PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 15 (29), pp. 111—119

AN INTERPRETATION OF SOME ASPECTS OF KARAMATA'S THEORY OF REGULAR VARIATION

E. Seneta

(Received March 3, 1973)

1. Introduction

Karamata's characterization theorem [5] in regard to regularly varying functions, in a modern setting, may be expressed as follows: *let $f(x)$ be a function defined, finite-valued and measurable on a semiinfinite interval* $[A, \infty)$, *and satisfying*

$$f(x+t)-f(x)\to \psi(t) \tag{1}$$

as $x\to\infty$, *for each* t *in a subset of positive measure of* $(-\infty, \infty)$, *where* $\psi(t)$ *is finite valued for such* t. *Then* (1) *holds for each* $t\in(-\infty, \infty)$, *with finite valued* ψ; *and in fact* $\psi(t)$ *has form* $\psi(t)=\rho t$, *for some constant* ρ, $-\infty<\rho<\infty$.

The extension of the convergence (1) from a set of positive measure to an interval can be carried out with the aid of a well-known theorem of Steinhaus [9], and subsequent extension to all t is trivial; while the form of $\psi(t)$ follows from the fact that it is a measurable solution of Cauchy's functional equation: $\psi(t+\tau)=\psi(t)+\psi(\tau)$, for all real t and τ.

Functions f of this kind have various remarkable properties; for a recent review of the most important of these see [1]. *In particular, there exists a constant* $B\geq A$ *such that* $f(x)$ *is bounded on any finite closed interval* $[a, b]\subset[B, \infty)$.

The aim of the present note is twofold. Firstly, it is intended to show that the characterization theorem remains true in essence if the measurability assumption on f is replaced by that of its boundedness on all finite intervals sufficiently far (this, as just mentioned, is in fact a weaker assumption), and that certain other aspects of the theory of regular variation also persist. Secondly, it is intended to show that a technique used in the paper of Karamata [5], where the characterization theorem first appears, is particularly appropriate to this end. Karamata in this context stated no explicit regularity assumption of f, such as continuity or measurability, nor was the form of ψ deduced from the properties of solutions of Cauchy's functional equation. This has caused subsequent uncertainty as to when his method was applicable, which we shall discuss later. Thus it is also hoped to make rigorous results stated in the first part (which bears the title *Premier groupe de théorèmes*) of this paper, simultaneously with the fulfilment of our primary aim, a generalization to some extent of the modern measure-theory based theory of regular variation.

2. Results. Lemma 1. *Let* $h(x)$ *be a real function defined and finite on* $[A, \infty)$ *for some* A. *Then a necessary and sufficient condition for*

$$h(x+1)-h(x) \to c \Rightarrow h(x)/x \to c \tag{2}$$

as $x \to \infty$, *where* $-\infty < c < \infty$, *is that* $h(x)$ *be bounded on each finite interval beyond some value* x_0.

Proof: Necessity. For each h such that $h(x+1)-h(x) \to c$, $|c|<\infty$, we have for $x \geq x_0(\varepsilon)$ say that $(c-\varepsilon)x \leq h(x) \leq (c+\varepsilon)x$ for arbitrarily chosen $\varepsilon>0$, so that $h(x)$ is bounded on finite intervals beyond $x_0(\varepsilon)$.

Sufficiency. The proposition (2) to be proved can be recognized as a continuous variable version of Cauchy's assertion that the Césaro limit of a sequence exists and is the same as the ordinary limit if this last exists ("exists" here is in reference to a finite number). We use a similar proof.

Assume for convenience and without loss of generality that $A=0$ and that $h(x)$ is bounded in finite intervals in $[0, \infty)$.

Let x be "large" and write

$$x = n(x) + \delta(x), \quad \text{where} \quad 0 \leq \delta(x) < 1$$

and $n(x) = [x] =$ the integral part of x. Then

$$h(x) = h(n(x)+\delta(x)) = h(n(x)+\delta(x)) - h(n(x)-1+\delta(x)) + h(n(x)-1+\delta(x)) \ldots$$
$$\ldots\ldots\ldots\ldots h(1+\delta(x)) - h(\delta(x)) + h(\delta(x)).$$

