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AUTOMORPHISMS OF EXACT SEQUENCES

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1. Introduction. Let F denote the following exact sequence

$$F: 0 \to A \xrightarrow{f} B \xrightarrow{hg} D \xrightarrow{j} 0$$

where hg represents the canonical factorization of the middle morphism of F into an epimorphism g followed by a monomorphis h. We shall take the term „endomorphism" of F to mean a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & A & \xrightarrow{f} & B & \xrightarrow{hg} & D & \xrightarrow{j} & E & \to 0 \\ & \| 1 & & \downarrow \beta & & \downarrow \gamma & & \| 1 & \\ 0 \to & A & \xrightarrow{f} & B & \xrightarrow{hg} & D & \xrightarrow{j} & E & \to 0 \end{array}$$

We shall compute the automorphism groups of F. It is shown by Pressman that End $(F) \simeq$ Hom (h, g) where the Hom is a functor on a category of morphisms with range the category of semigroups.

2. Notation. Let R denote a fixed ring with unit and $\mathcal{M}$ the category of left R-modules. Let $\mathcal{F}$ denote the category of all exact sequences F which begin with A and end with C, and whose morphisms are quadruples $(1, \beta, \gamma, 1)$ making the diagram

$$\begin{array}{ccccccccc} F: 0 \to & A & \to & B & \to & D & \to & C & \to 0 \\ & 1 \| & & \downarrow \beta & & \downarrow \gamma & & \| 1 & \\ F': 0 \to & A & \to & B' & \to & D' & \to & C & \to 0 \end{array}$$

commutative. Let $\mathcal{M}^2$ denote the abelian category whose objects are all morphisms of $\mathcal{M}$, and whose morphisms are all pairs $\begin{pmatrix} \rho \\ \sigma \end{pmatrix}: h \to g$ which gives rise to commutative squares

$$\begin{array}{ccc} \cdot & \xrightarrow{\rho} & \cdot \\ h \downarrow & & \downarrow g \\ \cdot & \xrightarrow[\sigma]{} & \cdot \end{array}$$

One should note that there is no way of adding endomorphisms of $\mathcal{F}$; one may compose them with each other. Pressman in [1], did compute $\mathrm{End}_{\mathcal{F}}(F)$

and asked about $\mathrm{AUT}_{\mathcal{F}}(F)$. In this note we throw some light on $\mathrm{AUT}_{\mathcal{F}}(F)$ which we believe the best one can say. We shall use Pressman's notation in the compulation of $\mathrm{End}_{\mathcal{F}}(F)$ and prove

Theorem. $\mathrm{AUT}_{\mathcal{F}}(F) \simeq$ *Subgroup of all invertible elements of* $\mathrm{Hom}_{\mathcal{M}^2}(h, g)$.

Proof. Suppose $(1, \beta, \gamma, 1)$ is an automorphism. Then we have $(1, \beta^{-1}, \gamma^{-1}, 1)$ such that $(1, \beta, \gamma, 1)\,(1, \beta^{-1}, \gamma^{-1}, 1) = (1, 1, 1, 1)$ the identity automorphism. The translation $(1, \beta, \gamma, 1)$ determines uniquely a translation $\begin{pmatrix} \rho \\ \sigma \end{pmatrix} : h \to g$; similarly the translation $(1, \beta^{-1}, \gamma^{-1}, 1)$ also determines $\begin{pmatrix} \rho' \\ \sigma' \end{pmatrix} : h \to g \in \mathrm{Hom}_{\mathcal{M}^2}(h, g)$ uniquely. Now $\beta^{-1}\beta = 1$, gives,

$$(1+\rho' g)(1+\rho g) = 1, \quad \text{i.e.} \quad (\rho+\rho')g + \rho' g \rho g = 0; \quad [\rho+\rho'+\rho' g\rho]g = 0,$$

which implies $\rho + \rho' + \rho' g \rho = 0$, since g is an epimorphism.

Similarly $\gamma\gamma^{-1} = 1 \Rightarrow \sigma + \sigma' + \sigma h \sigma' = 0$;

$$\beta\beta^{-1} = 1 \Rightarrow \rho' + \rho + \rho g \rho' = 0; \quad \gamma^{-1}\gamma = 1 \Rightarrow \sigma' + \sigma + \sigma' h \sigma = 0.$$

Now $\begin{pmatrix} \rho \\ \sigma \end{pmatrix} \begin{pmatrix} \rho' \\ \sigma' \end{pmatrix} = \begin{pmatrix} \rho+\rho'+\rho g \rho' \\ \sigma+\sigma'+\sigma h \sigma' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \rho' \\ \sigma' \end{pmatrix} \begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{pmatrix} \rho \\ \sigma \end{pmatrix}$ is invertible and $\begin{pmatrix} \rho' \\ \sigma' \end{pmatrix} = \begin{pmatrix} \rho \\ \sigma \end{pmatrix}^{-1}$.

Conversely given any invertible element $\begin{pmatrix} \rho \\ \sigma \end{pmatrix}$, we define $(1, \beta^{-1}, \gamma^{-1}, 1)$: $F \to F$ by $\beta^{-1} = 1 + \rho' g$, $\gamma^{-1} = 1 + h\sigma'$ where $\begin{pmatrix} \rho' \\ \sigma' \end{pmatrix} = \begin{pmatrix} \rho \\ \sigma \end{pmatrix}^{-1}$.

It is then easy to check

$$\beta\beta^{-1} = (1+\rho g)(1+\rho' g) = 1 + [(\rho+\rho'+\rho g \rho')g] = 1 + 0 = 1, \quad \text{since } \rho+\rho'+\rho g \rho' = 0,$$

and similarly, for the other three equations, i.e. $(1, \beta, \gamma, 1)$ is in fact an automorphism.

Now using the fact that the set of all invertible elements of a semigroup, is a group, we deduce from Pressman's result [1] $\mathrm{AUT}_{\mathcal{F}}(F) \simeq$ Group of all invertible elements of $\mathrm{Hom}_{\mathcal{M}^2}(h, g)$.

REFERENCE

[1] I. Pressman, *Endomorphisms of Exact Sequences*, Bull. Amer. Math. Soc., 77. 239—242 (1971).

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