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A CHARACTERIZATION OF CRAMÉR REPRESENTATION OF STOCHASTIC PROCESSES

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Let $X(t)$, $t\in[a, b]$ be a complex valued stochastic process with $EX(t)=0$ and $E|X(t)|^2<\infty$ for each $t\in[a, b]$. The interval $[a, b]$ may be finite or infinite. We consider the process $X(t)$, $t\in[a, b]$ as a curve in the Hilbert space H of all random variables with finite dispersion. The scalar product in H is defined by $(X, Y)=EX\overline{Y}$; $X, Y\in H$. Let $H(X; t)$ be the linear closure in H generated by $X(s)$, $s\in[a, t]$. We will suppose that $X(t)$, $t\in[a, b]$ is continuous from the left (which yields the separability of the spaces $H(X; t)$, $t\in[a, b]$) and regular (i. e. $\bigcap_{t>a} H(X; t)=0$).

Let $\boldsymbol{Z}(t)=\|Z(t)\|_{n=\overline{1, N}}$, $t\in[a, b]$ be a stochastic process (considered as a vector column), where $Z_n(t)$, $t\in[a, b]$, $n=\overline{1, N}$ are mutually orthogonal processes with orthogonal increaments. N may be finite or infinite. Put $\boldsymbol{F}(t)=E\boldsymbol{Z}(t)\boldsymbol{Z}^*(t)=\left\|F_{jk}(t)\right\|_{k=\overline{1, N}}^{j=\overline{1, N}}$, where $\boldsymbol{Z}^*(t)$ denotes the transposed matrix of $\boldsymbol{Z}(t)$.

Matrix function $\boldsymbol{F}(t)$, $t\in[a, b]$ has non-zero elements only on the principal diagonal and we denote them by $F_n(t)=F_{nn}(t)=E|Z_n(t)|^2$, $n=\overline{1, N}$

Let $L_2(\boldsymbol{F})$ be the Hilbert space of all complex valued vector row functions $f(t)=\|f_n(t)\|^{n=\overline{1, N}}$, $t\in[a, b]$ for which

$$\int_a^b f(u)\,d\boldsymbol{F}(u)\,f^*(u)<\infty.$$

The scalar product in $L_2(\boldsymbol{F})$ is defined by

$$<f_1, f_2>=\int_a^b f_1(u)\,d\boldsymbol{F}(u)\,f_2^*(u);\quad f_1, f_2\in L_2(\boldsymbol{F}).$$

Consider the class of all distribution functions F (defined on $[a, b]$ and being bounded, nondecreasing and continuous from the left). We write $F_1\succ F_2$ if and only if the measure induced by F_2 is absolutely continuous with respect to the measure induced by F_1. Let R be a class of the distribution functions equivalent with respect to the relation $\succ$. We will consider the arbitrary finite or infinite sequence of classes

$$R_1 \succ R_2 \succ \cdots R_N. \tag{1}$$

The fundamental result in [1] is the following:

For a given sequence (1) there exists the process $X(t)$, $t \in [a, b]$ such that

$$X(t) = \int_a^t g(t, u)\, dZ(u), \quad t \in [a, b], \tag{2}$$

where

1. The processes $Z_n(t)$, $n = \overline{1, N}$ in $Z(t) = \| Z_n(t) \|_{n=\overline{1,N}}$ are mutually orthogonal with orthogonal increaments and

$$E\,|Z_n(t)|^2 = F_n(t) \in R_n, \quad n = \overline{1, N}$$

2. $H(X; t) = \sum_{n=1}^{N} \oplus H(Z_n; t), \quad t \in [a, b]$

3. For $F(t) = EZ(t)\, Z^*(t)$, $t \in [a, b]$ then $g(t, u) \in L_2(F)$ as a function of $u \in [a, t]$ $(g(t, u) = 0$ for $u > t)$.

We call Cramér representation of the process $X(t)$ the representation (2) with 1, 2. and 3. The sequence (1) or $F(t)$, $t \in [a, b]$ is the spectral type of $X(t)$.

It follows from the theorem of complete system of unitary invariants of a selfadjoint operator in Hilbert space [5] that the correlation function of $X(t)$

$$B(s, t) = (X(s), X(t)) = EX(s)\overline{X(t)}, \quad s, t \in [a, b]$$

uniquely determines the spectral type of $X(t)$.

