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SOME RESULTS INVOLVING A GENERALIZED KAMPÉ DE FÉRIET FUNCTION

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1. In this note we derive certain new generating functions for a generalized Kampé de Fériet function in two arguments and consider their multidimensional extensions. We also evaluate two new Eulerian integrals of the second kind involving this function.

The following generalized Kampé de Fériet function has been studied by H. M. Srivastava and M. C. Daoust in [1]:

$$S^{A:B;B'}_{C:D;D'}\binom{x}{y} \equiv S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\Phi]:[(b):\psi];\ [(b'):\psi'];\\ [(c):\delta,\varepsilon]:[(d):\eta];\ [(d'):\eta'];\end{matrix}\ x,\ y\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{A}\Gamma(a_j+m\theta_j+n\Phi_j)\prod_{j=1}^{B}\Gamma(b_j+m\psi_j)\prod_{j=1}^{B'}\Gamma(b_j'+n\psi_j')}{\prod_{j=1}^{C}\Gamma(c_j+m\delta_j+n\varepsilon_j)\prod_{j=1}^{D}\Gamma(d_j+m\eta_j)\prod_{j=1}^{D'}\Gamma(d_j'+n\eta_j')}\ \frac{x^m}{m!}\ \frac{y^n}{n!},$$

where, for convergence,

$$(1.1)\quad \sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\eta_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j+1>0,\quad \sum_{j=1}^{C}\varepsilon_j+\sum_{j=1}^{D'}\eta_j'-\sum_{j=1}^{A}\Phi_j-\sum_{j=1}^{B'}\psi_j'+1>0.$$

For a complete set of these conditions, see sections 3 and 4 of a recent paper [2].

2. Following the simple method in [5], i.e. using only Taylor's theorem, we can prove that

$$(2.1)\qquad (1+t)^{-\lambda}\, S^{A:B;B'}_{C:D;D'}\binom{x}{y/(1+t)^{\xi}}$$

$$= \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\, S^{A:B;B'+1}_{C:D;D'+1}\left(\begin{matrix}[(a):\theta,\Phi]:[(b):\psi];\ [(b'):\psi],\ [\lambda+k:\xi];\\ [(c):\delta,\varepsilon]:[(d):\eta];\ [(d'):\eta'],\ [\lambda:\xi];\end{matrix}\ x,\ y\right)$$

and

$$(2.2)\qquad (1+t)^{-\lambda}\, S^{A:B;B'}_{C:C;D'}\binom{x/(1+t)^{\xi}}{y}$$

$$= \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\, S^{A:B+1;B'}_{C:D+1;D'}\left(\begin{matrix}[(a):\theta,\Phi]:[(b):\psi],\ [\lambda+k:\xi];\ [(b'):\psi'];\\ [(c):\delta,\varepsilon]:[(d):\eta],\ [\lambda:\xi];\ [(d'):\eta'];\end{matrix}\ x,\ y\right),$$

where $|t|<1$, λ is an arbitrary complex number, $\xi>0$, and θ's, Φ's, ψ's, etc., are positive constants such that (1.1) holds; the equality holds when $|x|$ and $|y|$ are constrained appropriately.

Proof of (2.1). Let

$$f(p)=p^{-\lambda} S_{C:D;D'}^{A:B;B'}\begin{pmatrix} x \\ yp^{-\xi} \end{pmatrix}$$

and

$$\varphi(p)=p^{-\lambda-n\xi}.$$

Since for each $k=0, 1, 2, \ldots$ the derivative $\varphi^{(k)}(p)$ is

$$\varphi^{(k)}(p)=(-1)^k p^{-\lambda-n\xi-k}\frac{\Gamma(\lambda+n\xi+k)}{\Gamma(\lambda+n\xi)},$$

it follows that

$$f^{(k)}(p)=(-1)^k p^{-\lambda-k}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+n\xi+k)}{\Gamma(\lambda+n\xi)}\frac{\prod_{j=1}^{A}\Gamma(a_j+m\theta_j+n\Phi_j)}{\prod_{j=1}^{C}\Gamma(c_j+m\delta_j+n\varepsilon_j)}$$

$$\times\frac{\prod_{j=1}^{B}\Gamma(b_j+m\psi_j)\prod_{j=1}^{B'}\Gamma(b_j'+n\psi_j')}{\prod_{j=1}^{D}\Gamma(d_j+m\eta_j)\prod_{j=1}^{D'}\Gamma(d_j'+n\eta_j')}\frac{x^m}{m!}\frac{(y/p^{\xi})^n}{n!}$$

$$=(-1)^k p^{-\lambda-k} S_{C:D;D'+1}^{A:B;B'+1}\left(\begin{matrix}[(a):\theta,\Phi]:[(b):\psi];\ [(b'):\psi'],\ [\lambda+k:\xi]; \\ [(c):\delta,\varepsilon]:[(d):\eta];\ [(d'):\eta'],\ [\lambda:\xi];\end{matrix}\ x,\ \frac{y}{p^{\xi}}\right).$$

Using Taylor's theorem, we get

$$f(p+t)=(p+t)^{-\lambda} S_{C:D;D'}^{A:B;B'}\begin{pmatrix} x \\ y/(p+t)^{\xi} \end{pmatrix}=\sum_{k=0}^{\infty}\frac{t^k}{k!}f^{(k)}(p)$$

$$=\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}p^{-\lambda-k} S_{C:D;D'+1}^{A:B;B'+1}\left(\begin{matrix}[(a):\theta,\Phi]:[(b):\psi];\ [(b'):\psi'],\ [\lambda+k:\xi]; \\ [(c):\delta,\varepsilon]:[(d):\eta];\ [(d'):\eta'],\ [\lambda:\xi];\end{matrix}\ x,\ \frac{y}{p^{\xi}}\right),$$

from which (2.1) follows at once.

