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SOME FUNDAMENTAL STRUCTURES ON CLASSES

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1. Introduction.

In the paper [1] we have proposed an axiomatized system Σ as a foundation for mathematics. Models for this system, as we have seen, are specters of classes consisting of classes of different levels and membership relations among them. In that manner any class on a level $i \in \mathcal{J}$, where $\mathcal{J}$ denotes the hierarchy of all levels, is an element of a class on the level $i+1 \in \mathcal{J}$. Its elements are classes of the level $i-1 \in \mathcal{J}$, called i-sets. There, we have also shown that there exists an initial specter whose elements are i-universes $\mathcal{U}_i$ and bonding relations are strict dominations. Each i-class t_i is a subclass of the universe $\mathcal{U}_i$, respectively an element of the universe $\mathcal{U}_{i+1}$. The inductive limit of the initial specter is the universe $\mathcal{U}$. The universe $\mathcal{U}$ is a proper universe and also a proper class. It is a wide frame which contains all objects of mathematics, all objects of all levels. The system of which $\mathcal{U}$ is a model is the limit system $\Sigma_{\rightarrow}$ of Σ.

In several subsequent papers we shall attempt to show that the system Σ provides an adequate framework for (all) mathematics. To attain this purpose we adopt two ways, the one devoted to the study of some mathematical structures in $\mathcal{U}$ and the other to the formalization of such investigations in a system falling under the scheme of Σ. Then we shall determine, within such a system, the position of existing logical systems.

The realization of our programme we originate by the study of some structures in $\mathcal{U}$. The first type structures that we mean to consider are structures among the same level objects, namely structures on a class t of a level i of $\mathcal{J}$. Such a type of structures we call horizontal structures in $\mathcal{U}$. We consider here certain types of these structures, those which are in a sense, the most natural in $\mathcal{U}$. These structures we take as basic ones and call them fundamental structures in $\mathcal{U}$. We present here an elementary theory of these structures. However, before it we explain what fundamental structures are. Let us consider a class t_i of some mathematical (i—1)-objects, where by i—1 is denoted their class-theoretical level. To study this class we introduce on it a new class p_i, the class of connections or rules for relating elements in it. If we introduce a structure on the class of rules p_i, then this structure will give some informations of the class t_i itself. A certain structure on the class p_i we call a fundamental structure of the class t_i, and denote it by $p_i\,(t_i)$. In future we shall often say simply a fundamental structure $p_i\,(t_i)$ without stressing the class of objects of which it is such a structure. In order that these structures be more powerful tools for our future needs we enrich them by some additional structures. The second part of this paper is devoted to such structures.

Now, since we have explained the notion of a fundamental structure on a class in $\mathcal{U}$ we shall pay more attention to the class of rules in it. Rules in a class may be of different natures depending on objects and informations we want to have about them. In abstract considerations of classes, when we want only to establish that there are connections among their objects satisfying certain laws, the nature of these rules is not significant. However, in concrete situations when we want to have more available informations of objects in a class and also a possibility to do something with them, for instance to construct new objects from the original ones, the nature of rules is an essential matter. Because of that we have to specify rules in a certain sense. Clearly, it has a sense to consider those rules only which naturally correspond to a considered class of objects. A rule which preserves basic properties characterizing the structure of objects in a class will naturally belong to it. Such a rule is then admissible for that class. We restrict all rules in $\mathcal{U}$ to these ones. Hence, a rule in $\mathcal{U}$ means an admissible rule for a particular class of objects. Furthermore, since we consider here the horizontal type structures, then we must also do some restrictions to rules in that sense. We shall restrict rules to those ones which have the same levels as considered objects. A rule in the universe will be of the level i if it connects i-objects, namely if its domain and codomain have the same ith level. Such rules we call horizontal rules in $\mathcal{U}$. Let us consider two i-classes t_i and s_i and a rule $\alpha : t_i \to s_i$. If this rule is such that with each element $t_{i-1} \in t_i$ it associates a single element $\alpha(t_{i-1}) \in s_i$, then its codomain has ith level as its domain itself. Such a rule we call a single-valued rule. It is obviously a horizontal rule in $\mathcal{U}$. Provided that s_i is not the i-universe $\mathcal{U}_i$, then α is a horizontal rule even if $\alpha(t_{i-1}) \subseteq s_i$. This follows immediately from the axioms of domination and codomination, since the codomain of this rule is a subclass of $\mathcal{P}(s_i)$. Such rules we call many-valued horizontal rules in $\mathcal{U}$. A many-valued rule to s_i may be viewed as a single-valued rule to $\mathcal{P}(s_i)$. We shall not consider many-valued rules in this paper, because it is too hard to make them to be admissible for classes of objects in $\mathcal{U}$. Hence, by the rules in $\mathcal{U}$ we mean single-valued horizontal rules being admissible for a class of objects in $\mathcal{U}$. We shall call them simply "rules". If they occur, many-valued horizontal rules will be particularly stressed. In the case of the preuniverse $\mathcal{U}^*$ it is not true that if the domain of a rule is on ith level that so is the codomain. In that case we must postulate it, namely require that all rules have this property. To horizontal rules we shall add an index to denote their levels. So, the rule α_i will mean a rule of ith level. Besides horizontal rules there are also in $\mathcal{U}$ rules whose domains and codomains are on different levels. Such rules we call vertical rules in $\mathcal{U}$. Their studies are without the scope of this paper. They will occur in considerations of vertical structures in $\mathcal{U}$. Then we shall study them in more details. To mention only that they may also be single-valued and many-valued rules.

In this paper we present an elementary theory of fundamental structures, define some special types of these structures and study their elementary properties. A part of results presented here we consider the well-known or essentially known, see [3], [4] and others. Here, we think, in the first place, of the Section 2. However, we consider that because of our future references it is better to collect together the results we need further, to attain a terminological compatibility of notions and to accommodate them to our new foundation. Hence, we consider this paper as an auxiliary one for our future intentions.

2. The elementary theory of fundamental structures.

In this section we shall present the elementary theory of fundamental structures (abbreviated as ETFS) in a volume that will be sufficient in future. In order to specify the mentioned theory, we have to specify its nonlogical symbols and its monlogical axioms. However, before doing this we shall do some considerations concerning the language of ETFS. This language we shall denote by **L**. We require that **L** contains two sorts of variables. To distinguish among them we may add to them some names. So, variables of the first sort we may call dots and of the second, arrows. In the sequel we shall specify some letters that we shall use as symbols for variables of both sorts. For the first sort we reserve the beginning of the Latin alphabet and also letters s and t; for the second, the end of that alphabet, from p up to the end, excluded s and t, sometimes Latin capitals and also a part of Grecian alphabet. Variables of both sorts we take to range over the universe $\mathcal{U}$, at this, the first sort over objects in $\mathcal{U}$ and the second over rules among these objects. Thus, we assume interpretations of the language **L** to be in the universe $\mathcal{U}$. In order to emphasize the class-theoretical levels of interpretations in $\mathcal{U}$ we add to every language, which is to be interpreted in $\mathcal{U}$ an idenx $i\in\mathcal{J}$, where $\mathcal{J}$ denotes the hierarchy of all levels in $\mathcal{U}$. Hence, L_i means that a language L has an interpretation on the ith level in $\mathcal{U}$ namely that the first sort variables of it are ranging over (i—1)-objects and the second over rules among them. However, we may also regard that the language L has variable symbols of the (i—1)-level and that, thus, we denote it by L_i. In order to specify languages of different levels we presuppose that there is a class $\mathcal{S}$ elements of which are symbols od different sorts and levels. We assume that starting symbols of different sorts in it are of the level —1. The class of all such symbols we denote by $\mathcal{S}_0$. Then we have different sort symbols of the levels 0, 1, . . . , and accordingly classes $\mathcal{S}_1$, $\mathcal{S}_2, \ldots$. These classes we take as the classes of variable symbols for languages L_i, $i\in\mathcal{J}$, and assume that all interpretations of L_i are horizontal, i.e. along the same levels. If they are in $\mathcal{U}$ then elements of $\mathcal{S}_{i+1}$ are ranging over the universe $\mathcal{U}_{i+1}$. Hence, a variable symbol t_i will mean an i-object and p_i a rule between two such objects.

The study of languages of different levels and relationships among them we leave for our final step, while we shall be formalizing our investigations and here proceed to specify the theory ETFS on a fixed level. We choose the level to be $i+1$. At first, we specify the nonlogical symbols of ETFS. They are the symbols of its language **L**, which is now provided with the index $i+1$. We take that the language $\mathbf{L}_{i+1}$ has the following nonlogical symbols, two 1-ary function symbols $\mathcal{D}_0$ and $\mathcal{D}_1$ and a 2-ary function symbol $\mathcal{C}$. Of course, we understand that $\mathbf{L}_{i+1}$ is a language with the equality symbol $=$. By the atomic formulas of $\mathbf{L}_{i+1}$ we mean expressions of the forms $\mathcal{D}_0(p_i)=a_i$, $\mathcal{D}_1(p_i)=b_i$, $\mathcal{C}(p_i, q_i)=r_i$ and $p_i=q_i$. The formula $\mathcal{C}(p_i, q_i)=r_i$ we shall write $\mathcal{C}(p_i, q_i, r_i)$. These formulas are to be read as follows: $\mathcal{D}_0(p_i)=a_i$, a_i is the domain of p_i; $\mathcal{D}_1(p_i)=b_i$, b_i is the codomain of p_i; $\mathcal{C}(p_i, q_i; r_i)$, r_i is the composition of p_i followed by q_i and $p_i=q_i$, p_i equals q_i.

