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ON THE USE OF STRUCTURAL FORMULAE OF KINEMATIC CHAINS

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The general structural formula could not be used for all kinematic chains (for example, for four-bar chain). Therefore, the known concepts of families of kinematic chains, based on common joint conditions of kinematic pairs, are introduced. However, there are kinematic chains whose actual degree of mobility (degree of freedom with respect to the frame) could not be obtained by any of structural formulae of these families. Then, one resorts to removal of links which include the passive conditions.

We showed that, in order to calculate the degree of mobility, it is not necessary to remove any link, and that it is possible to use, for a kinematic chain of ith family, anyone of the structural formulae of the jth family ($j \leq i$). It is necessary to take care of dependence among joint conditions of all kinematic pairs, which takes place when a kinematic chain has closed parts. Then, before the use of the structural formula, one should find the assembly of necessary kinematic pairs. The concept of necessity depends on a choice of possible structural formula.

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The degree of freedom for a motion of a kinematic chain with respect to one of its link (the frame), i.e. its degree of mobility W, is obtained by the structural formula

$$W = 6n - \sum_{i=1}^{5} ip_i, \tag{1}$$

n being the number of mobile links (mobile with respect to the frame), and p_i the number of kinematic pairs of the ith class. The cited formula is obtained when one takes into consideration that there are six degrees of freedom for a rigid body, and that every kinematic pair of the ith class represents the connection of two links (rigid bodies) which imposes to the motion of one of them, with respect to the other, i independent joint conditions.

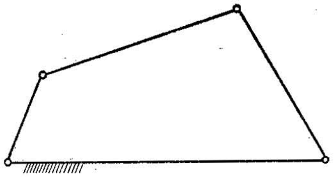

Fig. 1

However, this formula does not give in all cases of kinematic chains an actual degree of mobility. For example, there is one degree of mobility for a four-bar chain (fig. 1), while (1) gives the senseless result $W = -2$, because the number of mobile links is three, and there are four kinematic pairs of the V class (four pin joints).

For that reason, in literature[1] one resorts to a reasoning which follows. Among the joint conditions of kinematic pairs of kinematic chain from fig. 1 there are conditions which are common for all kinematic pairs. Namely, there exists a direction in which none of joints permits a translation, while the other two common joint conditions of all kinematic pairs are impossibilities of rotations about axes in the plane perpendicular to that direction. To obtain a formula which gives, in this situation, the actual degree of mobility, one proceeds from the fact that there would be three degrees of freedom, with respect to the frame, for each of the n mobile links if in each kinematic pair these exist only these three common joint conditions, and that the presence of each kinematic pair of the ith class subtracts from $3n$ only $i-3$, so that the structural formula, then, becomes

$$W=3n-\sum_{i=4}^{5}(i-3)\,p_i, \tag{2}$$

i.e.

$$W=3n-p_4-2p_5. \tag{3}$$

According to (3), for the four-bar chain degree of mobility one obtains the correct result $W=1$.

In a similar way, the degree of mobility of the kinematic chain whose kinematic pairs have i common joint conditions (the kinematic chain of the ith family) is given by the structural formula of the ith family

$$\begin{cases} W=(6-i)\,n-\sum\limits_{j=i+1}^{5}(j-i)\,p_j, & (i=0,\ 1,\ 2,\ 3,\ 4) \\ W=n, & (i=5). \end{cases} \tag{4}$$

The formulae (1) and (3) are special cases of (4).

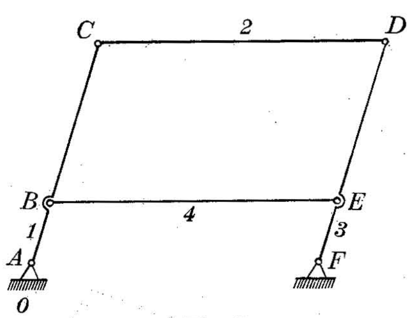

Fig. 2

However, there are kinematic chains whose actual degree of mobility cannot be obtained by any of the formulae (4). Such is the kinematic chain from fig. 2, for which we have $\overline{AF}=\overline{CD}$, $\overline{AC}=\overline{FD}$ and $\overline{AB}=\overline{FE}$. Obviously, there is one degree of mobility. However, its kinematic pairs have three common joint conditions, as well as the kinematic chain from fig. 1, and, therefore, (3) should be valid for it. Taking into account that $n=4$, $p_4=0$, $p_5=6$, (3) gives $W=0$, which is wrong. In literature[2] one finds the way out of this trouble in assertion that the link BE (or CD) imposes passive joint conditions to the motion of the kinematic chain, and so a removal of this link does not disturb the character of the kinematic chain motion. Therefore, it is said, the degree of mobility should be calculated on the basis of kinematic chain obtained from the studied one by elimination of the link which introduces these passive conditions.