We will say that the family of functions $g(t, u) \in L_2(F)$, $u \in [a, t]$ (where the parameter $t \in [a, b]$) is complete in $L_2(F)$ if for any fixed t from

$$\int_a^s g(s, u)\, dF(u)\, f^*(u) = 0 \qquad \text{for all } s \in [a, t]$$

follows $f(u) = 0$, $u \in [a, t]$ almost everywhere with respect to F $\Big($i. e.

$$\int_a^t f(u)\, dF(u)\, f^*(u) = 0\Big).$$

Theorem 1 ([3], see also [4]). *The process $X(t)$ $t \in [a, b]$ has Cramér representation* (2) *if and only if*

$$B(s, t) = \int_a^{\min\{s, t\}} g(s, u)\, dF(u)\, g^*(t, u), \quad s, t \in [a, b] \tag{4}$$

where the family of functions $g(t, u)$, $u \in [a, t]$ is complete in $L_2(F)$.

Proof. It follows immediately from Cramér representation that $B(s, t)$ has the form (4). To prove that the family $g(t, u)$ is complete observe that any element Y of $\sum_{n=1}^{N} \oplus H(Z_n;\, t)$ is of the form $Y = \int_a^t f(u)\, d\mathbf{Z}(u)$, $f \in L_2(\mathbf{F})$. As $Y \in H(X;\, t)$ it follows that $(X(s),\, Y) = 0$, for all $s \in [a,\, t]$ implies $Y = 0$, what is equivalent to the condition of completness of $\mathbf{g}(t,\, u)$.

Now, let $B(s,\, t)$ be of the form (4) with the complete family $\mathbf{g}(t,\, u)$. Then we can always find the process $\mathbf{Z}(t) = \| Z_n(t) \|_{n=\overline{1,N}}$, where $Z_n(t)$, $n = \overline{1,\, N}$ are mutually orthogonal and with orthogonal increaments, for which

$$E\,\mathbf{Z}(t)\,\mathbf{Z}^*(t) = \mathbf{F}(t)$$

with $\mathbf{F}(t)$ from (4). So the process

$$X(t) = \int_a^t \mathbf{g}(t,\, u)\, d\mathbf{Z}(u), \quad t \in [a,\, b] \tag{5}$$

has the correlation function (4). We only have to prove that the condition 2. of the Cramér representation holds. From (5) follows

$$H(X;\, t) \subset \sum_{n=1}^{N} \oplus H(Z_n;\, t), \quad t \in [a,\, b].$$

If $H(X;\, t) \supset \sum_{n=1}^{N} \oplus H(Z_n;\, t)$ there exists $Y: 0 \neq Y \in \sum_{n=1}^{N} \oplus H(Z_n;\, t)$ such that $(X(s),\, Y) = 0$ for all $s \in [a,\, t]$. As

$$Y = \int_a^t f(u)\, d\mathbf{Z}(u), \quad 0 = f \in L_2(\mathbf{F})$$

we have

$$(X(s),\, Y) = \int_a^s \mathbf{g}(s,\, u)\, d\mathbf{F}(u)\, f^*(u) = 0 \quad \text{for all } s \in [a,\, t],$$

which is the contradiction with the condition that $\mathbf{g}(t,\, u)$ is complete.

Example 1. Let A_n, $n = \overline{1,\, N}$ be disjoint sets such that $\bigcup_{n=1}^{N} A_n = [0,1]$ and let every A_n be dense in [0,1]. We consider the family $\mathbf{g}(t,\, u) = \| g_n(t,\, u) \|^{n=\overline{1,N}}$, $t \in [0,1]$, where

$$g_n(t,\, u) = \begin{cases} 1, & u \in [0,\, t], \quad u \in A_n. \\ 0, & \text{elsewhere.} \end{cases}$$

$$\text{Let } \mathbf{F}(t) = \| F_{jk}(t) \|_{k=\overline{1,N}}^{j=\overline{1,N}} = \begin{Vmatrix} t & & 0 \\ & \ddots & \\ 0 & & t \end{Vmatrix}, \quad t \in [0,1].$$

It is easy to show that the family $\mathbf{g}(t,\, u)$, $u \in [0,\, t]$ is complete in $L_2(\mathbf{F})$; if for all $s \in A_n \cap [0,\, t]$

$$\int_0^s \mathbf{g}(t,\, u)\, d\mathbf{F}(u)\, f^*(u) = \int_0^s f_n(u)\, du = 0,$$

then $f_n(u)=0$, $u\in[0, t]$ a. e. or $f(u)=\|f_n(u)\|^{n=\overline{1,N}}=0$, $u\in[0, t]$ a. e. with respect to F.