Denote by Θ_{i+1} the class of all formulas of $\mathbf{L}_{i+1}$. We assume that this class consists of the atomic formulas and those such that:

a) If $\theta_i\in\Theta_{i+1}$, then $\neg\theta_i\in\Theta_{i+1}$,

b) if $\Theta^*_{i+1}=\{\theta_i^\alpha;\ \alpha<c_\beta\}$ is a class of formulas from Θ_{i+1} then $\bigvee\Theta^*_{i+1}$, $\bigwedge\Theta^*_{i+1}$ are formulas of Θ_{i+1},

c) if θ_i, $\lambda_i\in\Theta_{i+1}$, then $\theta_i\Rightarrow\lambda_i\in\theta_{i+1}$, and

d) if $\theta_i\in\Theta_{i+1}$ and $p_i\in p_{i+1}$ then $\forall p_i\,\theta_i$ and $\exists p_i\,\theta_i\in\Theta_{i+1}$.

For the time being we shall say nothing more about this class. Later, after introducing topological structures in $\mathcal{U}$ we shall consider this question once more to show what is the structure on Θ_{i+1}. Then we shall consider the class of formulas of an arbitrary language L, introduce on it a particular kind of considered structures and distinguish some classes of its elements.

We are now able to give a precise description of the notion of a fundamental structure on $(i+1)$-level in $\mathcal{U}$. It is a structure for $\mathbf{L}_{i+1}$, namely a system $\langle p_{i+1}(t_{i+1});\ \mathcal{D}_0, \mathcal{D}_0, \mathcal{C}\rangle$ consisting of the following things: a class $p_{i+1}(t_{i+1})$ of i-objects and rules among them, called the domain of the structure, and for the function symbols $\mathcal{D}_0, \mathcal{D}_1$ and $\mathcal{C}$ of $\mathbf{L}_{i+1}$, two 1-ary functions $\mathcal{D}_0, \mathcal{D}_1$ of $p_{i+1}(t_{i+1})$ to itself with values in the subclass t_{i+1} of $p_{i+1}(t_{i+1})$, and a 2-ary function $\mathcal{C}$ of $p_{i+1}(t_{i+1})$ to itself which assigns, to some pairs of elements of $p_{i+1}(t_{i+1})$ single elements of it. Note that in the definition of $\mathcal{C}$ is relaxed the condition for every, namely $\mathcal{C}$ is not defined for all pairs but for some pairs of $p_{i+1}(t_{i+1})$. From now on we shall identify the notations for a fundamental structure and its domain.

If we define on a fundamental structure $p_{i+1}(t_{i+1})$ a one-one transformation op which assigns, to each rule p_i of $p_{i+1}(t_{i+1})$, a rule p_i^{op} in such a way that $op(\mathcal{D}_{0,1}(p_i) = \mathcal{D}_{1,0}(p_i^{\mathrm{op}})$ and $op(\mathcal{C}(p_i, q_i; r_i)) = \mathcal{C}(q_i^{\mathrm{op}}, p_i^{\mathrm{op}}; r_i^{\mathrm{op}})$, then of $p_{i+1}(t_{i+1})$ we obtain the opposite or the dual fundamental structure $p_{i+1}^{\mathrm{op}}(t_{i+1})$. This structure differs from $p_{i+1}(t_{i+1})$ in reversed directions of rules and the order of the composition. Obviously, the transformation op satisfies the condition $op \circ op = 1$. We can also do the above stressed transformations throughout the formulas of $\mathbf{L}_{i+1}$ and obtain the dual language $\mathbf{L}_{i+1}^{\mathrm{op}}$ of $\mathbf{L}_{i+1}$. A structure for $\mathbf{L}_{i+1}^{\mathrm{op}}$ is then the dual fundamental structure of a structure for $\mathbf{L}_{i+1}$.

Now we shall specify the nonlogical axioms of ETFS and accordingly particular kinds of fundamental structures. At first, we formulate axioms that will be the basic ones. These axioms can be found in [3] and also in a form in [2]. They are divided in the following two groups:

I. Technical axioms

1. $\mathcal{D}_n(\mathcal{D}_m(p_i)) = \mathcal{D}_m(p_i), \quad n,m = 0,1,$

2. $\exists r_i \mathcal{C}(p_i, q_i; r_i) \Leftrightarrow \mathcal{D}_1(p_i) = \mathcal{D}_0(q_i),$

3. $\mathcal{C}(p_i, q_i; r_i) \Rightarrow \mathcal{D}_0(p_i) = \mathcal{D}_0(r_i) \wedge \mathcal{D}_1(r_i) = \mathcal{D}_1(q_i),$

4. $\mathcal{C}(p_i, q_i; r_i) \wedge \mathcal{C}(p_i, q_i; r_i') \Rightarrow r_i = r_i'.$

II. Principal axioms

A_1. Axiom of identity

$$\mathcal{C}(\mathcal{D}_0(p_i), p_i; p_i) \wedge \mathcal{C}(p_i, \mathcal{D}_1(p_i); p_i).$$

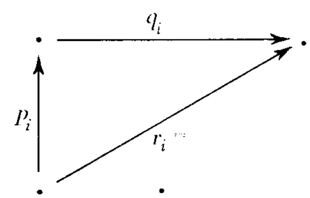
A_2. Axiom of associativity

$$\mathcal{C}(p_i, q_i; u_i) \wedge \mathcal{C}(q_i, r_i; w_i) \wedge \mathcal{C}(p_i, w_i; x_i) \wedge \mathcal{C}(u_i, r_i; y_i) \Rightarrow x_i = y_i.$$

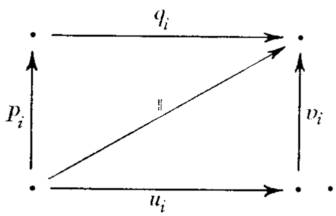
A structure $p_{i+1}(t_{i+1})$ for $\mathbf{L}_{i+1}$ in which all the above axioms are valid we call a fundamental semigroupoid. In that case ETFS means the elementary theory of fundamental semigroupoids, and Θ_{i+1} the class of its formulas. A formula of Θ_{i+1} is valid if it is a logical consequence of the above axioms.

Let us see now what the above axioms state. We first regard the axioms of the group I. The axiom 1. specifies the functions $\mathcal{D}_0$ and $\mathcal{D}_1$ to be retractions of $p_{i+1}(t_{i+1})$ to t_{i+1}, namely functions which leave the elements of t_{i+1} fixed. The remaining axioms of I. state that $\mathcal{C}(p_i, q_i)$ is defined for any two elements $p_i, q_i \in \in p_{i+1}(t_{i+1})$ such that $\mathcal{D}_1(p_i) = \mathcal{D}_0(q_i)$ and that, if it is defined, it is defined in a unique way. The axioms of the group II. state that, $\mathcal{D}_0(p_i)$ and $\mathcal{D}_1(p_i)$ are the left and the right identity of p_i, respectively, and that $\mathcal{C}$ is associative.

Now we shall introduce some abbreviations for certain formulas of $\mathbf{L}_{i+1}$ to make them shorter. So, the formula $a_i \xrightarrow{p_i} b_i$ will mean $\mathcal{D}_0(p_i) = b_i$ and $\mathcal{D}_1(p_i) = b_i$ for a p_i, and the formula $p_i \cdot q_i = r_i$ will mean $\mathcal{C}(p_i, q_i; r_i)$. We need in future the formula $\mathcal{C}(p_i, q_i; r_i) \wedge \mathcal{D}_0(p_i) = \mathcal{D}_0(r_i) \wedge \mathcal{D}_1(p_i) = \mathcal{D}_0(q_i) \wedge \mathcal{D}_1(q_i) = \mathcal{D}_1(r_i)$. For it we introduce the abbreviation $\mathcal{C}_{\text{com}}(p_i, q_i; r_i)$. We mention now the following abbreviations: $\mathcal{C}(p_i, q_i \,\|\, u_i, v_i)$ for $\mathcal{C}(p_i, q_i; r_i) \wedge \mathcal{C}(u_i, v_i; r_i') \wedge r_i = r_i'$ and also $\mathcal{C}_{\text{com}}(p_i, q_i \,\|\, u_i, v_i)$ for $\mathcal{C}_{\text{com}}(p_i, q_i; r_i) \wedge \mathcal{C}_{\text{com}}(u_i, v_i; r_i') \wedge r_i = r_i$. Sometimes, we shall represent a formula $p_i \cdot q_i = r_i$, and accordingly $\mathcal{C}(p_i, q_i; r_i)$, graphically by a diagram



Then the formula $\mathcal{C}_{\text{com}}(p_i, q_i; r_i)$ will mean that this diagram is commutative. Likewise, the formula $\mathcal{C}_{\text{com}}(p_i, q_i \,\|\, u_i, v_i)$ will mean the commutativity of the diagram



We mention some further abbreviated formulas, those which we need in future. These formulas can be found in [3] and also in any book on category theory.

Mono (p_i) means $\forall r_i \forall r_i' (\mathcal{C}(r_i, p_i \,\|\, r_i', p_i) \Rightarrow r_i = r_i')$.
Epi (p_i) means $\forall u_i \forall u_i' (\mathcal{C}(p_i, u_i \,\|\, p_i, u_i') \Rightarrow u_i = u_i')$.
Retr (p_i) means $\exists q_i \mathcal{C}(p_i, q_i; \mathcal{D}_0(p_i))$.
Coretr (p_i) means $\exists q_i \mathcal{C}(q_i, p_i; \mathcal{D}_1(p_1))$.
Iso (p_i) means Retr $(p_i) \wedge$ Coretr (p_i).
$a_i \simeq b_i$ means $\exists p_i (\mathcal{D}_0(p_i) = a_i \wedge \mathcal{D}_1(p_i) = b_i \wedge \text{Iso}(p_i))$.

In a similar way we can express many other notions concerning fundamental semigroupoids. However, we shall not do it here but later when such notions will arise.

