A CHARACTERIZATION OF CRAMÉR REPRESENTATION OF STOCHASTIC PROCESSES

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Let $X(t), t \in [a, b]$ be a complex valued stochastic process with $EX(t) = 0$ and $E|X(t)|^2 < \infty$ for each $t \in [a, b]$. The interval $[a, b]$ may be finite or infinite. We consider the process $X(t), t \in [a, b]$ as a curve in the Hilbert space $H$ of all random variables with finite dispersion. The scalar product in $H$ is defined by $(X, Y) = EXY; X, Y \in H$. Let $H(X; t)$ be the linear closure in $H$ generated by $X(s), s \in [a, t]$. We will suppose that $X(t), t \in [a, b]$ is continuous from the left (which yields the separability of the spaces $H(X; t), t \in [a, b]$) and regular (i.e. $\bigcap_{t>a} H(X; t) = 0$).

Let $Z(t) = \|Z(t)\|_{n=1}^{N}, t \in [a, b]$ be a stochastic process (considered as a vector column), where $Z_n(t), t \in [a, b], n = 1, N$ are mutually orthogonal processes with orthogonal increments. $N$ may be finite or infinite. Put $F(t) = EZ(t)Z^*(t) = F_{jk}(t)\|_{k, j=1}^{N}$, where $Z^*(t)$ denotes the transposed matrix of $Z(t)$.

Matrix function $F(t), t \in [a, b]$ has non-zero elements only on the principal diagonal and we denote them by $F_{nn}(t) = E|Z_n(t)|^2, n = 1, N$.

Let $L_2(F)$ be the Hilbert space of all complex valued vector row functions $f(t) = \|f_n(t)\|_{n=1}^{N}, t \in [a, b]$ for which

$$\int_a^b |f(u)|^2 dF(u) < \infty.$$ 

The scalar product in $L_2(F)$ is defined by

$$<f_1, f_2> = \int_a^b f_1(u) dF(u) f_2^*(u); f_1, f_2 \in L_2(F).$$

Consider the class of all distribution functions $F$ (defined on $[a, b]$ and being bounded, nondecreasing and continuous from the left). We write $F_1 \succ F_2$ if and only if the measure induced by $F_2$ is absolutely continuous with respect to the measure induced by $F_1$. Let $R$ be a class of the distribution functions
equivalent with respect to the relation $\succ$. We will consider the arbitrary finite or infinite sequence of classes

$$R_1 \succ R_2 \succ \cdots \succ R_N.$$  

The fundamental result in [1] is the following:

For a given sequence (1) there exists the process $X(t), t \in [a, b]$ such that

$$X(t) = \int_a^t g(t, u) \, dZ(u), \quad t \in [a, b],$$  

where

1. The processes $Z_n(t), n = 1, N$ in $Z(t) = \| Z_n(t) \|_{n=1, N}$ are mutually orthogonal with orthogonal increments and

$$E \left| Z_n(t) \right|^2 = F_n(t) \subseteq R_n, \quad n = 1, N$$

2. $H(X; t) = \sum_{n=1}^N \oplus H(Z_n; t), \quad t \in [a, b]$  

3. For $F(t) = E Z(t) Z^*(t), \quad t \in [a, b]$ then $g(t, u) \in L_2(F)$ as a function of $u \in [a, t]$ ($g(t, u) = 0$ for $u > t$).

We call Cramér representation of the process $X(t)$ the representation (2) with 1, 2, and 3. The sequence (1) or $F(t), t \in [a, b]$ is the spectral type of $X(t)$.

It follows from the theorem of complete system of unitary invariants of a selfadjoint operator in Hilbert space [5] that the correlation function of $X(t)$

$$B(s, t) = (X(s), X(t)) = EX(s) \overline{X(t)}, \quad s, t \in [a, b]$$

uniquely determines the spectral type of $X(t)$.

