $h(x) \doteqdot p^{-2m-1} g(1/p)$

(iii) $x^{-m-1} g(1/x)$ *be* $R_{u,v}^{\lambda}$, *then*

$x^m h\left(\frac{1}{x}\right)$ *will be* $\tilde{\omega}_{u,v}^{\lambda}(x)$ *transform of* $x^{-m-1} f(x)$, *provided the conditions of the theorem 2 (a) are satisfied.*

We can prove these theorems by taking $\varphi(p) \doteqdot x^m w_{u,v}^{\lambda}(1/x)$ and proceeding as in the proof of the theorems (1 a, 1 b).

Theorem 3(a): *Let* (i) $f(x) \doteqdot g(p)$

(ii) $x^{2m-1/2} f(1/x) \doteqdot \psi(p)$

(iii) $x^{2m-1} f(1/x^2)$ *be* $R_{u,v}^{\lambda}$, *then*

$x^{-2m} \psi(1/x^2)$ *will be* $\tilde{w}_{u,v}^{\lambda}(x)$ *transform of* $x^{2m-1} g(x^2)$, *provided that* $x^{-m} f(x)$, $x^{2m-1/2} f(1/x)$, $x^{m-1} g(x)$ *are bounded and absolutely integrable in* $(0, \infty)$.

Theorem 3(b): *Let* (i) $f(x) \doteqdot g(p)$

(ii) $h(x) = p^{1/2-2m} g(1/p)$

(iii) $x^{2m-1} g(x^2)$ *be* $R_{u,v}^{\lambda}$, *then*

$x^{-2m} h(x^2)$ *is the* $\tilde{w}_{u,v}^{\lambda}(x)$ *transform of* $x^{2m-1} f(1/x^2)$, *provided the conditions of the theorem 3 (a) are satisfied.*

We can prove these theorems by taking $\varphi(p) \doteqdot x^m \tilde{w}_{u,v}^{\lambda}(\sqrt{x})$ and proceeding as in the proof of theorems (1 a, 1 b).

Theorem 4(a): *Let* (i) $f(x) \doteqdot g(p)$

(ii) $x^{2m+1/2} f(1/x) \doteqdot \psi(p)$

(iii) $x^{-2m-1} f(x^2)$ *be* $R_{u,v}^{\lambda}$ *then*

$x^{2m}\psi(x^2)$ *will be* $\tilde{w}_{u,v}^{\lambda}(x)$ *transform of* $x^{-2m-1} g(1/x^2)$, *provided that* $x^{-2m-1} f(x^2)$, $x^{2m+1/2} f(1/x)$ *and* $x^{-2m-1} g(1/x^2)$ *are bounded and absolutely integrable in* $(0, \infty)$.

Theorem 4(b): *Let* (i) $f(x) \doteqdot g(p)$

(ii) $h(x) \doteqdot p^{-2m-1/2} g(1/p)$

(iii) $x^{-2m-1} g(1/x^2)$ *be* $R_{u,v}^{\lambda}$, *then*

$x^{2m} h(1/x^2)$ *is the* $\tilde{w}_{u,v}^{\lambda}(x)$ *transform of* $x^{-2m-1} f(x^2)$, *provided the conditions of the theorem 4* (a) *are satisfied.*

We can prove these theorems by taking $\varphi(p) \doteqdot x^m \tilde{w}_{u,v}^{\lambda}(1/\sqrt{x})$ and proceeding as in the theorems (1 a, 1 b).

Theorem 5: *Let* $f(x)$ *be bounded and integrable in* $(0, \infty)$. *Then a sufficient condition for* $f(x)$ *to be* $R_{u,v}^{u+v+1/2}$ *is that it should be of the form*

$$f(x) = \frac{\Gamma(5/4 + u/2)\, x^{-\frac{1}{2}(u+v+1)}}{\Gamma(1 + u/2 - v/2)\, \Pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{4}sx^2} M_{-\frac{u+v+3}{4}, \frac{u-v}{4}}\left(\frac{1}{2} sx^2\right)$$

$$X x^{-\left(\frac{u+v}{4}+1\right)} \varphi(s)\, ds \text{ where } \varphi(s) = \varphi(1/s).$$

Proof: Let

$$\chi(x) = \int_0^\infty (sx)^{\frac{u+v-1}{2}} e^{-\frac{1}{2}s^2x^2} W_{\frac{1}{4}(1-u-v), \frac{u-v}{4}}\left(\frac{1}{2} sx^2\right) f(x)\, dx. \tag{2.7}$$

Assuming that $f(x)$ is $R_{u,v}^{u+v+1/2}$, we have

$$\chi(s) = \int_0^\infty (sx)^{\frac{u+v-1}{2}} e^{-\frac{1}{4}s^2x^2} W_{\frac{1-u-v}{4}, \frac{u-v}{4}}\left(\frac{1}{2} sx^2\right) dx \int_0^\infty \tilde{w}_{u,v}^{u+v+1/2}(xy) f(y)\, dy.$$