Now let $\nu \geq 2$ be a positive integer such that for $x \geq \nu \equiv \nu(\varepsilon)$

$$|h(x)-h(x-1)-c| < \varepsilon$$

for arbitrary fixed $\varepsilon>0$. Then for x such that $n(x) \geq \nu$,

$$h(x) - cn(x) = \sum_{r=\nu}^{n(x)} \{h(r+\delta(x)) - h(r-1+\delta(x)) - c\}$$
$$+ \sum_{r=1}^{\nu-1} \{h(r+\delta(x)) - h(r-1+\delta(x)) - c\}$$
$$+ h(\delta(x)).$$

Thus using the triangle inequality

$$\left| \frac{h(x)-cn(x)}{n(x)} \right| \leq \left(\frac{n(x)-\nu+1}{n(x)} \right) \varepsilon$$
$$+ (1/n(x)) \sum_{r=1}^{\nu-1} |h(r+\delta(x)) - h(r-1+\delta(x)) - c|$$
$$+ |h(\delta(x))|/n(x).$$

Thus letting $x \to \infty$, keeping in mind the arbitrariness of ε, and the boundedness of $h(x)$ on $[r, r+1]$ for each $r=0, 1, 2, \ldots$ we obtain

$$\frac{h(x)-cn(x)}{n(x)} \to 0. \tag{3}$$

Now

$$\frac{h(x)}{x} - c = \frac{h(x) - cn(x) - c\delta(x)}{n(x) + \delta(x)} \to 0$$

as $x \to \infty$, on account of (3), which completes the proof.

Definition 1. A function $R(x)$, defined finite and positive on some interval $[C, \infty)$, $C > 0$, is said to be weakly regularly varying (at infinity) if both $R(x)$ and $1/R(x)$ are bounded on any finite subinterval of an interval $[D, \infty)$ where $D \geq C$, and

$$\lim_{x \to \infty} \frac{R(\lambda x)}{R(x)} = \varphi(\lambda) \tag{4}$$

for each λ in a fixed closed interval $[a, b]$, $0 < a < b < \infty$, where $\varphi(\lambda)$ is finite and positive on this interval.

We remind the reader that the definition of *ordinary* regular variation replaces the boundedness assumptions on $R(x)$ by assumption of its measurability.

Lemma 2. *The relation* (4) *in fact holds for each* $\lambda > 0$, *and for some finite* $\varphi(\lambda) > 0$ *for each* $\lambda > 0$

Proof: Standard. The boundedness properties of the definition do not in fact need to be invoked.

Theorem 1 (Weak Characterization Theorem). *The function* $\varphi(\lambda)$, *finite, defined and positive for* $\lambda > 0$, *in the definition of a weakly regularly varying function, has the form* λ^ρ *for some* ρ, $-\infty < \rho < \infty$.

Proof: Let R be a weakly regularly varying function. Put $f(x) = \log R(e^x)$. Then since for each $\lambda > 0$, $R(\lambda x)/R(x) \to \varphi(\lambda) > 0$. it follows that as $x \to \infty$

$$f(x + \mu) - f(x) \to \log \varphi(e^\mu) \tag{5}$$

for any fixed μ in $-\infty < \mu < \infty$, where f is a bounded function on finite intervals beyond a certain point (from the definition of weak regular variation).

We now consider 3 cases:

Case 1 $\mu = 0$; then $\varphi(1) = 1$, clearly.

Case 2 $\mu > 0$; then putting $t = x/\mu$, (5) yields

$$f(\mu(t+1)) - f(\mu t) \to \log \varphi(e^\mu)$$

as $t \to \infty$, so appealing to Lemma 1

$$\frac{f(\mu t)}{t} \to \log \varphi(e^\mu)$$

i.e.

$$\frac{f(x)}{x} \to \mu^{-1} \log \varphi(e^\mu)$$

as $x \to \infty$. Hence for $\mu > 0$

$$\mu^{-1} \log \varphi(e^\mu) \equiv \text{const} = \rho \text{ say}$$

i.e.

$$\varphi(\lambda) = \lambda^\rho \qquad \text{for } \lambda > 1.$$

Case 3. $\mu<0$. Put $y=x+\mu$; thus

$$f(y)-f(y-\mu)\to\log\varphi(e^{\mu})$$

as $y\to\infty$ i.e.

$$f(y+|\mu|)-f(y)\to-\log\varphi(e^{\mu})$$

as $y\to\infty$. From Case 2 applied to the left hand side of this we have that the limit is

$$\rho|\mu|=-\rho\mu=-\log\varphi(e^{\mu})$$

i.e.

