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ON EQUALITY OF INDICES

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The following two theorems appear in [2, p. 13].

Theorem 1: Inequality on Indices; $[A \cup B : B] \geqslant [A : A \cap B]$

Theorem 2: Equality of Indices: If $[A \cup B : B]$ and $[A \cup B : A]$ are finite and relatively prime, then

$$[A \cup B : B] = [A : A \cap B] \text{ and } [A \cup B : A] = [B : A \cap A].$$

In this paper we propose to tackle the problem of finding the most general conditions for the validity of equality of indices. For this purpose we first study the conditions of above theorem 2. We take up the case when $0(A \cup B)$ is finite

We know that

$$[A \cup B : A] = \frac{0(A \cup B)}{0(A)} \text{ and } [A \cup B : B] = \frac{0(A \cup B)}{0(B)}.$$

Since A, B are subgroups of $A \cup B$, their orders will divide the order of $A \cup B$, and so let

$$0(A \cup B) = K\,0(A) = K'\,0(B).$$

From the condition of theorem, it follows that

$$(k, k') = 1.$$

Now, let

$$\mu = \frac{0(B)}{K} = \frac{0(A)}{K'}.$$

Then

$$\left(\frac{0(B)}{\mu}, \frac{0(A)}{\mu}\right) = 1$$

and as $0(A \cap B)$ divides both $0(A)$ and $0(B)$, we must have

$$\mu = \nu\,0(A \cap B)$$

for some integer ν, Thus we have

$$0(A \cup B) = \frac{0(A)\,0(B)}{\nu\,0(A \cap B)}$$

Now, we first give a new proof of a counting principle given in [1, p. 39].

A counting principle: If A and B are finite subgroups of G of orders $0(A)$ and $0(B)$, respectively, then

$$0(AB)=\frac{0(A)\,0(B)}{0(A\cap B)}.$$

P r o o f. Denote $A\cap B$ by D and let

$$A=1.D+x_2D+\cdots+x_nD,$$
$$B=D.1+Dy_2+\cdots+xy_m$$

be two coset decompositions of A and B. Then

$$AB=(1D+x_2D+\cdots+x_nD)(D1+Dy_2+\cdots+Dy_m)$$
$$=(1+x_2+\cdots+x_n)D(1+y_2+\cdots+y_m).$$

We show that x_iDy_j's are disjoint. For this let us assume on the contrary that x_iDy_j and x_kDy_l have an element in common $(x_i\neq x_k\, y_j\neq y_l)$. Then we have

$$x_i d_1 y_j=x_k d_2 y_l \quad (d_1, d_2\in D)$$

or

$$x_k^{-1}x_i d_1=d_2 y_l y_j^{-1}.$$

Since the first is in A and second is in B, we have that both belong to D. Thus we have for some $d_3, d_4\in D$,

$$x_i=x_k d_3, \quad y_j=d_4 y_l$$

and hence we find that

$$x_iD=x_kD \text{ and } Dy_j=Dy_l,$$

which is contrary to the fact that x_iD's and Dy_j's are distinct.

Hence

$$0(AB)=n\,0(D)\,m=\frac{0(A)}{0(D)}\cdot\frac{0(B)}{0(D)}\cdot 0(D)=\frac{0(A)\,0(B)}{0(D)}.$$

Now as $AB\subseteq A\cup B$, we have $0(AB)\leqslant 0(A\cup B)$, or

$$\frac{0(A)\,0(B)}{0(A\cap B)}\leqslant\frac{0(A)\,0(B)}{\nu\,0(A\cap B)}$$

so that $\nu=1$ and $AB=A\cup B$. Interchanging A and B, we have $BA=B\cup A=$ $=A\cup B$. Hence we have $AB=BA$ in this case i.e. A and B commute.

As it will be shown below, this condition plays an important role in our problem.