On changing the order of integration we have

$$\chi(s) = \int_0^\infty f(y)\, dy \int_0^\infty (sx)^{\frac{u+v-1}{2}} e^{-\frac{s^2x^2}{4}} W_{\frac{1-u-v}{4}, \frac{u-v}{4}}\left(\frac{1}{2} sx^2\right) \tilde{w}_{u,v}^{u+v+1/2}(xy)\, dx$$

$$= \int_0^\infty f(y)\, dy \int_0^\infty x^{\frac{u+v-1}{2}} e^{-x^2/4} W_{\frac{1}{4}(1-u-v), \frac{u-v}{4}}(x^2/2)\, \tilde{w}_{u,v}^{u+v+1/2}(xy/s) \frac{dx}{s}.$$

Since $x^{\frac{u+v-1}{2}} e^{-x^2/4} W_{\frac{1}{4}(1-u-v), \frac{u-v}{4}}(x^2/2)$ is $R_{u,v}^{u+v+1/2}$, [3], we have

$$\chi(s) = \frac{1}{s} \int_0^\infty (y/s)^{\frac{u+v-1}{2}} e^{-y^2/4s^2} W_{\frac{1-u-v}{4}, \frac{u-v}{4}}(y^2/2s^2) f(y)\, dy = \frac{1}{s} \chi(1/s).$$

Let $\varphi(s) = s^{1/4} \chi(\sqrt{s})$ then $\varphi(s) = \varphi(1/s)$.
From (2.7) we have

$$\chi(\sqrt{s}) = 2^{\frac{u+v-3}{4}} \int_0^\infty (su)^{\frac{u+v-1}{4}} e^{-su/2} W_{\frac{1-u-v}{4}, \frac{u-v}{4}}(su) f(\sqrt{2u})\, du/\sqrt{2u}. \tag{2.8}$$

Applying inversion formula we have

$$f(x) = \frac{x\Gamma(u/2+5/4)}{\Gamma(1+u/2-v/2)\,\Pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{4}sx^2} M_{-\left(\frac{u+v+3}{4}\right), \frac{u-v}{4}}\left(\frac{1}{2} sx^2\right) (sx^2)^{-\frac{v+u+3}{4}} \times \chi(\sqrt{s})\, ds$$

or

$$f(x) = \frac{\Gamma(5/4+u/2)\, x^{-\frac{u+v+1}{2}}}{\Pi i \Gamma\left(1+\frac{u-v}{2}\right)} \int_{c-i\infty}^{c+i\infty} e^{sx^2/4} M_{-\frac{u+v+3}{4}, \frac{u-v}{4}}\left(\frac{1}{2} sx^2\right) \times s^{-\left(\frac{u+v}{4}+1\right)} \varphi(s)\, ds$$

where $\varphi(s) = \varphi(1/s)$. Thus the theorem is proved.

3. Examples: (1) Let $\varphi(s) = \frac{\sqrt{s}}{1+s}$, then we have from theorem 5

$$f(x) = \frac{\Gamma(5/4+u/2)\, x^{-u/2-v/2-1/2}}{\Pi i \Gamma\left(1+\frac{u-v}{2}\right)} \int_{c-i\infty}^{c+i\infty} e^{sx^2/4} M_{-\frac{u+v+3}{4}, \frac{u-v}{4}}(sx^2/4) \frac{s^{-(u+v+2)/4}}{1+s} ds$$

is $R_{u,v}^{u+v+1/2}$. On taking $v = 0$ and evaluating the integral we have

$$x^{-(1+u)/2} e^{-x^2/4} M_{\frac{u+3}{4}, \frac{u}{4}}(x^2/4) \text{ is } R_{u,0}^{u+1/2}.$$

(2) Let $\varphi(s) = \frac{s}{1+s^2}$, then we have from theorem 5

$$f(x) = \frac{\Gamma(5/4+u/2)\, x^{-u/2-v/2-1/2}}{\Pi i \Gamma(1+u/2-v/2)} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{4}sx^2} M_{-\frac{u+v+3}{4}, \frac{u-v}{4}}\left(\frac{1}{2} sx^2\right) \times \frac{s^{-(u+v)/4}}{1+s^2} ds \text{ is } R_{u,v}^{u+v+1/2}.$$

On putting $v=1$ and evaluating the integral by residue theorem we have

$$x^{-u/2-1}\left[-\frac{1}{\sqrt{2}}(1+i)\,e^{-ix^2/4}\,M_{\frac{u}{4}+1,\,\frac{u-1}{4}}\left(\frac{1}{2}\,ix^2\right)+e^{ix^2/4}\,M_{-\left(\frac{u}{4}+1\right),\,\frac{u-1}{4}}\left(\frac{1}{2}\,ix^2\right)\right]$$

is $R_{u,1}^{u+3/2}$.

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