$$\varphi(\lambda)=\lambda^{\rho}\qquad \text{for } 0<\lambda<1.$$

The reader is reminded that in the case of ordinary regular variation the same result about φ is true, by first transforming to $f(x)=\log R(e^{x})$ and then invoking the proposition given in § 1 of the present paper. He may wish to note, further, that the crucial implication (2) is no longer necessarily true if one assumes $h(x)$ only measurable, rather than assuming its boundedness on finite intervals, as the following simple example shows:
With

$$h(x)=|\operatorname{cosec}\pi x|,\quad x\neq m\ \ (m \text{ an integer}),\quad =1 \ \text{ otherwise},$$

$$h(x+1)-h(x)=0 \quad \text{for all } x;$$

but if we take the sequence $\{x_n\}$, where $x_n=n+n^{-1}$, then

$$h(x_n)/x_n=|\operatorname{cosec}\pi n^{-1}|/(n+n^{-1})=\{|\sin\pi n^{-1}|(n+n^{-1})\}^{-1}\sim\pi^{-1}$$

as $n\to\infty$.

Definition 2. A weakly regularly varying function (which has been shown to be characterized by the value of the index ρ) will be called weakly slowly varying when $\rho=0$.

We enumerate some of the properties of weakly slowly varying functions which will be familiar from the 'ordinary' theory of regular variation.

Lemma 3. *Suppose L, L_1 and L_2 are weakly slowly varying (at infinity) Then*

1°. *For any $\gamma>0$, $x^{\gamma}L(x)\to\infty$, $x^{-\gamma}L(x)\to 0$ as $x\to\infty$.*

2°. $\log L(x)/\log x\to 0$ *as* $x\to\infty$.

3°. *$L^{\alpha}(x)$ for any α in $-\infty<\alpha<\infty$, $L_1(x)L_2(x)$, $L_1(x)+L_2(x)$ are weakly slowly varying.*

Proof: 1° and 2°: Putting $f(x)=\log L(e^{x})$, we have since $L(\lambda x)/L(x)\to 1$ as $x\to\infty$ for each $\lambda>0$, that

$$f(x+\mu)-f(x)\to 0$$

as $x\to\infty$, $-\infty<\mu<\infty$. From the boundedness of f on finite intervals, we obtain from Lemma 1 (proceeding as in the proof of Theorem 1) that

$$\frac{f(x)}{x}\to 0 \qquad \text{as } x\to\infty$$

which is tantamount to $\log L(x)/\log x\to 0$ as $x\to\infty$, which proves 2°.

Consider now for $\gamma>0$,

$$\log\{x^{\pm\gamma} L(x)\} = \pm\gamma\log x + \log L(x) = \pm\gamma\log x\left[1+\frac{\log L(x)}{\pm\gamma\log x}\right] \sim \pm\gamma\log x.$$

3°. We prove only the proposition about $L_1(x)+L_2(x)$, since the proofs pertaining to the other operations are standard. We have for positive fixed λ

$$\frac{L_1(\lambda x)+L_2(\lambda x)}{L_1(x)+L_2(x)} = \frac{L_1(\lambda x)}{L_1(x)}\left\{\frac{L_1(x)}{L_1(x)+L_2(x)}\right\} + \frac{L_2(\lambda x)}{L_2(x)}\left\{\frac{L_2(x)}{L_1(x)+L_2(x)}\right\}$$

$$= (1+\varepsilon_1(x,\lambda))\left\{\frac{L_1(x)}{L_1(x)+L_2(x)}\right\} + (1+\varepsilon_2(x,\lambda))\left\{\frac{L_2(x)}{L_1(x)+L_2(x)}\right\}$$

where $\varepsilon_i(x,\lambda)\to 0$ as $x\to\infty$, $i=1,2$

$$= 1 + \sum_{i=1}^{2}\varepsilon_i(x,\lambda)\left\{\frac{L_i(x)}{L_1(x)+L_2(x)}\right\}\to 1$$

as $x\to\infty$, for each fixed $\lambda>0$, since

$$0 < \frac{L_i(x)}{L_1(x)+L_2(x)} < 1, \quad i=1,2.$$

Also $L_1(x)+L_2(x)$ is bounded away from zero and infinity for any finite interval sufficiently far, in virtue of this property for each component.