So

$$B(s, t)=\int_0^{\min\{s, t\}} g(s, u)\, dF(u)\, g^*(t, u)=\begin{cases}\min\{s, t\}, & \text{if both } s \text{ and } t \text{ are in the same set } A_n,\ n=\overline{1, N}\\ 0, & \text{elsewhere}\end{cases}$$

is the correlation function of the process $X(t)$, $t\in[0,1]$ with the spectral type F. For example,

$$X(t)=\int_0^t g(t, u)\, dZ(u), \quad t\in[0,1],$$

where $Z_n(t)$, $n=\overline{1, N}$ in $Z(t)=\|Z_n(t)\|_{n=\overline{1,N}}$ are independent Brownian motion processes.

Remark. The Theorem 1. holds under weaker conditions than those that the functions $F_n(t)$, $n=\overline{1, N}$ in F are ordered according to $\succ$. Namely, let $G(t)=\|G_{jk}(t)\|_{j=\overline{1,L}}^{k=\overline{1,L}}$ be the matrix function which having the only non-zero elements distribution functions $G_{nn}(t)$, $n=\overline{1, L}$ (L may be infinite). For the process $X(t)$, $t\in[a, b]$ with the correlation function

$$B(s, t)=\int_a^{\min\{s, t\}} h(s, u)\, dG(u)\, h^*(t, u); \quad s, t\in[a, b]$$

where the family of functions $h(t, u)$, $u\in[a, t]$ (parameter $t\in[a, b]$) is complete in $L_2(G)$, the spectral type can be found as follows. Starting with $G(t)$, the procedure for determining F in Cramér representation of $X(t)$ is equivalent to the so called regularizing transposition from [2], Ch. VII (Note that $N\leqslant L$).

It is shown in [1] that the process $X(t)$ with given spectral type F^X can be found so to be continuous (in quadratic mean). Let $X(t)$ be regular everywhere (i. e. $F_1^X(t)$ in F^X is absolutely continuous). According to [3] this is not essential restriction. Now we define

$$Y(t)=\int_a^t \varphi(t, u)\, X(u)\, du, \quad t\in[a, b] \tag{6}$$

Theorem 2. *If the family of functions $\varphi(t, u)$, $u\in[a, t]$ in (6), is complete in L_2, then the processes $Y(t)$ and $X(t)$ have the same spectral type.*

Proof. We need only to show $H(Y; t)=H(X; t)$ for all $t\in[a, b]$. Since, according to (6), $H(Y; t)\subset H(X; t)$ suppose that $H(Y; t)\neq H(X; t)$, then there exists Z: $0\neq Z\in H(X; t)$ such that $(Y(s), Z)=0$ for all $s\in[a, t]$.

Being

$$Z=\int_a^t f(v)\, dZ^X(v), \quad f\in L_2(F^X),$$

we have

$$(Y(s), Z) = E\int_a^s \varphi(s, u)\, X(u)\, du \cdot \overline{\int_a^t f(v)\, dZ^X(v)} =$$

$$= E\int_a^s \varphi(s, u)\left[\int_a^u g^X(u, x)\, dZ^X(x)\right] du \cdot \overline{\int_a^t f(v)\, dZ^X(v)} =$$

$$\int_a^s \varphi(s, u)\left[\int_a^u g^X(u, v)\, dF^X(v)\, f^*(v)\right] du = 0$$

for all $s \in [a, t]$. Since $\varphi(t, u)$ is complete in L_2 it follows that

$$\int_a^u g^X(u, v)\, dF^X(v)\, f^*(v) = 0$$

almost everywhere on $[a, t]$. Being $X(t)$ regular everywhere it follows that the last equality holds for all $u \in [a, t]$. From the completness of $g^X(u, v)$ in $L_2(F^X)$ we conclude that $f(v) = 0$ with respect to F^X. This contradiction concludes the proof.

Corrolary. Let

$$B^Y(s, t) = \int_a^s \int_a^t \varphi(s, u)\, \overline{\varphi(t, v)}\, B^X(u, v)\; du\, dv, \quad s, t \in [a, b]$$

where the process $X(t)$ is as in Theorem 2. If $\varphi(t, u)$ is complete in L_2 then $F^Y = F^X$.

Example 2. The process

$$Y(t) = \int_a^t X(u)\, du, \quad t \in [a, b]$$

has the spectral type F^X of the process $X(t)$.

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