We can specify some another fundamental structures by further specifications of nonlogical axioms. If we add to the group II. of the basic class of axioms the following axiom

A_3. Existence of inverses

$$\forall p_i \exists q_i (\mathcal{C}(p_i, q_i; \mathcal{D}_0(p_i)) \wedge \mathcal{C}(q_i, p_i; \mathcal{D}_1(p_i))),$$

then a structure for $\mathbf{L}_{i+1}$ in which all these axioms are valid is a fundamental groupoid. Obviously, a fundamental groupoid $p_{i+1}(t_{i+1})$ is a fundamental semigroupoid in which the formula $\forall p_i \operatorname{Iso}(p_i)$ is valid.

So far we have involved two kinds of fundamental structures, a fundamental semigroupoid and a fundamental groupoid. We can further obtain special cases of these structures by requiring that $\mathcal{D}_0 = \mathcal{D}_1$ and that both are constant functions. In that case a fundamental semigroupoid is reduced to a fundamental semigroup and a fundamental groupoid to a fundamental group. If the common value of the constant functions we denote by e_i, then the fundamental structures we may denote by $p_{i+1}(e_i)$. The object e_i is the left and the right identity of every element of $p_{i+1}(e_i)$. Now, $\mathcal{C}(p_i, q_i)$ is defined for any two elements of $p_{i+1}(e_i)$.

Let $p_{i+1}(t_{i+1})$ be either a fundamental semigroupoid or a fundamental groupoid and a_i an object in it. Denote by $p_{i+1}(a_i)$ the class of all elements of $p_{i+1}(t_{i+1})$ of which a_i is the left and the right identity. Then we have an obvious

Proposition 1. *If $p_{i+1}(t_{i+1})$ is a fundamental semigroupoid, then $p_{i+1}(a_i)$ is a fundamental subsemigroup of it. In the case that $p_{i+1}(t_{i+1})$ is a fundamental groupoid, $p_{i+1}(a_i)$ is a fundamental subgroup of it.* ∎

In the remainder of this paper we shall consider fundamental semigroupoids only because their considerations are less elementary and restrictive than of other fundamental structures. Thus, from now on, by ETFS we mean an elementary theory of fundamental semigroupoids.

Now we shall briefly consider classes of fundamental semigroupoids. In order that these classes be capable for studies we have to define rules for relating objects in them. Certainly, the rules will be those ones which naturaly belong to these classes, i.e. the rules preserving some intrinsic or basic properties of fundamental semigroupoids. A rule between two fundamental semigroupoids which preserves validity of atomic formulas we call a homomorphism. There are two sorts of these rules. The ones, preserving directions of rules, called covariant homomorphisms, and the others, reversing directions, called contravariant homomorphisms. We define here covariant homomorphisms only, since contravariant homomorphisms are then easily obtained as covariant ones to the opposite fundamental semigroupoids of the fundamental semigroupoids staying for their targets.

Definition 1. By a *covariant homomorphism* between fundamental semigroupoids $p_{i+1}(t_{i+1})$ and $q_{i+1}(s_{i+1})$ we understand a rule $F_{i+1}: p_{i+1}(t_{i+1}) \to q_{i+1}(s_{i+1})$ which assigns, to each element $p_i \in p_{i+1}(t_{i+1})$, an element $F_{i+1}(p_i) \in q_{i+1}(s_{i+1})$ in such a way that always:

$$\text{if } \mathcal{D}_{0,1}(p_i) = a_i, \text{ then } \mathcal{D}'_{0,1}(F_{i+1}(p_i)) = F_{i+1}(a_i), \text{ and}$$

$$\text{if } \mathcal{C}(p_i, q_i; r_i), \text{ then } \mathcal{C}'(F_{i+1}(p_i), F_{i+1}(q_i); F_{i+1}(r_i)),$$

where the primes denote the interpretations of function symbols of $\mathbf{L}_{i+1}$ in $q_{i+1}(s_{i+1})$.

A class of $(i+1)$-fundamental semigroupoids provided with a class of homomorphisms among its objects also allow the structure of a fundamental semigroupoid. Hence, it follows that, we can also formulate ETFS on the level $i+2$. This theory we denote by ETFS_{i+2}. If we take that two-sort variables of the language $\mathbf{L}_{i+2}$ of ETFS_{i+2} are ranging over $(i+1)$-fundamental semigroupoids and homomorphisms among them, then models for such a theory are $(i+2)$-fundamental semigroupoids which have $(i+1)$-fundamental semigroupoids as their objects. Now, as well as for ETFS_{i+1}, we can write down some abbreviated formulas whose means are special to this particular case of ETFS_{i+2}. Let F_{i+1} be an arrow od $\mathbf{L}_{i+2}$ of ETFS_{i+2} such that $\Delta_0(F_{i+1})=p_{i+1}(t_{i+1})$ and $\Delta_1(F_{i+1})=q_{i+1}(s_{i+1})$, where Δ_0 and Δ_1 are function symbols denoting respectively, domains and codomains of arrows, then we have the following formulas:

Inject (F_{i+1}) means $\forall p_i \forall p'_i (F_{i+1}(p_i)=F_{i+1}(p'_i) \rightarrow p_i=p'_i)$.

Surj (F_{i+1}) means $\forall q_i \exists p_i (F_{i+1}(p_i)=q_i)$.

Bij (F_{i+1}) means Inject (F_{i+1}) & Surj (F_{i+1}).

The structure of a fundamental semigroupoid can be involved on almost each class of objects in $\mathcal{U}$. It is sufficient to define proper rules for that purpose. Thus, we can say that this structure is the most natural structure in $\mathcal{U}$. To show this we indicate some more examples. For instance, we can involve this structure on the class of rules in a fundamental semigroupoid, then on the class of rules among these rules and so on. We are interested here to define rules under which a class of rules in an $(i+2)$-fundamental semigroupoid, objects of which are $(i+1)$-fundamental semigroupoids, allows the structure of a fundamental semigroupoid. Let $\mathrm{Hom}_{i+2}(p_{i+1}(t_{i+1}), q_{i+1}(s_{i+1}))$ be a class of homorphisms between fundamental semigroupoids $p_{i+1}(t_{i+1})$ and $q_{i+1}(s_{i+1})$. The rules among elements in this class we call natural rules. Their definition is as follows.

Definition 2. By a *natural rule* between homomorphisms $F_{i+1}, G_{i+1} \in \mathrm{Hom}_{i+2}(p_{i+1}(t_{i+1}), q_{i+1}(s_{i+1}))$ we understand a rule $\eta_{i+1}: F_{i+1} \rightarrow G_{i+1}$ which assigns, to each object $t_i \in p_{i+1}(t_{i+1})$ a rule $\eta_{i+1}(t_i): F_{i+1}(t_i) \rightarrow G_{i+1}(t_i)$ of $q_{i+1}(s_{i+1})$ in such a way that for every rule $p_i: a_i \rightarrow b_i \in p_{i+1}(t_{i+1})$ the formula $\mathcal{C}_{\mathrm{com}}(\eta_{i+1}(a_i), G_{i+1}(p_i) \,||\, F_{i+1}(p_i), \eta_{i+1}(b_i))$ holds in $q_{i+1}(s_{i+1})$.

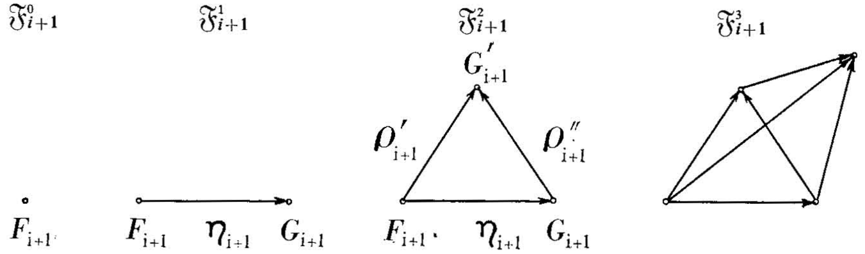
If we now want to introduce the structure of a fundamental semigroupoid on the class of natural rules we have to define rules among them. We shall indicate such rules. By a rule between natural rules $\eta_{i+1}: F_{i+1} \rightarrow G_{i+1}$ and $\eta'_{i+1}: F'_{i+1} \rightarrow G'_{i+1}$, where F_{i+1}, F'_{i+1}, G_{i+1} and G'_{i+1} are all from $\mathrm{Hom}_{i+2}(p_{i+1}(t_{i+1}), q_{i+1}(s_{i+1}))$, we understand a rule pair $\rho_{i+1}=(\rho'_{i+1}, \rho''_{i+1})$, where $\rho'_{i+1}: F_{i+1} \rightarrow F'_{i+1}$ and $\rho''_{i+1}: : G_{i+1} \rightarrow G'_{i+1}$, which assigns, to each object $t_i \in p_{i+1}(t_{i+1})$, a pair of rules $(\rho'_{i+1}(t_i), \rho''_{i+1}(t_i))$ of $q_{i+1}(s_{i+1})$ such that the formula $\mathcal{C}_{\mathrm{com}}(\eta_{i+1}(t_i), \rho''_{i+1}(t_i) \,||\, \rho'_{i+1}(t_i), \eta'_{i+1}(t_i))$ holds in $q_{i+1}(s_{i+1})$. Obviously, this assignment is such that for every $p_i \in p_{i+1}(t_{i+1})$ certain formulas hold in $q_{i+1}(s_{i+1})$. For instance, the formulas $\mathcal{C}_{\mathrm{com}}(\rho'_{i+1}(a_i), F'_{i+1}(p_i) \,||\, F_{i+1}(p_i), \rho'_{i+1}(b_i))$ and $\mathcal{C}_{\mathrm{com}}(\rho''_{i+1}(a_i), G'_{i+1}(p_i) \,||\, G_{i+1}(p_i), \rho''_{i+1}(b_i))$ hold. If we assume here that the natural rule η'_{i+1} is the identity natural rule $1'_{i+1}: : F'_{i+1} \rightarrow F'_{i+1}$, then we have the definition of a rule between the natural rule η_{i+1} and the homonorphism F_{i+1}. This rule assigns, to each object $t_i \in p_{i+1}(t_{i+1})$ a pair of rules of $q_{i+1}(s_{i+1})$ such that the formula $\mathcal{C}_{\mathrm{com}}(\eta_{i+1}(t_i), \rho''_{i+1}(t_i); \rho'_{i+1}(t))$ holds. Certainly, we can proceed in an obvious way to define rules among just defined rules, then among these ones and so on. However, we omit to do it.