We will say that the family of functions $g(t, u) \in L_2(F), \quad u \in [a, t]$ (where the parameter $t \in [a, b]$) is complete in $L_2(F)$ if for any fixed $t$ from

$$\int_a^s g(s, u) \, dF(u) f^*(u) = 0 \quad \text{for all} \quad s \in [a, t]$$

follows $f(u) = 0, \quad u \in [a, t]$ almost everywhere with respect to $F$ (i.e.

$$\int_a^t f(u) \, dF(u) f^*(u) = 0.$$  

Theorem 1 ([3], see also [4]). The process $X(t), t \in [a, b]$ has Cramér representation (2) if and only if

$$\min_{[a, t]} B(s, t) = \int_a^t g(s, u) \, dF(u) g^*(t, u), \quad s, t \in [a, b]$$

where the family of functions $g(t, u), \quad u \in [a, t]$ is complete in $L_2(F)$.

Proof. It follows immediately from Cramér representation that $B(s, t)$ has the form (4). To prove that the family $g(t, u)$ is complete observe that any
element $Y$ of $\sum_{n=1}^{N} \oplus H(Z_n; t)$ is of the form $Y = \int_{a}^{t} f(u) \, dZ(u)$, $f \in L_2(F)$. As $Y \in H(X; t)$ it follows that $(X(s), Y) = 0$, for all $s \in [a, t]$ implies $Y = 0$, what is equivalent to the condition of completeness of $g(t, u)$.

Now, let $B(s, t)$ be of the form (4) with the complete family $g(t, u)$. Then we can always find the process $Z(t) = \| Z_n(t) \|_{n=1, N}$, where $Z_n(t)$, $n = 1, N$ are mutually orthogonal and with orthogonal increments, for which

$$EZ(t) Z^*(t) = F(t)$$

with $F(t)$ from (4). So the process

$$X(t) = \int_{a}^{t} g(t, u) \, dZ(u), \quad t \in [a, b]$$

has the correlation function (4). We only have to prove that the condition 2. of the Cramér representation holds. From (5) follows

$$H(X; t) \subseteq \sum_{n=1}^{N} \oplus H(Z_n; t), \quad t \in [a, b].$$

If $H(X; t) \supseteq \sum_{n=1}^{N} \oplus H(Z_n; t)$ there exists $Y \neq 0 \in \sum_{n=1}^{N} \oplus H(Z_n; t)$ such that $(X(s), Y) = 0$ for all $s \in [a, t]$. As

$$Y = \int_{a}^{t} f(u) \, dZ(u), \quad 0 = f \in L_2(F)$$

we have

$$(X(s), Y) = \int_{a}^{s} g(s, u) \, dF(u) f^*(u) = 0 \quad \text{for all } s \in [a, t],$$

which is the contradiction with the condition that $g(t, u)$ is complete.

**Example 1.** Let $A_n, \ n = 1, N$ be disjoint sets such that $\bigcup_{n=1}^{N} A_n = [0,1]$ and let every $A_n$ be dense in $[0,1]$. We consider the family $g(t, u) = \| g_n(t, u) \|_{n=1, N}$, $t \in [0,1]$, where

$$g_n(t, u) = \begin{cases} 1, & u \in [0, t], \quad u \in A_n. \\ 0, & \text{elsewhere} \end{cases}$$

Let $F(t) = \| F_n(t) \|_{n=1, N} = \| t \ 0 \| \quad 0 \ t \ |\|_{n=1, N}, \quad t \in [0,1].$

It is easy to show that the family $g(t, u)$, $u \in [0, t]$ is complete in $L_2(F)$; if for all $s \in A_n \cap [0, t]$}

$$\int_{a}^{s} g(t, u) \, dF(u) f^*(u) = \int_{a}^{s} f_n(u) \, du = 0,$$
then \( f_n(u) = 0, \ u \in [0, t] \) a.e. or \( f(u) = \| f_n(u) \|^{n-1, N} = 0, \ u \in [0, t] \) a.e. with respect to \( F \).

So

\[
B(s, t) = \int_0^1 g(s, u) dF(u) g^*(t, u) = \begin{cases} 
\min \{ s, t \}, & \text{if both } s \text{ and } t \text{ are in the same set } A_n, \ n = 1, N \\
0, & \text{elsewhere}
\end{cases}
\]

is the correlation function of the process \( X(t), \ t \in [0,1] \) with the spectral type \( F \). For example,

\[
X(t) = \int_0^t g(t, u) dZ(u), \ t \in [0,1],
\]

where \( Z_n(t), \ n = 1, N \) in \( Z(t) = \| Z_n(t) \|^{n=1, N} \) are independent Brownian motion processes.