L e m m a 4. *Let f be a function defined on, finite on, and bounded on every finite subinterval of, some interval $[A,\infty)$, and suppose as $x\to\infty$*

(6) $$\lim_{x\to\infty}(f(x+t)-f(x)) = \psi(t)$$

exists and is finite for all t in a dense subset D of the reals. Suppose also that

(7) $$\liminf_{x\to\infty}(f(x+t)-f(x)) \ge \omega(t)$$

for each real t, where $\omega(t)\to 0$ as $t\to 0$. Then the limit in (6) *exists for all t, $-\infty<t<\infty$, and in fact $\psi(t)=Kt$, for some constant K.*

P r o o f: For any real t, recognising that $\omega(t)$ may not be finite valued for some t, *it is useful to note that*:

$$\begin{aligned}\omega(t) &\le \liminf_{x\to\infty}(f(x+t)-f(x))\\ &\le \limsup_{x\to\infty}(f(x+t)-f(x))\\ &= \limsup_{x\to\infty}(f(x)-f(x-t))\\ &= -\liminf_{x\to\infty}(f(x-t)-f(x))\\ &\le -\omega(-t).\end{aligned}$$

Now, if $t\in D$, $t\neq 0$, then (6) holds, and we may write that as $x\to\infty$

$$f(t(1+(x/t))) - f(t(x/t)) \to \psi(t)$$

and by use of Lemma 1 as in the proof of Theorem 1, we deduce that for *all* $t \in D$

$$\psi(t) = Kt$$

for some constant K.

Suppose we can find a point t_0 such that the limit in (6) does not exist (or is not finite). Since D is dense in the reals, we can find a sequence $\{e_i\}$ of positive numbers, such that as $i \to \infty$, $e_i \downarrow 0$; and $t_0 + e_i \in D$ for each $i = 1, 2, \ldots$

Now, for any fixed i,

$$\limsup_{x\to\infty} \left(f(x+t_0) - f(x)\right) = \limsup_{x\to\infty} \left(f(x+t_0+e_i) - f(x+e_i)\right).$$

But

$$f(x+t_0+e_i) - f(x) + f(x) - f(x+e_i) \le K(t_0+e_i) + f(x) - f(x+e_i) + e_i$$

for $x \ge x_i\,(\equiv x_i(e_i))$. Thus

$$\limsup_{x\to\infty} \left(f(x+t_0) - f(x)\right) \le K(t_0+e_i) - \liminf_{x\to\infty} \left(f(x+e_i) - f(x)\right) + e_i$$

$$\le K(t_0+e_i) - \omega(e_i) + e_i$$

the last step following from an inequality established at the outset of the proof. Letting $i \to \infty$, and making use of the fact that $\omega(t) \to 0$ as $t \to 0$, we have that

$$\limsup_{x\to\infty} \left(f(x+t_0) - f(x)\right) \le Kt_0 .$$

We may show that, similarly,

$$\liminf_{x\to\infty} \left(f(x+t_0) - f(x)\right) \ge Kt_0 .$$

T h e o r e m 2. *Suppose R is a function defined positive and, with its reciprocal, bounded on each finite subinterval of some interval $[C, \infty)$; and that*

$$\lim_{x\to\infty} \frac{R(\lambda x)}{R(x)} = \varphi(\lambda) \tag{8}$$

for a set of positive λ such that the corresponding numbers $\log \lambda$ are dense in $(-\infty, \infty)$, and, for such λ, $\varphi(\lambda)$ is finite and positive. Then R is weakly regularly varying if there exists a function $v(t)$ such that for each $\lambda > 0$

$$\liminf_{x\to\infty} \left\{\frac{R(x\lambda) - R(x)}{R(x)}\right\} \ge v(\lambda)$$

where $v(\lambda) \to 0$ as $\lambda \to 1-$.

P r o o f: We have

$$\liminf_{x\to\infty} \frac{R(x\lambda)}{R(x)} \ge 1 + v(\lambda)$$

and transforming as usual to $f(x) = \log R(e^x)$ we have

$$\liminf_{x\to\infty} \{f(x+t) - f(x)\} \ge \omega(t)$$

where $\omega(t) \equiv \log\left(1 + v(e^t)\right)$ for t close to 0, $\to 0$ as $t \to 0$, and (6) also holds with $\psi(t) = \log \varphi(e^t)$ for a set of t dense in $(-\infty, \infty)$. An application of Lemma 4 completes the proof. This theorem is a non-measure-theoretic version of a statement of Karamata [5], p. 58, studied by Heiberg [4].