T h e o r e m 3 *If A and B are finite subgroups of a group G, then in order that the equality of indices*

$$[A\cup B:B]=[A:A\cap B]$$

and

$$[A\cup B:A]=[B:A\cap B]$$

hold, it is necessary and sufficient that A and B commute i.e. $AB=BA$.

P r o o f. In this finite case the proof is facilitated by the following expressions:

$$[A\cup B:B]=\frac{0(A\cup B)}{0(B)}\quad [A\cup B:A]=\frac{0(A\cup B)}{0(A)}$$

$$[A:A\cap B]=\frac{0(A)}{0(A\cap B)}\quad [B:A\cap B]=\frac{0(B)}{0(A\cap B)}$$

$$0(AB)=0(BA)=\frac{0(A)\,0(B)}{0(A\cap B)}.$$

Let $AB=BA$. Then $A\cup B=AB=BA$, from which it follows that $A\cup B$ is of finite order. Thus, what we have to prove reduces to

$$\frac{0(A\cup B)}{0(B)}=\frac{0(A)}{0(A\cap B)}\quad\text{and}\quad\frac{0(A\cup B)}{0(A)}=\frac{0(B)}{0(A\cap B)}$$

i.e. $0(AB)=\dfrac{0(A)\,0(B)}{0(A\cap B)}$ in both cases as $A\cup B=AB$, and which is true.

Again, let

$$[A\cup B:B]=[A:A\cap B]$$

i.e.

$$[A\cup B:B]=\frac{0(A)}{0(A\cap B)}=\frac{0(AB)}{0(B)}$$

Hence the index of B in $A\cup B$ is finite and so we have $0(A\cup B)=0(B)\,0(AB)/0(B)=0(AB)$, so that as $AB\subseteq A\cup B$, we have $AB=A\cup B$. Similarly $BA=A\cup B$ whence we have $AB=BA$, which completes the proof.

To prove the result for general A and B i. e. when they are not necessarily finite, the above method fails and we are to follow another course. We first of all extend the Lagrange's theorem for a group G to the set AB where A and B are two subgroups of a group. We state our result in the

T h e o r e m 4. *If A and B are subgroups of a group then the set AB can be partitioned into cosets of A (or B) such that they all have the same number of elements and that they are mutually exclusive.*

P r o o f. If A is equal to AB, we have nothing to prove. If not let a residual element in AB be x_α. Then $Ax_\alpha\subseteq AB$. If A and Ax_α together are equal to AB, we are done otherwise we continue likewise and thus have cosets A, Ax_α, $Ax_\beta, \ldots$.. Since an arbitrary $x\in AB$ also belongs to Ax, these cosets together must cover AB. Now we show that they are mutually exclusive.

Let, on the contrary, two non-indentical cosets, say, Ax_δ and Ax_λ, have an element in common, so that $a_1x_\delta=a_2x_\lambda$

$$\text{whence}\quad Aa_1x_\delta=Aa_2x_\lambda\qquad\text{i.e.}\quad Ax_\delta=Ax_\lambda$$

so that they are indentical contradicting our hypothesis. Similarly for B. Hence the result.

Let us call the cardinality of the set of dinstinct cosets (right or left) of AB w.r.t. A the index of A in AB denoted by $[AB:A]$.

Obviously we have

$$0(AB)=[AB:A]\,0(A).$$

From above theorem taking A or B to be the whole group G, Lagrange's theorem follows.

Now we prove the following theorem which will straightway lead to the solution of our problem.

Theorem 5 (Equality of Indices): *If A and B are two subgroups of a group G, then we have*

$$[AB:B]=[A:A\cap B]$$

such that the cosets have a $1\cdot 1$ correspondence and have the form xB and $xA\cap B$, respectively, $x\in A$.

Proof: – Call $A\cap B=D$. Let AB be decomposed into cosets $1.B$, $x_\alpha B, x_\beta B,\ldots.$ Then we assert that the cosets $1.D, x_\alpha D, x_\beta D,\ldots$ are all distinct in A. For if $x_\lambda D=x_\mu D\,(\lambda\neq\mu)$, then $x_\lambda=x_\mu d$ for some $d\in D$ and then $x_\lambda B=x_\mu dB=x_\mu B$ contrary to our assumption. Hence there are at least as many distinct cosets of $A\cap B$ in A as there are of B in AB. Hence $[AB:B]\leqslant[A;A\cap B]$.