Remark. The Theorem 1 holds under weaker conditions than those that the functions \( F_n(t), \ n = 1, N \) in \( F \) are ordered according to \( \succ \). Namely, let \( G(t) = \| G_{jk}(t) \|^{j=1, L, \ k=1, L} \) be the matrix function which having the only non-zero elements distribution functions \( G_{nn}(t), \ n = 1, L \) (\( L \) may be infinite). For the process \( X(t), \ t \in [a, b] \) with the correlation function

\[
B(s, t) = \int_a^L h(s, u) dG(u) h^*(t, u); \ s, t \in [a, b]
\]

where the family of functions \( h(t, u), u \in [a, t] \) (parameter \( t \in [a, b] \)) is complete in \( L_2(G) \), the spectral type can be found as follows. Starting with \( G(t) \), the procedure for determining \( F \) in Cramér representation of \( X(t) \) is equivalent to the so called regularizing transposition from [2], Ch. VII (Note that \( N < L \)).

It is shown in [1] that the process \( X(t) \) with given spectral type \( F^X \) can be found so to be continuous (in quadratic mean). Let \( X(t) \) be regular everywhere (i.e. \( F^X(t) \) in \( F^X \) is absolutely continuous). According to [3] this is not essential restriction. Now we define

\[(6)\]

\[
Y(t) = \int_a^t \varphi(t, u) X(u) \, du, \ t \in [a, b]
\]

Theorem 2. If the family of functions \( \varphi(t, u), u \in [a, t] \) in (6), is complete in \( L_2 \), then the processes \( Y(t) \) and \( X(t) \) have the same spectral type.

Proof. We need only to show \( H(Y; t) = H(X; t) \) for all \( t \in [a, b] \). Since, according to (6), \( H(Y; t) \subset H(X; t) \) suppose that \( H(Y; t) \not= H(X; t) \), then there exists \( Z: 0 \not= Z \in H(X; t) \) such that \( (Y(s), Z) = 0 \) for all \( s \in [a, t] \).

Being

\[
Z = \int_a^t f(v) dZ^X(v), \ f \in L_2(F^X),
\]
we have
\[
\begin{align*}
(Y(s), Z) &= \mathbb{E} \int_{a}^{s} \varphi(s, u) \, X(u) \, du \cdot \int_{a}^{t} f(v) \, dZ^X(v) = \\
&= \mathbb{E} \int_{a}^{s} \varphi(s, u) \left[ \int_{a}^{u} g^X(u, x) \, dZ^X(x) \right] du \cdot \int_{a}^{t} f(v) \, dZ^X(v) = \\
&= \int_{a}^{s} \varphi(s, u) \left[ \int_{a}^{u} g^X(u, v) \, dF^X(v) \, f^*(v) \right] du = 0
\end{align*}
\]
for all \( s \in [a, t] \). Since \( \varphi(t, u) \) is complete in \( L_2 \) it follows that
\[
\int_{a}^{u} g^X(u, v) \, dF^X(v) \, f^*(v) = 0
\]
almost everywhere on \([a, t]\). Being \( X(t) \) regular everywhere it follows that the last equality holds for all \( u \in [a, t] \). From the completeness of \( g^X(u, v) \) in \( L_2(F^X) \) we conclude that \( f(v) = 0 \) with respect to \( F^X \). This contradiction concludes the proof.

Corollary. Let
\[
B^Y(s, t) = \int_{a}^{s} \int_{a}^{t} \varphi(s, u) \, \overline{\varphi(t, v)} \, B^X(u, v) \, du \, dv, \quad s, t \in [a, b]
\]
where the process \( X(t) \) is as in Theorem 2. If \( \varphi(t, u) \) is complete in \( L_2 \) then \( F^Y = F^X \).

Example 2. The process
\[
Y(t) = \int_{a}^{t} X(u) \, du, \quad t \in [a, b]
\]
has the spectral type \( F^X \) of the process \( X(t) \).

REFERENCES