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1) I. I. Artobolevski, Teorija mehanizmov, „Nauka“, Moskva 1965, pp. 70—90.

2) Op. cit.

Such explanation, in our opinion, has two defects.

First, the kinematic chain from fig. 2 and a kinematic chain originating from it by elimination of the link BE are not the same ones, disregarding the same character of motions of corresponding links, and it is natural to insist on a treatment which would determine the degree of mobility without eliminaton of any link.

Secondly, an existence of passive joint conditions, i.e. restrictions unnecessarily imposed, is, just, the reason of abandonment of (1) and adoption of (4), what we shall prove. (If so, this shows that an existence of passive joint conditions does not provoke the need of abandonment of certain links, but only the need of eliminating the influence of passive joint conditions on results obtained by structural formulae.)

We saw that the degree of mobility of the kinematic chain from fig. 1 cannot be obtained from (1). Its degree of mobility is given by (3), but it is not explained why (1) does not give a correct result. We shall show that a wrong result is a consequence of unnecessity of assembly of kinematic pairs as shown in fig. 1. Namely, the joint conditions of given kinematic pairs, which are, by definition of joint conditions, *independent* when one considers anyone of kinematic pairs separately, are *dependent* in the scope of kinematic chain. In other words, the joint conditions of one of the kinematic pairs represent restrictions, some of which are already imposed by other kinematic pairs. These conditions are unnecessary.

To find one assembly of necessary kinematic pairs in the kinematic chain from fig. 1, let us observe a kinematic chain which differs from the first one in such a way that the links 0 and 3 are not in a kinematic pair (fig. 3) and ask the question: with what kind of pair we have to join the links 0 and 3 in order that a motion of the link 3, with respect to the link 0, be a rotation about fixed axis through the point D of the link 0 parallel to the axes A, B, and C? Our task is to find a kinematic pair which provides a required motion and being of the lowest class. The wrong result we obtained from (1) tells us that this kinematic pair is not a pin joint.

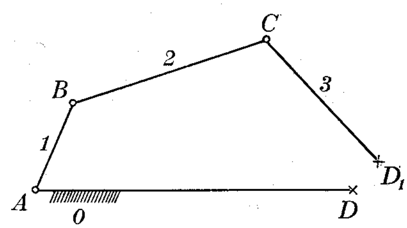

Fig. 3

Indeed, on account of parallelism of the axes A, B, and C, it follows that the links 1, 2, and 3 perform plane motions with respect to the frame. To find a necessary connection between the links 0 and 3 which would provide to the link 3 the same motion with respect to the frame as their connection in a pin joint with an axis perpendicular to the plane of motion which passes through the point D of the frame and the point D_1 of the link 3, it is sufficient to connect the two links in a kinematic pair which would make impossible the motion, with respect to the frame, of the point D_1 of the link 3 in the mentioned plane. Taking the point D_1 as a pole rigidly connected to the link 3 which determines its translation, it is seen that the kinematic pair should make impossible two translations (in the mentioned plane), and permit one translation and all rotations. The required kinematic pair is, therefore, of the II class, and we can realize it by a sphere into the circular cylinder with an axis perpendicular to the plane of motion, which pair, by a temporary convention for the scheme of such a kinematic pair, is represented in fig. 4. Applying (1) in the case of kinematic chain from fig. 4 we obtain, on account of $n=3$, $p_2=1$, $p_5=3$, that there is $W=6\cdot 3-2\cdot 1-5\cdot 3=1$ degree of mobility, i.e. the result which should be obtained for the kinematic chain from fig. 1.

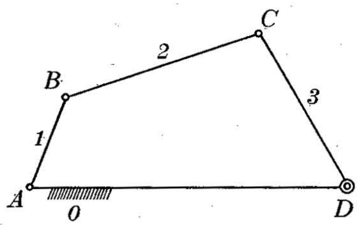

Fig. 4

To a situation in which, before the use of (1), one should carry out an analysis of necessity of an assembly of kinematic pairs in a kinematic chain, one comes with closed kinematic chains. Then, a dependence of joint conditions of kinematic pairs could exist, as in case of kinematic chain from fig. 1.

It is possible to avoid such an analysis in many cases, as we have seen, and to find a degree of mobility of a kinematic chain, basing a calculation on a given assembly of kinematic pairs, without searching their necessary assembly. Of course, then (1) is not valid, but one of the formulae (4) would be valid.

However, there are kinematic chains whose degree of mobility could not be obtained by any of the formulae (4), basing a calculation on a given assembly of kinematic pairs, but should resort to a search of assembly of necessary kinematic pairs.