By means of the above defined rules we can form certain new rules. These rules will be certain triples composed by the mentioned rules. At first we have a homomorphism $F_{i+1} \in \mathrm{Hom}_{i+2}(p_{i+1}(t_{i+1}), q_{i+1}(s_{i+1}))$. This rule we can view as a triple $(F_{i+1}, 1_{i+1}, F_{i+1})$ which assigns to each object of $p_{i+1}(t_{i+1})$ a subclass of $q_{i+1}(s_{i+1})$ consisting of a single object and the identity natural rule. Hence, it is a single-valued rule. We have further the triple $(F_{i+1}, \eta_{i+1}, G_{i+1})$, where F_{i+1} and G_{i+1} are two homomorphisms of $\mathrm{Hom}_{i+2}(p_{i+1}(t_{i+1}), q_{i+1}(s_{i+1}))$ and η_{i+1} is a natural rule between them. This triple is obviously a homomorphism of $p_{i+1}(t_{i+1})$ to $q_{i+1}(s_{i+1})$ which assigns to each object of $p_{i+1}(t_{i+1})$ a subclass of $q_{i+1}(s_{i+1})$ consisting of two objects and a rule between them. Hence, it is a many-valued homomorphism. Furthermore we have triples $(\mathfrak{F}_{i+1}, \rho^2_{i+1}, H_{i+1})$ and $(\mathfrak{F}_{i+1}, \zeta^2_{i+1}, \mathfrak{F}'_{i+1})$, where $\mathfrak{F}_{i+1}$ and $\mathfrak{F}'_{i+1}$ are triples consisting of homomorphisms and natural rules between them, and ρ^2_{i+1} means a rule pair. Obviously, these triples are many--valued homomorphisms of $p_{i+1}(t_{i+1})$ to $q_{i+1}(s_{i+1})$. In a similar way we can form various kinds of many-valued homomorphisms (rules) of $p_{i+1}(t_{i+1})$ to $q_{i+1}(s_{i+1})$. For further needs we form here two classes of these rules. Firstly we form the class

$$\mathfrak{F}_{i+1} = \{\mathfrak{F}^{\alpha}_{i+1} \mid \alpha \leqslant c_{q(s)}\},$$

such that $\mathfrak{F}^{\alpha+1}_{i+1} = (\mathfrak{F}^{\alpha}_{i+1}, \rho^{\alpha+1}_{i+1}, G^{(k\alpha)}_{i+1})$ and $\mathfrak{F}^0_{i+1}$ is a rule of $\mathrm{Hom}_{i+2}(p_{i+1}(t_{i+1}), q_{i+1}(s_{i+1}))$. Here, $\rho^{\alpha+1}_{i+1}$ means a rule $(\alpha+1)$-tiple and $G^{(k\alpha)}_{i+1}$ a rule of $\mathrm{Hom}_{i+2}(p_{i+1}(t_{i+1}), q_{i+1}(s_{i+1}))$. Provided that all $G^{(k\alpha)}_{i+1}$ are constant rules we have the following pictures of elements of $\mathfrak{F}_{i+1}$ for $\alpha \leqslant 3$

(*)



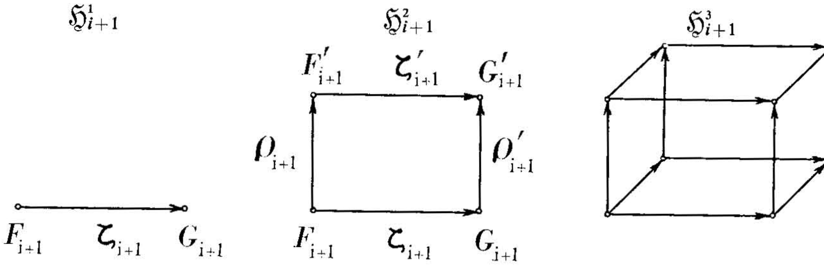
We distinguish this case denoting $\mathfrak{F}_{i+1}$ by $\mathfrak{F}^*_{i+1}$.

The next class of many-valued rules which we shall form here is the class

$$\mathfrak{H}_{i+1} = \{\mathfrak{H}^{\beta}_{i+1} \mid \beta \leqslant c_{(s)}\},$$

such that $\mathfrak{H}^{\beta+1}_{i+1} = (\mathfrak{H}^{\beta}_{i+1}, \rho^{2\beta}_{i+1}, {}^*\mathfrak{H}^{\beta}_{i+1})$, where $\rho^{2\beta}_{i+1}$ denotes a rule 2^{β}- tiple. If we assume that $\mathfrak{H}^0_{i+1} = F_{i+1}$ and ${}^*\mathfrak{H}^0_{i+1} = G_{i+1}$, then the pictures of the elements of this class for $\beta \leqslant 3$ are

(**)



The above classes $\mathfrak{F}_{i+1}$ and $\mathfrak{H}_{i+1}$ we can also view as many-valued rules of $p_{i+1}(t_{i+1})$ to $q_{i+1}(s_{i+1})$ such that $\mathfrak{F}_{i+1}(a_i) = \{\mathfrak{F}^{\alpha}_{i+1}(a_i) \mid \alpha \leqslant c_{q(s)}\}$ and $\mathfrak{H}_{i+1}(a_i) = \{\mathfrak{H}^{\beta}_{i+1}(a_i) \mid \beta \leqslant c_{q(s)}\}$ for $a_i \in p_{i+1}(t_{i+1})$. We have assumed here that $q_{i+1}(s_{i+1})$ is not $(i+1)$-universe, and because of that we have that codomains of $\mathfrak{F}_{i+1}$ and $\mathfrak{H}_{i+1}$ have $(i+1)$-level, namely that they are horizontal rules in $\mathcal{U}$.

Now we shall involve some notions in a fundamental semigroupoid to show what the above two rules $\mathfrak{F}_{i+1}$ and $\mathfrak{H}_{i+1}$ represent. At first we involve the notion of a simplex. By a *simplex* in a fundamental semigroupoid $p_{i+1}(t_{i+1})$ we understand a subclass $q_{i+1}(t_{i+1})$ of it such that the following two conditions are fulfilled:

i) For each pair of objects a_i, b_i of $q_{i+1}(s_{i+1})$ there is a rule $q_i \in q_{i+1}(s_{i+1})$ such that $a_i \xrightarrow{q_i} b_i$ or $b_i \xrightarrow{q_i} a_i$, and

ii) for any two objects in $q_{i+1}(s_{i+1})$, the rule obtained by following a path between the two objects in the directions of the arrows is independent of the choice of path.

The condition ii) means commutativity of any diagram in $q_{i+1}(s_{i+1})$. Thus, a simplex in $p_{i+1}(t_{i+1})$ is a connected subclass for which the formulas meaning commutativity of its diagrams hold. By an n-simplex, where n is a cardinal, we understand a simplex with $n+1$ nonisomorphic objects. Since a singleton subclass of $p_{i+1}(t_{i+1})$ consisting of a single object and the identity rule is also a simplex a o-simplex, then the picture of an n-simplex for $n \leqslant 3$ is the same as (*) except the notations of dots. The number n we call the dimension of the simplex. By a *face* of a simplex we mean a subsimplex of it. The *boundary* of an n-simplex is the union of all its faces of dimension $<n$.

Besides the above notions we have also the notion of a *simplicial complex* in a fundamental semigroupoid. It is a class $\mathcal{K}_{i+1}$ of simplexes in it such that, if $q_{i+1}(s_{i+1}) \in \mathcal{K}_{i+1}$ then any face of it also belongs to $\mathcal{K}_{i+1}$, and any two simplexes of $\mathcal{K}_{i+1}$ meet in a common face (possibly empty). Hence, $\mathcal{K}_{i+1}$ is a hereditary class. Its level is $i+1$, since we consider that $q_{i+1}(s_{i+1})$ is not $(i+1)$-universe.

If we now look at the many-valued rule $\mathfrak{F}^*_{i+1}$ we see that this rule assigns to each object of its domain a simplicial complex. Its each n-component $\mathfrak{F}^*_{i+1}$ assigns to each object an n-simplex. Thus, we may say that $\mathfrak{F}_{i+1}$ carries the structure of a simplicial complex. Similarly we have that the many-valued rule $\mathfrak{H}_{i+1}$ carries the structure of an other complex called a *cell complex*. Its each n-component represents an *n-cell.*

A further study of the rules $\mathfrak{F}_{i+1}$ and $\mathfrak{H}_{i+1}$ and related notions we leave for an other paper. Then we shall show that these rules allow some structures on themselves and shall emphasize certain elements in them.

Now, after a small digression, we proceed to involve fundamental semigroupoids with some additional structures, respectively fundamental semigroupoid, possessing certain special properties. These fundamental semigroupoids will play a central role in our further studies of structures in $\mathcal{U}$. We devote the remainder of this paper to them.