Similarly on using the function

$$p^{-\lambda-m\xi}$$

instead of $\varphi(p)$ and proceeding on the similar lines, relation (2.2) can be proved.

3. As an immediate consequence of the well-known formulas ([6], p. 292)

$$\int_0^{\infty} x^{\beta-1}e^{-x}L_n^{\alpha}(x)\,dx=\frac{\Gamma(\alpha-\beta+n+1)\,\Gamma(\beta)}{n!\,\Gamma(\alpha-\beta+1)},\quad \mathrm{Re}(\beta)>0;$$

$$\int_0^{\infty} x^{\alpha}e^{-x}\{L_n^{\alpha}(x)\}^2\,dx=\frac{\Gamma(\alpha+n+1)}{n!},\quad \mathrm{Re}(\alpha)>0,$$

where $L_n^\alpha(z)$ denote Laguerre polynomial, we get the following two new Eulerian integrals of the second kind which involve generalized Kampé de Fériet function:

$$\int_0^\infty e^{-t}\, t^{\beta-1}\, L_r^\alpha\,(t)\, S_{C:D;\,D'}^{A:B;\,B'}\binom{xt^\mu}{yt^\nu}\,dt$$

$$=\frac{1}{r!}\, S_{C+1:D;\,D'}^{A+2:B;\,B'}\left(\begin{matrix}[(a):\theta,\,\Phi],\ [\alpha-\beta+r+1:-\mu,\,-\nu],\ [\beta:\mu,\,\nu]:[(b):\psi];\ [(b'):\psi'];\\ [(c):\delta,\,\varepsilon].\ [\alpha-\beta+1:-\mu,\,-\nu]:[(d):\eta];\ [(d'):\eta'];\end{matrix}\ x,\,y\right),$$

$$r=0,\,1,\,2,\,\ldots,\ \mathrm{Re}\,(\beta)>0,\ \mu>0,\ \nu>0;$$

$$\int_0^\infty e^{-t}\, t^{\alpha}\,\{L_r^\alpha\,(t)\}^2\, S_{C:D;\,D'}^{A:B;\,B'}\binom{xt^\mu}{yt^\nu}\,dt$$

$$=\frac{1}{r!}\, S_{C:D;\,D'}^{A+1:B;\,B'}\left(\begin{matrix}[(a):\theta,\,\Phi],\ [\alpha+r+1:\mu,\,\nu]:[(b):\psi];\ [(b'):\psi'];\\ [(c):\delta,\,\varepsilon]:[(d):\eta];\ [(d'):\eta'];\end{matrix}\ x,\,y\right),$$

$$r=0,\,1,\,2,\,\ldots,\ \mathrm{Re}\,(\alpha)>0,\ \mu>0,\ \nu>0;$$

provided that the double series on the right-hand sides converge.

4. The method of the section 2 can be extended to obtain various new generating functions for the generalized Lauricella function

$$S_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\equiv S_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}\left(\begin{matrix}[(a):\theta',\,\ldots,\,\theta^{(n)}]:\\ [(c):\psi',\,\ldots,\,\psi^{(n)}]:\end{matrix}\right.$$
$$\left.\begin{matrix}[(b'):\Phi'];\ \ldots;\ [(b^{(n)}):\Phi^{(n)}];\\ [(d'):\delta'];\ \ldots;\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ x_1,\,\ldots,\,x_n\right)$$

defined by the multiple hypergeometric series (see [4] and [3])

$$\sum_{m_1=0}^\infty\cdots\sum_{m_n=0}^\infty\frac{\prod_{j=1}^{A}\Gamma\left(a_j+\sum_{i=1}^{n}m_i\theta_j^{(i)}\right)\prod_{j=1}^{B'}\Gamma\,(b_j'+m_1\Phi_j')\cdots\prod_{j=1}^{B^{(n)}}\Gamma\,(b_j^{(n)}+m_n\Phi_j^{(n)})}{\prod_{j=1}^{C}\Gamma\left(c_j+\sum_{i=1}^{n}m_i\psi_j^{(i)}\right)\prod_{j=1}^{D'}\Gamma\,(d_j'+m_1\delta_j')\cdots\prod_{j=1}^{D^{(n)}}\Gamma\,(d_j^{(n)}+m_n\delta_j^{(n)})}$$

$$\cdot\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}.\tag{4.2}$$

For a detailed discussion of the convergence conditions of the series in (4.2) see section 5 in [2].

Now, our multidimensional extensions of the generating relations (2.1) and (2.2) are

$$(1+t)^{-\lambda}\, S_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}\begin{pmatrix}x_1\\ \vdots\\ x_i/(1+t)\\ \vdots\\ x_n\end{pmatrix}=\sum_{k=0}^\infty\frac{(-t)^k}{k!}\, S_{C:D';\ldots;D^{(i)}+1;\ldots;D^{(n)}}^{A:B';\ldots;B^{(i)}+1;\ldots;B^{(n)}}\tag{4.3}$$

$$\left(\begin{matrix}[(a):\theta',\,\ldots,\,\theta^{(n)}]:[(b'):\Phi'];\ldots;[(b^{(i)}):\Phi^{(i)}],\ [\lambda+k:\xi];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [(c):\psi',\,\ldots,\,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(i)}):\delta^{(i)}],\ [\lambda:\xi];\ \ldots;\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ x_1,\ldots,x_n\right),$$

where $|t|<1$, λ is an arbitrary complex number, $\xi>0$, $|x_i|$, $i=1,\dots,n$ are constrained appropriately, and

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D(i)}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B(i)}\Phi_j^{(i)}\geqq 0,\quad i=1,\dots,n.$$

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