Corollary. *If R is assumed monotone on* $[C, \infty)$, (8) *need be assumed to hold only for two positive values* λ_1, λ_2, $(\neq 1)$ *such that* $\log \lambda_1 / \log \lambda_2$ *is irrational; then R is in fact regularly varying in the ordinary sense.*

Proof: If we transform as usual to $f(t) = \log R(e^t)$ we have that as $x \to \infty$

$$\lim_{x \to \infty} (f(x+t) - f(x)) = \psi(t)$$

for $t = t_j = \log \lambda_j$, $j = 1, 2$, where $\psi(t_j)$ is finite for $j = 1, 2$. It is easy to see that

$$\lim_{x \to \infty} (f(x+t) - f(x)) = \psi(t)$$

exists and is finite for $t \in D$, where $D = \{t;\ t = qt_1 + rt_2,\ q,\ r \text{ integers}\}$. By Kronecker's theorem (e.g. [3], Chapter XXIII, Theorem 438) D is dense in $(-\infty, \infty)$.

We now merely sketch the following step which has occurred earlier; we deduce from Lemma 1 that for all $t \in D$, $\psi(t) = Kt$ for some constant K (since f is monotone and finite valued it is bounded on all finite intervals sufficiently far). Now let $t_0 \in (-\infty, \infty)$, $t_0 \notin D$, $t \neq 0$. Then we can find positive null sequences $\{e_i^{(1)}\}$, $\{e_i^{(2)}\}$ such that $t_0 + e_i^{(1)}$, $t_0 - e_i^{(2)} \in D$, since D is dense in $(-\infty, \infty)$. If f is increasing, for any fixed i

$$f(x + t_0 + e_i^{(1)}) - f(x) \geq f(x + t_0) - f(x) \geq f(x + t_0 - e_i^{(2)}) - f(x)$$

with the inequalities reversed if f is decreasing. In the former case (and similarly for the latter)

$$\begin{aligned} K(t_0 + e_i^{(1)}) &\geq \limsup_{x \to \infty} (f(x + t_0) - f(x)) \\ &\geq \liminf_{x \to \infty} (f(x + t_0) - f(x)) \\ &\geq K(t_0 - e_i^{(2)}). \end{aligned}$$

Letting $i \to \infty$, we see that in fact

$$\lim_{x \to \infty} (f(x+t) - f(x)) = Kt$$

for *each* $t \in (-\infty, \infty)$. This completes the proof; since f is in fact measurable, being monotone, R is in fact regularly varying in the ordinary sense, as well as weakly regularly varying.

This Corollary is stated without proof on p. 58 of Karamata [5].

Theorem 3. *Suppose* $\{\theta_n\}$ *is a sequence of positive numbers such that* $\limsup \theta_n = \infty$, $\theta_{n+1}/\theta_n < K$ $(1 < K < \infty)$; *and R is defined, finite, positive on, and, with its reciprocal, bounded on finite subintervals of,* $[C, \infty)$, *for some* $C > 0$; *and that for each* $\lambda > 0$

$$\lim_{n \to \infty} R(\theta_n \lambda) / R(\theta_n) = \lambda^\rho \tag{9}$$

for some finite ρ, *the convergence being uniform in* λ *for any interval* $[a, b]$, $0 < a < b < \infty$. *Then R is weakly regularly varying* (*with index* ρ).

Proof: As proof of Theorem A of [8]; where instead of the boundedness on finite intervals, measurability of R was assumed. (It is not possible to replace λ^{ρ} by $\varphi(\lambda)$, even if it is assumed $\varphi(\lambda)$ is positive, finite, strictly monotone and continuous, as shown in [8]).

3. Conclusion

Karamata's own approach to proving the characterization theorem for regularly varying functions was not via a consideration solutions of Cauchy's functional equation. Instead, the key step of his argument occurs in the following statement of Karamata [5] on p. 56:

"Ce théorème est une conséquence immédiate du théorème classique de Cauchy, à savoir: *De la relation*

$$c(x+1)-c(x)\to c \qquad (x\to\infty)$$

il résulte

$$\frac{c(x)}{x}\to c \qquad (x\to\infty)$$

quel que soit le nombre c fini ou infini."