3. Fundamental structures with additional structures.

To make a fundamental semigroupoid capable for our future purposes we require that it allows certain creations in itself. We consider here those creations which have geometrical features. It means that the notions which are to be created in a fundamental semigroupoid will have shapes of some geometrical figures. We borrow for them corresponding names from geometry. In such a way we have the notions, a cylinder, a cone and so on. By allowing creations of these notions in a fundamental semigroupoid we obtain additional structures on it, or special organizations of it. Such a special organization of a fundamental semigroupoid will reflect itself through certain particular arrangement of objects in it with respect to its rules. It will give a possibility for constructions of new objects from the original ones in a fundamental semigroupoid. Thus, the additional structure that we mean to impose on a fundamental semigroupoid will have a constructive character for the class of objects in it. In order to introduce such a structure we have to define the notions which are to be created. At first, we define the notion of a cylinder in a fundamental semigroupoid. Let $p_{i+1}(t_{i+1})$ be a fundamental semigroupoid and $q_{i+1}(s_{i+1})$ and $r_{i+1}(a_{i+1})$ two subclasses of it, then we have

Definition 3. By a *cylinder* in $p_{i+1}(t_{i+1})$ with the lower basis $q_{i+1}(s_{i+1})$ and the upper one in $r_{i+1}(a_{i+1})$ we mean a triple $(q_{i+1}(s_{i+1}),\ \varphi_{i+1},\ r_{i+1}(a_{i+1}))$, where φ_{i+1} is a class of rules of $p_{i+1}(t_{i+1})$ among objects of $q_{i+1}(s_{i+1})$ and $r_{i+1}(a_{i+1})$, one for each object of $q_{i+1}(s_{i+1})$, with the property that, for every q_{i+1}-rule $q_i:$ $: s_i \to s_i'$ there is an r_{i+1}-rule $r_i: a_i \to a_i'$ and φ_{i+1}-rules $\varphi_i: s_i \to a_i$ and $\varphi_i': s_i' \to a_i'$ such that the formula $\mathcal{C}_{\text{com}}(\varphi_i,\ r_i \,\|\, q_i,\ \varphi_i')$ holds.

In the opposite direction of rules of the class φ_{i+1} we have the case of a *co-cylinder* in $p_{i+1}(t_{i+1})$ with the notions — the lower and the upper cobasis.

If the class $r_{i+1}(a_{i+1})$ contains a single object only, then the notion of a cylinder is reduced to the notion of a *cone*, and a cocylinder to a *cocone*. The single object in the upper basis, in that case, we call a vertex and in the lower cobasis a covertex.

In the case that the classes $q_{i+1}(s_{i+1})$ and $r_{i+1}(a_{i+1})$ are discrete, namely that they contain identities only, we have notions — a discrete cylinder, a discrete cone and so on. In that case the commutativities of the diagrams are to be dropped from the definitions.

In the sequel we shall identify a cone $(q_{i+1}(s_{i+1}),\ \varphi_{i+1},\ \{a_i\})$ with the class φ_{i+1} only. At this we shall consider that from the context is obvious what are the basis and the vertex of it. In that case we shall call the class φ_{i+1} a cone over the class $q_{i+1}(s_{i+1})$ with the vertex a_i.

A cylinder in $p_{i+1}(t_{i+1})$ with the lower basis $q_{i+1}(s_{i+1})$ and the upper one in $r_{i+1}(a_{i+1})$ can be viewed as the image of a many-valued homomorphism $\mathfrak{F}_{i+1} =$ $= (I_{i+1},\ \eta_{i+1},\ C_{i+1})$ of $p_{i+1}(t_{i+1})$ to itself, where I_{i+1} is the inclusion homomorphism of $q_{i+1}(s_{i+1})$ to $p_{i+1}(t_{i+1})$ and C_{i+1} a homomorphism with values belonging to $r_{i+1}(a_{i+1})$. If C_{i+1} is a constant homomorphism, then the image of the above homorphism, that we shall denote now by ${}_c\mathfrak{F}_{i+1}$, is a cone in $p_{i+1}(t_{i+1})$. Analogously we can conceive a cocylinder and a cocone in $p_{i+1}(t_{i+1})$. So , a cocylinder in $p_{i+1}(t_{i+1})$ is the image of a many-valued homomorphism $\bar{\mathfrak{F}}_{i+1} = (C_{i+1},\ \bar{\eta}_{i+1},\ I_{i+1})$. Throughout the paper we shall use both kinds of definitions.

If in $\mathfrak{F}_{i+1} = (I_{i+1},\ \eta_{i+1},\ C_{i+1})$, given above, the natural rule η_{i+1} is a natural equivalence, then the homomorphism $\mathfrak{F}_{i+1}$ represents a *contraction* of $q_{i+1}(s_{i+1})$ into $r_{i+1}(a_{i+1})$ in $p_{i+1}(t_{i+1})$, and ${}_c\mathfrak{F}_{i+1}$ a contraction of $q_{i+1}(s_{i+1})$ in $p_{i+1}(t_{i+1})$.

Proposition 2. *Any two homomorphisms of an arbitrary fundamental semigroupoid to a contractive fundamental semigroupoid are naturaly equivalent.*

Proof. Let $q_{i+1}(s_{i+1})$ be a contractive fundamental semigroupoid and suppose that I_{i+1} is naturally equivalent to C_{i+1}, i.e. $I_{i+1} \simeq C_{i+1}$, where C_{i+1} is a constant homomorphism of $q_{i+1}(s_{i+1})$ to itself. Let $F_{i+1},\ G_{i+1}: p_{i+1}(t_{i+1}) \to$ $\to q_{i+1}(s_{i+1})$ be arbitrary homomorphisms. Then $F_{i+1} = I_{i+1} \circ F_{i+1} \simeq C_{i+1} \circ F_{i+1}$, and similarly, $G_{i+1} \simeq C_{i+1} \circ G_{i+1}$. Since $C_{i+1} \circ F_{i+1} = C_{i+1} \circ G_{i+1}$, it follows that $F_{i+1} \simeq G_{i+1}$. ∎

Corollary. *If $q_{i+1}(s_{i+1})$ is contractive, then any two constant homomorphism of $q_{i+1}(s_{i+1})$ to itself are naturaly equivalent, and the identity homomorphism is naturally equivalent to any constant homomorphism of $q_{i+1}(s_{i+1})$ to itself.* ∎

If we define a homomorphisms F_{i+1} of a fundamental semigroupoid $p_{i+1}(t_{i+1})$ to a fundamental semigroupoid $q_{i+1}(s_{i+1})$ to be *inessential* if it is naturally equivalent to a constant homomorphism $C_{i+1}: p_{i+1}(t_{i+1}) \to q_{i+1}(s_{i+1})$, then from the above proposition we have that every homomorphism of a fundamental semigroupoid to a contractive fundamental semigroupoid is inessential.

A homomorphism F_{i+1} with values in $p_{i+1}(t_{i+1})$ we shall say to form or *create* cylinders in $p_{i+1}(t_{i+1})$ if there is a natural rule $\eta_{i+1}: I_{i+1} \to F_{i+1}$, where I_{i+1} is the identity homomorphism of $p_{i+1}(t_{i+1})$ to itself. In the similar way we have creations of cones, cocylinders and cocones in $p_{i+1}(t_{i+1})$.

Now we shall consider a class of cylinders or cones in $p_{i+1}(t_{i+1})$. Let $\mathfrak{F}_{i+1}(q_{i+1}(s_{i+1}))$ and $\mathfrak{F}'_{i+1}(q'_{i+1}(s'_{i+1}))$ be two arbitrary cylinders in $p_{i+1}(t_{i+1})$ over the subclasses $q_{i+1}(s_{i+1})$ and $q'_{i+1}(s'_{i+1})$. By a rule $\rho_{i+1}: \mathfrak{F}_{i+1}(q_{i+1}(s_{i+1})) \to \to \mathfrak{F}'_{i+1}(q'_{i+1}(s'_{i+1}))$ for their relating we mean a class of p_{i+1}-rules, one for each object of the lower and one for each object of the upper basis of $\mathfrak{F}_{i+1}(q_{i+1}(s_{i+1}))$, with the property that, for every $\eta_{i+1}(s_i) \in \mathfrak{F}_{i+1}(q_{i+1}(s_{i+1}))$ and an $\eta'_{i+1}(s'_i) \in \in \mathfrak{F}'_{i+1}(q'_{i+1}(s'_{i+1}))$ the formula $\mathcal{C}_{\text{com}}(\eta_{i+1}(s_i), p'_i \| p_i, \eta'_{i+1}(s'_i))$ holds. Thus, the rule ρ_{i+1} is a class of pair rules. We call it a *cylinder-rule*. If the considered two cylinders are over the same class in $p_{i+1}(t_{i+1})$, for instance $q_{i+1}(s_{i+1})$ then the rule ρ_{i+1} is a class of p_{i+1}-rules, one for each object of the upper basis of $\mathfrak{F}_{i+1}(q_{i+1}(s_{i+1}))$. Such a rule we call a *cylinder-rule over a class*. In the case of cones we have notions — a *cone-rule* and also a *cone-rule over a class*. This last rule consists of a single p_{i+1}-rule connecting vertices of the cones. The classes of cylinders as well as cones in $p_{i+1}(t_{i+1})$ with defined rules allow the structures of fundamental semigroupoids.

In a similar way we have the notions concerning cocylinders and cocones. They are a *cocylinder-rule*, a *cocone-rule* and the same over a class. The classes of cocylinders and cocones in $p_{i+1}(t_{i+1})$ with such rules also allow the structures of fundamental semigroupoids.

Because in the remainder of this paper we need, in the main, notions of cones and cocones, then we shall devote most care just to them. Let $\mathbf{C}_{i+2}(q_{i+1})$ be the class of all cones and cone-rules over a class $q_{i+1}(s_{i+1})$ in $p_{i+1}(t_{i+1})$. To shorten notations we denote elements of this class simply as the homomorphisms. Thus, instead of ${}_c\mathfrak{F}_{i+1}(q_{i+1}(s_{i+1}))$ we shall write for short ${}_c\mathfrak{F}_{i+1}$. If there exists a cone ${}_c\mathfrak{F}_{i+1}$ in the class $\mathbf{C}_{i+2}(q_{i+1})$ such that for every other cone ${}_c\mathfrak{F}^{\alpha}_{i+1}$ of $\mathbf{C}_{i+2}(q_{i+1})$ there is a cone-rule $\rho^{\alpha}_i: {}_c\mathfrak{F}_{i+1} \to {}_c\mathfrak{F}^{\alpha}_{i+1}$, then such a cone in $\mathbf{C}_{i+2}(q_{i+1})$ we call initial or *first cone* (abbreviated as *fc*). It is not necessarily unique. The uniqueness is determined by the following

Proposition 3. *The first cone* ${}_c\mathfrak{F}_{i+1}$ *in* $\mathbf{C}_{i+2}(q_{i+1})$ *is unique if every cone-rule* ρ^{α}_i *is unique.*∎

Let φ_{i+1} be *fc* over $q_{i+1}(s_{i+1})$ with the vertex t_i. The vertex t_i we shall call the *sequent* of the class of objects of $q_{i+1}(s_{i+1})$ with respect to the class of rules φ_{i+1}. If the class $q_{i+1}(s_{i+1})$ contains a single object only, then the vertex t_i of *fc* we call the *successor* of that object, of course, with respect to a rule φ_i. The uniqueness of successors follows from the above proposition.