We shall show this on the kinematic chain from fig. 2. Its kinematic pairs have three common joint conditions, because there exists the direction in which none of joints permits a translation, while the other two common joint conditions are impossibilities of rotations about axes in the plane perpendicular to that direction, and, therefore, we have the case of plane kinematic chain. In searching how many degrees of mobility it has, one could use (3). But even remaining joint conditions, conditions which are not common for all kinematic pairs, are not independent in the scope of this kinematic chain. The kinematic pair of the links 3 and 4, for example, imposes to their mutual relative motions in the plane of kinematic chain two joint conditions (two restrictions: taking the point E as a pole determining a translation, it does not permit, in the relative motion, any of the two translations in the plane of motion), although for realization of a given kinematic chain the joint of the links 3 and 4 may permit a translation of the link 3, with respect to the link 4, in the direction FD. Forbiddance of this translation is the joint condition which is not independent in the assembly of all joint conditions of the considered plane kinematic chain, because this condition, in the scope of the kinematic chain, is imposed by the assembly of all other kinematic pairs. Connecting the links 3 and 4 in a kinematic pair of the IV class which permits a rotation about the axis perpendicular to the plane of motion and a translation in the direction FD we obtain the asked assembly of necessary kinematic pairs (necessary in the scope of plane kinematic chains, i.e. with respect to (3)). Then we have $n=4$, $p_4=1$, $p_5=5$, and (3) gives $W=3\cdot 4-1\cdot 1-2\cdot 5=1$, which is in agreement with the actual degree of mobility of the kinematic chain from fig. 2.

From this illustration it is seen that an assembly of necessary kinematic pairs depends on a choice among permitted structural formulae. In the case of the same kinematic chain, the kinematic chain from fig. 2, the assembly of necessary kinematic pairs with respect to (1) is different: it is sufficient to connect the links 3 and 4 in a kinematic pair of the I class (which permits three independent rotations and two independent translations, and does not permit the translation perpendicular to FD in the plane of motion), and the link 3 and the frame in a kinematic pair of the II class (in the same way as the links 0 and 3 in the fig. 4). Then we have $n=4$, $p_1=1$, $p_2=1$, $p_5=4$, and (1) gives $W=6\cdot 4-1\cdot 1-2\cdot 1-5\cdot 4=1$, as before.

Now, we can ask a question: does exist, perhaps, some structural formula which gives an actual degree of mobility of the kinematic chain from fig. 2, calculated on the basis of the given assembly of kinematic pairs, without reduction of this assembly to some other, necessary one? Asking this question, one should have in mind that a structural formula is a formula which gives a degree of mobility of a kinematic chain on the basis of a number of its links and numbers of kinematic pairs of particular classes.

The answer is: no, there is not such structural formula. To prove this it is sufficient to cite one kinematic chain which would have the same number of mobile links ($n=4$) and the same number of kinematic pairs of the same class ($p_5=6$) and type as the kinematic chain from fig. 2, and whose degree of mobility would differ from the degree of mobility of the kinematic chain from fig. 2. Such kinematic chain is shown on fig. 5, for which we have $\overline{AF}=\overline{CD}$, $\overline{AC}=\overline{FD}$ and $\overline{AB}\neq\overline{FE}$. Obviously, there is zero degree of mobility. This result is, also, obtained from (3), $W=3\cdot 4-2\cdot 6=0$. The reason because (3), in application to the kinematic chain from fig. 5, gives directly the true result is in the fact that the assembly of given kinematic pairs is, with respect to (3), simultaneously the assembly of necessary kinematic pairs.

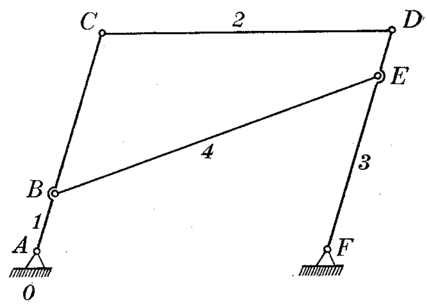

Fig. 5

From all what is said it follows:

The degree of mobility of a kinematic chain, whose kinematic pairs have $k$common joint conditions ($k=0, 1, 2, 3, 4, 5$), is obtained by the structural formula

$$(4)\qquad \begin{cases} W=(6-i)\,n-\sum\limits_{j=i+1}^{5}(j-i)\,p_i, & (i=0,\,1,\,2,\,3,\,4),\\ W=n, & (i=5), \end{cases}$$

for $i\leqslant k$, the assembly of kinematic pairs being reduced to an assembly of necessary kinematic pairs with respect to the chosen i.