That this last proposition is false in general has been recognized by several authors; we find for example, on p. 3 of [2] the following statement: "We follow Karamatas's second paper ... but avoid an error in his treatment, due to the application of an incorrect theorem of Cauchy," It in fact appears that no such "theorem of Cauchy" exists; and what was probably in Karamata's mind was an application of a continuous-variable version of the classical and well-known result of Cauchy concerning *sequences* viz. that if $\{a(n)\}$, $n=1,2,\dots$ is a convergent sequence, then the sequence $\{b(n)\}$ $n=1,2,\dots$ of Césaro sums, i.e, where

$$b(n)=n^{-1}\sum_{k=1}^{n}a(k)$$

converges also, and to the same limit, as mentioned in the proof of Lemma 1. This view is supported by reference to Karamata's much later book [6] p. 234, where this result on *sequences* is given as Cauchy's theorem.

On the other hand, it is not uninteresting that precisely a Karamata-like statement of the above form, is used without commentary in a proof given by Pólya and Szegö [7], p. 68, Exercise 152 and p. 231, Solution 152.

There is little doubt however that the above idea of Karamata in regard to proving the characterization theorem is most elegant; and most applicable in a certain setting, as we have shown. Another example of its applicability is outlined in the appendix.

4. Appendix

The following proposition for a positive sequence $\{c(n)\}$, $n\geq 0$ (with $c(x)=c([x])$, $x\geq 0$) is due to R. S. Slack, and was mentioned in a letter to the author from G. E. H. Reuter. Slack's result occurs in a branching process context, and his method of proof is not known to the present author.

Theorem. *If* (1) $c(n)>0$ *and* $\{c(n)\}$ *is monotone;* (2) $\lim_{n\to\infty} c(kn)/c(n) = \varphi(k)$ *exists, with* $\varphi(k)>0$, *for each positive integer* k; (3) $c(n+1)/c(n) \to 1$ *as* $n\to\infty$, *then the function* $c(x)$, $x\geq 0$ *is regularly varying.*

We sketch a proof depending heavily on Lemma 1; and just for the case $c(n)\downarrow$.

(a) (1), (2) and (3) imply that

$$c(xk)/c(x) \to \varphi(k), \text{ integer } k\geq 2,$$

as $x\to\infty$ through *all* real values;

(b) $\varphi(k)=k^{-\beta}$, $\beta\geq 0$;

(c) $c(xr)/c(x)\to r^{-\beta}$ for every positive rational r, and hence by monotonicity of $c(x)$ for every real $\lambda>0$.

The only non-obvious step is the deduction of (b), which is obtained by writing $d(x)=\log c(e^x)$ and deducing from (a) via Lemma 1 that as $x\to\infty$

$$d(x)/x \to \log\varphi(k)/\log k,$$

so the right-hand side is a constant independent of k, which we may call $-\beta$. ($\beta\geq 0$ only because we assume $c(n)\downarrow$).

REFERENCES

[1] R. Bojanić and E. Seneta, *Slowly varying functions and asymptotic relations*, J. Math. Anal. Appl., 34 (1971), 302—315.

[2] L. de Haan, *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes*, Math. Centre Tracts 32, Amsterdam, 1970.

[3] E. M. Hardy and G. H. Wright, *An Introduction to the Theory of Numbers.* (3rd edn) Clarendon Press, Oxford, 1954.

[4] C. H. Heiberg, *A proof of a conjecture of Karamata*, Publ. Inst. Math., Acad S:rbe Sci., Nouv. ser. 12 (26), (1971) 41—44.

[5] J. Karamata, *Sur un mode de croissance régulière, Théorèmes fondamentaux.* Bull. Soc. Math. France, 61 (1933), 55—62.

[6] J. Карамата (J. Karamata) *Теорија и Пракса Stieltjes-ова Интеграла.* Српска Академија Наука, Посебна издања, Књига CLIV. Математички институт, Књига 1. Београд (Beograd), 1949.

[7] G. Pólya and G. Szegö *Aufgaben und Lehrsätze der Analysis*, Erster Band: Reihen, Integralrechnung, Funktionentheorie. 4 Auflage, Springer-Verlag, Heidelberg, 1970. (1 Auflage: 1925).

[8] E. Seneta, *Sequential criteria for regular variation*, Quart. J. Math. Oxford (2), 22 (1971), 565—570.

[9] H. Steinhaus, *Sur les distances des points de mesure positive*, Fundamenta Math., 1 (1920), 93—104.

Department of Statistics, Fine Hall
Princeton University
Princeton, N. J. 08540
U.S.A.

and (permanent address)

Department of Statistics, S.G.S.,
Australian National University
P.O. Box 4, Canberra A.C.T. 2600
Australia