We can also consider the class $\overline{\mathbf{C}}_{i+2}(q_{i+1})$ of all cocones and cocone-rules over the class $q_{i+1}(s_{i+1})$ in $p_{i+1}(t_{i+1})$. If, in this class, there exists a cocone ${}_{cc}\mathfrak{F}_{i+1}$ such that for every other cocone ${}_{cc}\mathfrak{F}^{\beta}_{i+2}$ of $\overline{\mathbf{C}}_{i+2}(q_{i+1})$ there is a cocone-rule $\zeta^{\beta}_i: : {}_{cc}\mathfrak{F}^{\beta}_{i+1} \to {}_{cc}\mathfrak{F}_{i+1}$, then such a cocone in $\overline{\mathbf{C}}_{i+2}(q_{i+1})$ we call terminal or *last cocone* (abbreviated as *lcc*). The covertex of this cocone we call the *presequent* of the class of objects of $q_{i+1}(s_{i+1})$ with respect to the class of rules connecting the objects of $q_{i+1}(s_{i+1})$ with this object. If the class $q_{i+1}(s_{i+1})$ consists of a single object only, then the covertex is its *predecessor*. The uniqueness of these notions is to be determined by the following

Proposition 4. *The last cocone* ${}_{cc}\mathfrak{F}_{i+1}$ in $\bar{\mathbf{C}}_{i+2}(q_{i+1})$ *is unique if every cocone-rule* ζ_i^β *is unique*.▮

Now we shall specify fundamental semigroupoids with additional structures. Here, by a fundamental semigroupoid with additional structure we mean a system $\langle p_{i+1}(t_{i+1}); {}_{c}\mathfrak{F}_{i+1}, {}_{cc}\mathfrak{F}_{i+1}\rangle$ consisting of a fundamental semigroupoid $p_{i+1}(t_{i+1})$ and rules ${}_{c}\mathfrak{F}_{i+1}$ and ${}_{cc}\mathfrak{F}_{i+1}$ of $p_{i+1}(t_{i+1})$ to itself which assign, to some subclasses of $p_{i+1}(t_{i+1})$ respectively, cones and cocones. If ${}_{c}\mathfrak{F}_{i+1}$ and ${}_{cc}\mathfrak{F}_{i+1}$ are defined on $p_{i+1}(t_{i+1})$, then we say that $p_{i+1}(t_{i+1})$ allows cone and cocone formations on its subclasses. Certainly, these formations are not unique and they are not defined for every subclass of $p_{i+1}(t_{i+1})$. Depending, if one or both these formations are defined on a fundamental semigroupoid and if they are defined for certain or any its subclass we shall have particular cases of that fundamental semigroupoid. To define these cases we need some preliminary considerations.

Let c_u be a number associated to the universe $\mathcal{U}_{i+1}$ by which we can number members in $\mathcal{U}_{i+1}$. By a c_t-indexed class or simply a c_t-class we denote an $(i+1)$-class t having c_t members, i.e. an $(i+1)$-class which when considered as a family has an index class whose cardinality is equal to c_t. Obviously, $c_t \leqslant c_u$. In the case of a $c_{q(s)}$-class we have that $c_{q(s)} = c_q + c_s$, where c_q corresponds to the class of rules and c_s to the class of objects of the class $q_{i+1}(s_{i+1})$. If we consider further a c_α-subclass of $p_{i+1}(t_{i+1})$ we shall think of a class having c'_α rules and c''_α objects of $p_{i+1}(t_{i+1})$. For instance, a subclass of $p_{i+1}(t_{i+1})$ consisting of two objects and (all) rules between them, or a class consisting of all objects of $p_{i+1}(t_{i+1})$, and so on.

We are now able to define announced cases of a fundamental semigroupoid.

Definition 4. By a $u^{c_\beta}-(d^{c_\beta}-)$ *semigroupoid* we mean a fundamental semigroupoid allowing in itself cone (cocone) formations on every its c_α-subclass, for $c_\alpha < c_\beta$.

Equivalently, a u^{c_β}-semigroupoid is a system $\langle p_{i+1}(t_{i+1}); {}_{c}\mathfrak{F}_{i+1}\rangle$ in which ${}_{c}\mathfrak{F}_{i+1}$ is defined for every c_α-subclass of $p_{i+1}(t_{i+1})$, where $c_\alpha < c_\beta$. Here, ${}_{c}\mathfrak{F}_{i+1}$ is then a rule of $p_{i+1}(t_{i+1})$ to itself which assigns, to each c_α-subclass, $c_\alpha < c_\beta$, a cone. Hence, a u^{c_β}-semigroupoid is c_β-upper directed. We correlate the notion of a u^{c_β}-limit to such a semigroupoid. The same we have for a d^{c_β}-semigroupoid.

Definition 5. By a $u^{c_\beta}-(d^{c_\beta}-)$ *limit* in a $u^{c_\beta}-(d^{c_\beta}-)$ semigroupoid we mean *fc* (*lcc*) over a c_β-subclass of it.

Clearly, these limits do not necessarily exist in the above semigroupoids, and if they exist they are not necessarily unique. However, there are semigroupoids which possess these limits. They are then complete semigroupoids.

Definition 6. A $u^{c_\beta}-(d^{c_\beta})$ semigroupoid we shall say to be $u^{c_\beta}-(d^{c_\beta}-)$ *complete* if it possesses $u^{c_\beta}-(d^{c_\beta}-)$ limits of every its c_β-subclass.

Finally, we define the most important class of semigroupoids that we shall often employ in future. They are those fundamental semigroupoids which allow both *fc* and *lcc* formations on its subclasses. We call them *l*-semigroupoids.

Definition 7. By an *l^{c_β}-semigroupoid* we mean a fundamental semigroupoid allowing *fc* and *lcc* formations on every its c_α-subclass where $c_\alpha \leqslant c_\beta$. If being an l^{c_β}-semigroupoid is valid for every c_β, then we have an *l-semigroupoid.*

Obviously, an l^{c_β}-semigroupoid is both u^{c_β}-and d^{c_β}-complete. The following proposition is a criterion for determining when a fundamental semigroupoid will be an *l*-semigroupoid.

Proposition 5. *A fundamental semigroupoid $p_{i+1}(t_{i+1})$ which allows fc (or lcc) formation on any its subclass is an l-semigroupoid.*

Proof. The proof is simple. Let $q_{i+1}(s_{i+1})$ be an arbitrary subclass of $p_{i+1}(t_{+1})$. According to hypothesis its *lcc* is in $p_{i+1}(t_{i+1})$, we have to show that its *fc* is also in $p_{i+1}(t_{i+1})$. Let $\mathcal{V}_{i+1}(q_{i+1}(s_{i+1}))$ be the class of vertices of all cones over $q_{i+1}(s_{i+1})$ in $p_{i+1}(t_{i+1})$. Certainly, *lcc* of this class is in $p_{i+1}(t_{i+1})$. Its covertex is the vertex of *fc* over $q_{i+1}(s_{i+1})$.∎

Now we shall proceed to specify some distinguished objects in an l^{c_β}-semigroupoid $p_{i+1}(t_{1+1})$. Firstly, we define the first and the last object in it.

Definition 8. By the *first* (the *last*) object in $p_{i+1}(t_{i+1})$ we mean the presequent (the sequent) of all objects of $p_{i+1}(t_{i+1})$ and denote it by $o_i(1_i)$.

Hence, if the object $o_i(1_i)$ exists in $p_{i+1}(t_{i+1})$, then for every $t_i \in t_{i+1}$ there is a rule $p_i: o_i \rightarrow t_i (p_i: t_i \rightarrow 1_i)$. The above objects are not defined to be unique in $p_{i+1}(t_{i+1})$. Their uniquess is determined by the propositions 3 and 4. Certainly, an *l*-semiqroupoid has both these objects.

Now we shall define an object in an l^{c_β}-semigroupoid $p_{i+1}(t_{i+1})$ that we shall need later in our work. This object is a strictly first object in $p_{i+1}(t_{i+1})$.

Definition 9. An object o_i^s of $p_{i+1}(t_{i+1})$ is *strictly first* if it is a first object in $p_{i+1}(t_{i+1})$ such that, for all $t_i \in p_{i+1}(t_{i+1})$ there is no rule $t_i \rightarrow o_i^s$ for t_i nonisomorphic to o_i^s.

Certainly, there can exist only one such object in $p_{i+1}(t_{i+1})$. Thus, if $p_{i+1}(t_{i+1})$ contains the object o_i^s, then it is a unique first object in it. There is an obvious statement conceıning the object o_i^s that we give here without a proof.

Proposition 6. *For every object $t_i \in p_{i+1}(t_{i+1})$ the rule $o_i^s \rightarrow t_i$ is mono.*∎

If the presequent of two object in $p_{i+1}(t_{i+1})$ is just the object o_i^s, then these objects are disjoint. The sequent of such objects is a disjoint sum. A subclass $q_{i+1}(s_{i+1})$ of $p_{i+1}(t_{i+1})$ will be disjoint if $o_i^s \notin q_{i+1}(s_{i+1})$ and if the presequent of any two its objects is the object o_i^s.