Again let A be decomposed into the distinct cosets $1.D, y_\alpha D, y_\beta D,\ldots.$ Then we assert that the cosets $1.B, y_\alpha B, y_\beta B,\ldots$ are all distinct in AB. For if $y_\lambda B=y_\mu B\,(\lambda\neq\mu)$, then $y_\lambda=y_\mu b$ for some $b\in B$. Obviously $y_\lambda. y_\mu$ both belong to A, so that $b=y_\mu^{-1}y_\lambda\in A$, i.e. $b\in D$. So the cosets $y_\lambda D$ and $y_\mu D$ have in common the element $y_\lambda=y_\mu b$, contrary to the assumption. Hence there are at least as many distinct cosets of B in AB as there are of $A\cap B$ in A. Hence $[AB:B]\geqslant[A:A\cap B]$. Thus it follows that $[AB:B]=[A:A\cap B]$.

Likewise, it can be shown that

$$[AB:A]=[B:A\cap B].$$

Now, we tackle our problem in the following:

Theorem 6 (Equality of Indices): – *A necessary and sufficient condition that*

$$[A\cup B:B]=[A:A\cap B] \text{ and } [A\cup B:A]=[B:A\cap B].$$

where the indices are finite, may hold is that A and B commute i.e. $AB=BA$.

Proof. Let $AB=BA$. Then each of these is also equal to $A\cup B$ and the theorem reduces to the equalities $[AB:B]=[A:A\cap B]$ and $[AB:A]=[B:A\cap B]$ which hold due to theorem 5.

Again, let the equalities hold. Then by theorem 5 we have

$$[A\cup B:B]=[AB:B] \text{ and } [A\cup B:A]=[AB:A].$$

It follows that in each case $A\cup B=AB$. Interchanging A and B and using the same argument we have $A\cup B=BA$. Hence $AB=BA$ and the proof is completed.

Theorem 6 may not hold when the indices are not finite for $[A\cup B:B]$ may be equal to $[AB:B]$ without $A\cup B$ being equal to AB. Nevertheless the sufficient part of the theorem holds in general which we state as

Theorem 7 (Equality of Indices): *If* $AB = BA$, *then*

$$[A \cup B : B] = [A : A \cap B]$$

and

$$[A \cup B : A] = [B : A \cap B].$$

The proof follows along the same lines as in theorem 6 without assuming the indices to be finite.

Now finally we remove the difficulty of non-applicability of theorem 6 for nonfinite indices. For this we use necessary part of theorem 5. Let all distinct cosets of $A \cap B$ in A be represented in the form $\{xA \cap B : x \in X \text{(say)}\}$. Then cosets of B in AB have the form $\{xB : x \in X\}$. If A and B commute, then the same will be cosets of B in $A \cup B$, and if the same are cosets of B in $A \cup B$, then $AB = A \cup B$. Similarly for BA. And we will have that $AB = BA$.

Here we have denoted the set of cosets with curled brackets.

Theorem 8 (Equality of Indices): – *Let*

$$\{A : A \cap B\} = \{xA \cap B : x \in X\} \text{ and } \{B : A \cap B\} = \{yA \cap B : y \in Y\}$$

then a necessary and sufficient condition that

$$[A \cup B : B] = [A : A \cap B] \ \{A \cup B : B\} = \{xB : x \in X\}$$

$$[A \cup B : A] = [B : A \cap B] \ \{A \cup B : A\} = \{yA : y \in Y\}$$

hold is that A and B commute.

From theorems 6 and 2 follows the result that:

If $[A \cup B : A]$ and $[A \cup B : B]$ are finite and relatively prime, then A and B commute and $A \cup B = AB = BA$.

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