For a further specification of objects in an l^{c_β}-semigroupoid we need a new notion. This notion is reducibility of rules in it.

Definition 10. A rule p_i in an l^{c_β}-semigroupoid $p_{i+1}(t_{i+1})$ we shall say to be *reducible* through an object if there exist in $p_{i+1}(t_{i+1})$ an object c_i and rules α_i, β_i such that $\mathcal{C}_{\text{com}}(\alpha_i, \beta_i; p_i)$ holds.

That p_i is reducible through the object c_i we denote it by $p_i \mid c_i$. By means of this notion we shall define certain objects in an l^{c_β}-semigroupoid. Let $p_{i+1}^{o_i}(t_{i+1})$ be an l^{c_β}-semigroupoid with the object o_i. We define in it distinguished objects called atoms.

Definition 11. By an *atom* in $p_{i+1}^{o_i}(t_{i+1})$ we mean an object a_i such that the following conditions are fulfilled:

i) $a_i \neq o_i$, and

ii) for $p_i : o_i \to a_i \in p_{i+1}^{o_i}(t_{i+1})$ does not exist any object $b_i \in p_{i+1}^{o_i}(t_{i+1})$ nonisomorphic either o_i or a_i, such that $p_i \mid b_i$.

For an l^{c_β}-semigroupoid $p_{i+1}(t_{i+1})$ we shall say to be an *atomic* l^{c_β}-semigroupoid if for any object b_i and rule $p_i : o_i \to b_i$ of $p_{i+1}^{o_i}(t_{i+1})$ there is an atom a_i such that $p_i \mid a_i$.

If $p_{i+1}(t_{i+1})$ is an l^{c_β}-semigroupoid with o_i, then we have the following

Proposition 7. *A successor of* o_i *is an atom in* $p_{i+1}(t_{i+1})$.

Proof. Let a_i be a successor of o_i, there is *fc* $o_i \to a_i$. Then, for every other object b_i and a rule q_i such that $q_i : o_i \to b_i$, there is a rule $\alpha_i : a_i \to b_i$ such that $\mathcal{C}_{\text{com}}(p_i, \alpha_i; q_i)$ holds. Thus, there is no object in $p_{i+1}(t_{i+1})$ through which p_i would be reducible. Its existence would lead to a contradiction that a_i is a successor of o_i. ▮

Dual notions to the above ones are a *coatom* and a *coatomic* l^{c_β}-semigroupoid. The definition of these notions is quite obvious. It is to be given by using an l^{c_β}-semigroupoid $p_{i+1}(t_{i+1})$ with the object 1_i. An analogous proposition to the above one is

Proposition 8. *A predecessor of* 1_i *is a coatom in* $p_{i+1}(t_{i+1})$. ▮

If there is no atom or coatom in an l^{c_β}-semigroupoid, it is then an atomless or a coatomless l^{c_β}-semigroupoid, respectively.

Now we shall distinguish certain subclasses in a fundamental semigroupoid, respectively an l^{c_β}-semigroupoid, with common notation $p_{i+1}(t_{i+1})$, which have some particular properties. These subclasses we shall call filters and ideals in $p_{i+1}(t_{i+1})$. We define them as follows.

Definition 12. By a *c_γ-filter* (*c_γ-ideal*) in $p_{i+1}(t_{i+1})$ with the object $o_i(1_i)$ we mean a subclass $q_{i+1}(s_{i+1})$ of $p_{i+1}(t_{i+1})$ which satisfies the following conditions:

1) whenever it contains a c_α-subclass of $p_{i+1}(t_{i+1})$, where $c_\alpha < c_\gamma$, it contains its cocone (cone), and

2) if both $p_i \in p_{i+1}(t_{i+1})$ and $\mathcal{D}_0(p_i)$ belong to $q_{i+1}(s_{i+1})$, then $\mathcal{D}_1(p_i)$ also belongs to $q_{i+1}(s_{i+1})$ (if both $p_i \in p_{i+1}(t_{i+1})$ and $\mathcal{D}_1(p_i)$ belong to $q_{i+1}(s_{i+1})$, then $\mathcal{D}_0(p_i)$ also belongs to $q_{i+1}(s_{i+1})$).

We shall consider here filters only because ideals are dual notions. According to 1) a filter is a d^{c_γ}-directed subclass of $p_{i+1}(t_{i+1})$. It is also antiresidual, if we call so the condition 2). If we add the condition $o_i \notin q_{i+1}(s_{i+1})$ to the above conditions, then such a filter we call a *proper* filter.

Any nonempty class $q_{i+1}(s_{i+1})$ can always be extended to a class verifying the condition of antiresiduality. Let $\overline{q}_{i+1}(\overline{s}_{i+1})$ be a class such that $\overline{q}_{i+1} \supseteq q_{i+1}$ and the class of object $\overline{s}_{i+1}$ consists of all objects $\overline{s}_i$ for which there exist objects $s_i \in s_{i+1}$ with $s_i \to \overline{s}_i \in \overline{q}_{i+1}$. Then $\overline{q}_{i+1}(\overline{s}_{i+1})$ is determined by $q_{i+1}(s_{i+1})$ and $\overline{q}_{i+1}(\overline{s}_{i+1})$ obeys the condition 2). If $\overline{q}_{i+1} = q_{i+1}$, then the extension is full. If the class $\overline{q}_{i+1}(\overline{s}_{i+1})$ is a filter, then $q_{i+1}(s_{i+1})$ is its basis.

Definition 13. A class $q_{i+1}(s_{i+1})$ is a *c_γ-filter basis*, or simply a c_γ-basis, if $\overline{q}_{i+1}(\overline{s}_{i+1})$ is a c_γ-filter. Then we say that the filter $\overline{q}_{i+1}(\overline{s}_{i+1})$ is generated by the class $q_{i+1}(s_{i+1})$.

Now we shall see what properties the class $q_{i+1}(s_{i+1})$ must have to be a filter basis. To show this we have to introduce previously a new notion. A subclass $q_{i+1}(s_{i+1})$ of a fundamental semigroupoid we say to be a *lower cobasis-subclass* if cocylinders of any subclass of $p_{i+1}(t_{i+1})$ have their lower cobasises in $q_{i+1}(s_{i+1})$. Certainly, we assume that $p_{i+1}(t_{i+1})$ allows cocylinder formations. Hence, there is a homomorphism $_{cc}\mathfrak{F}_{i+1}$ of $p_{i+1}(t_{i+1})$ to itself which assigns, to each subclass of $p_{i+1}(t_{i+i})$, a cocylinder with the lower cobasis in $q_{i+1}(s_{i+1})$. Thus, $_{cc}\mathfrak{F}_{i+1}$ is of the form $(F_{i+1}, \eta_{i+1}, I_{i+1})$, where I_{i+1} is the identity homomorphism of $p_{i+1}(t_{i+1})$ and F_{i+1} a homomorphis of $p_{i+1}(t_{i+1})$ to $q_{i+1}(s_{i+1})$. Hence, for every object t_i of $p_{i+1}(t_{i+1})$ there is an object s_i of $q_{i+1}(s_{i+1})$ and one rule $\varphi_i : s_i \to t_i$ of $p_{i+1}(t_{i+1})$ such that if $t_i \xrightarrow{p_i} t'_i$, then there is a $s_i \xrightarrow{q_i} s'_i$ and p_{i+1}-rules φ_i and φ'_i such that $\mathcal{C}_{\text{com}}(\varphi_i, p_i \| q_i, \varphi'_i)$ holds. By means of the above notion we state the following.

Proposition 9. *A class $q_{i+1}(s_{i+1})$ is a c_γ-basis if $\overline{q}_{i+1}(\overline{s}_{i+1})$* is a class *which consists of all those objects and rules such that the following conditions are fulfilled*:

a) $q_{i+1}(s_{i+1})$ *is lower cobasis-subclass of $\overline{q}_{i+1}(\overline{s}_{i+1})$ and*

b) $q_{i+1}(s_{i+1})$ *is d^{c_γ}-directed subclass of $\overline{q}_{i+1}(\overline{s}_{i+1})$.*

Proof. It is sufficient to consider a c_α-subclass $\overline{r}_{i+1}(\overline{a}_{i+1})$ of $\overline{q}_{i+1}(\overline{s}_{i+1})$ with $c_\alpha < c_\gamma$. According to a), there is a cocylinder over it with the lower cobasis in $q_{i+1}(s_{i+1})$. Denote this cobasis by $r_{i+1}(a_{i+1})$. Then, by b) a cocone over $r_{i+1}(a_{i+1})$ is in $q_{i+1}(s_{i+1})$. Let φ_{i+1} be such a cocone with the covertex b_i. Then, for every $\overline{r}_i : \overline{a}_i \to \overline{a}'_i \in \overline{r}_{i+1}(\overline{a}_{i+1})$ we have the existence of an $r_i : a_i \to a'_i \in r_{i+1}(a_{i+1})$ and rules $\varphi_i : b_i \to a_i$ and $\varphi'_i : b'_i \to a'_i$, then $p_i : a_i \to \overline{a}_i$ and $p'_i : a'_i \to \overline{a}'_i$ such that the formulas $\mathcal{C}_{\text{com}}(\varphi_i, r_i; \varphi'_i)$ and $\mathcal{C}_{\text{com}}(p_i, \overline{r}_i \| r_i, p'_i)$ hold. Hence, one can further deduce that the formula $\mathcal{C}_{\text{com}}(\psi_i, \overline{r}_i; \psi'_i)$, where $\psi_i : b_i \to \overline{a}_i$ and $\psi'_i : b_i \to \overline{a}'_i$, also holds. The class ψ_{i+1} of the rules ψ_i is a cocone over $\overline{r}_{i+1}(\overline{a}_{i+1})$. Thus, $\overline{q}_{i+1}(\overline{s}_{i+1})$ is closed with respect to cocone formations on its c_α-subclasses, where $c_\alpha < c_\gamma$, i.e. it is a d^{c_γ}-directed class. It is also antiresidual.∎

It is obvious that, a basis of a filter is not unique and also that, a filter is a basis for itself.

Let $r_{i+1}(a_{i+1})$ be a c_β-filter in a fundamental semigroupoid $p_{i+1}(t_{i+1})$. A homomorphism of $p_{i+1}(t_{i+1})$ to an other fundamental semigroupoid is c_β-filter basis preserving if it preserves c_β-directedness. If it preserves c_α-cones, where $c_\alpha < _\beta$, then it will preserve d^{c_β}-directedness. It means that it has to preserve cardinalities of c_α-classes and cocones over them. A bijective homomorphism has such a property. If we do not require that cardinalities of filters are preserved, then we can relax the condition on the homomorphism to be bijective.

As we have already said our next paper will be devoted to an introduction of topological structures in $\mathcal{U}$. For that purpose we need some further considerations. Let $f_{i+2}(p_{i+1})$ be the class of all filters in a fundamental semigroupoid $p_{i+1}(t_{i+1})$. Clearly, $f_{i+2}(p_{i+1})$ is a subclass of $\mathcal{P}(p_{i+1}(t_{i+1}))$, the class of all subclasses of $p_{i+1}(t_{i+1})$. Hence, provided $p_{i+1}(t_{i+1})$ is not $(i+1)$-universe, it is class-theoretically an $(i+1)$-class. In that case we can write it with the index $i+1$, i.e. as $f_{i+1}(p_{i+1})$. However, this is not primary here, therefore, we retain the notation $f_{i+2}(p_{i+1})$. In what follows we shall specify, in a sense, filters in $f_{i+2}(p_{i+1})$. We shall distinguish two kinds of filters. Those which have their d-limits in $p_{i+1}(t_{i+1})$ and those which have not them. Filters which have the limits in $p_{i+1}(t_{i+1})$ we call complete filters. There are two possibilities, that the limits are in the filters and out of them. If they are in, then such filters we call *inner d-limit-filters*. However, if the filters have their d-limits out we call them *outer d-limit-filters*. Thus, we differ complete filters according to that where they have their d-limits. If $p_{i+1}(t_{i+1})$ is an l^{c_β}-semigroupoid, then every c_α-filter of $f_{i+2}(p_{i+1})$, where $c_\alpha < c_\beta$, is an inner d-limit-filter.

Let us consider a filter of $f_{i+2}(p_{i+1})$, whose basis is reduced to a singleton class consisting of an object and the identity rule. Such a filter we denote by $\overleftarrow{a}_i$, provided a_i is that single object, and call it a *principal filter*. Clearly, it is generated by this object. An inner d-limit-filter is obviously a principal filter.

Not every filter of $f_{i+2}(p_{i+1})$ has its d-limit in $p_{i+1}(t_{i+1})$. However, we can do some filters to have the limits in $p_{i+1}(t_{i+1})$ by inserting a class of objects into $p_{i+1}(t_{i+1})$ in such a way that each inserted object with a subclass of rules of $p_{i+1}(t_{i+1})$ forms a d-limit of a filter in $p_{i+1}(t_{i+1})$. Such filters are then complete in $p_{i+1}(t_{i+1})$. In that way we obtain in $p_{i+1}(t_{i+1})$ a number of filters having their d-limits in $p_{i+1}(t_{i+1})$. These completed filters will be basic ones for our further work. A filter of $f_{i+2}(p_{i+1})$ completed in such a way by an object a_i we shall denote by $\mathfrak{P}_{i+1}(a_i)$ and call it, for the time being, a *prominent filter* for the object a_i. The completeness of other filters of $f_{i+2}(p_{i+1})$ is then to be determined by their relating to these completed filters. For that purpose we need a class of rules on $f_{i+2}(p_{i+1})$. Since $f_{i+2}(p_{i+1})$ is a subclass of the class of all subclasses of $p_{i+1}(t_{i+1})$, we can involve in it inclusion rules. However, instead of an inclusion rule I_{i+1} it is more convenient to involve an opposite rule $\vdash_{i+1}$ such that $q_{i+1}(s_{i+1}) \vdash_{i+1} q'_{i+1}(s'_{i+1})$ iff $q'_{i+1}(s'_{i+1}) \xrightarrow{I_{i+1}} q_{i+1}(s_{i+1})$. We assume this as the definition of this rule. Thus, we have

Definition 13. For any two filters f_{i+1}, f'_{i+1} of $f_{+2}(p_{i+1})$ we define $f_{i+1} \vdash_{i+1} f'_{i+1}$ iff $f'_{i+1} \subseteq_{i+1} f_{i+1}$.

We shall always employ, in future, the above defined connections of filters. Since every filter is generated by a base, then we can stipulate relations of filters by examining their bases.

P r o p o s i t i o n 10. *If* $q_{i+1}(s_{i+1})$ *and* $q'_{i+1}(s'_{i+1})$ *are filters and* $r_{i+1}(a_{i+1})$ *and* $r'_{i+1}(a'_{i+1})$ *their bases, respectively, then* $q_{i+1}(s_{i+1}) \vdash_{i+1} q'_{i+1}(s'_{i+1})$ *iff* $q'_{i+1} \subseteq q_{i+1}$ *and for every* $a'_i \in r'_{i+1}(a'_{i+1})$ *there is an object* $a_i \in r_{i+1}(a_{i+1})$ *and one rule* $q_i \in q_{i+1}$ *such that* $a_i \xrightarrow{q_i} a'_i$.

P r o o f. Let $q_{i+1}(s_{i+1}) \vdash_{i+1} q'_{i+1}(s'_{i+1})$, then $q'_{i+1} \subseteq q_{i+1}$. If $a'_i \in r'_{i+1}(a'_{i+1})$, then $a'_i \in q'_{i+1}(s'_{i+1})$. Hence $a'_i \in q_{i+1}(s_{i+1})$. Thus, there exists an $a_i \in r_{i+1}(a_{i+1})$ and one rule $q_i \in q_{i+1}$ such that $a_i \xrightarrow{q_i} a'_i$ Conversely, if $s'_i \in q'_{i+1}(s'_{i+1})$, then there exists an $a'_i \in r'_{i+1}(a'_{i+1})$ and one rule $q'_i \in q'_{i+1}$ such that $a'_i \xrightarrow{q'_i} s'_i$. If the conditions of the proposition are fulfilled, then there are an $a_i \in r_{i+1}(a_{i+1})$ and one rule $q_i \in q_{i+1}$ such that $a_i \xrightarrow{q_i} a'_i$. Hence, $a'_i \in q_{i+1}(s_{i+1})$. Since $q'_i \in q_{i+1}$ and $a'_i \in q_{i+1}(s_{i+1})$, then because of antiresiduality we have $s'_i \in q_{i+i}(s_{i+1})$. Thus, $q_{i+1}(s_{i+1}) \vdash_{i+1} q'_{i+1} s'_{i+1})$.∎

Suppose that we have prominent filters in $p_{i+2}(t_{i+2})$, emboded by inserting a class of objects s_{i+1} into $p_{i+1}(t_{i+1})$. Since we have introduced the rules into $f_{i+2}(p_{i+1})$ then we can relate other filters to prominent ones to determine their *d*-limit objects. However, we shall not enter into this question here because it will be subject of our next paper devoted to the study of topological structures in $\mathcal{U}$. We shall only formulate here, without any comment, the following

D e f i n i t i o n 14. For a filter $f_{i+1} \in f_{i+2}(p_{i+1})$ we shall say to *converge* to an object s_i of s_{i+1} if $f_{i+1} \vdash_{i+1} \mathfrak{P}_{i+1}(s_i)$, where $\mathfrak{P}_{i+1}(s_i)$ is a prominent filter for s_i.

This convergence, in the general, is not unique. In the above mentioned paper we shall see under which conditions it will be unique.

The remainder of this paper we devote to the study of the class $f_{i+2}(p_{i+1})$, if we provide it with the class $\vdash_{i+2}$ of the rules $\vdash_{i+1}$, then we have the following

P r o p o s i t i o n 11. *The class* $f_{i+2}(p_{i+1})$ *together with the class* $\vdash_{i+2}$ *is an l-semigroupoid.*

P r o o f. According to the Proposition 5. it is enough to show that an arbitrary nonempty subclass $f'_{i+2}(p_{i+1})$ of $f_{i+2}(p_{i+1})$ has *fc* or *lcc* in $f_{i+2}(p_{i+1})$. Let $q_{i+1}(s_{i+1})$ be a class which is a subclass of every filter $q'_{i+1}(s'_{i+1}) \in f'_{i+2}(p_{i+1})$. We show first that $q_{i+1}(s_{i+1})$ is a filter. Let $r_{i+1}(a_{i+1})$ be a c_α-subclass of $q_{i+1}(s_{i+1})$, then it is also *a* c_α-subclass of every $q'_{i+1}(s'_{i+1}) \in f'_{i+2}(p_{i+1})$. Hence, if every $q'_{i+1}(s'_{i+1})$ is a c'_β-filter, where $c'_\beta > c_\alpha$, then a cocone over $r_{i+1}(a_{i+1})$ belongs to every $q'_{i+1}(s'_{i+1})$, and thus also to $\bigcap q'_{i+1}(s'_{i+1})$. Thus, if we define $q_{i+1}(s_{i+1})$ as $\bigcap q'_{i+1}(s'_{i+1})$ of all $q'_{i+1}(s'_{i+1}) \in f'_{i+2}(p_{i+1})$, then it is obvious that $q_{i+1}(s_{i+1})$ is a c_β-filter for $c_\beta = \min\{c'_\beta\}$. Hence, for every $q'_{i+1}(s'_{i+1}) \in f'_{i+2}(p_{i+1})$ we have $q'_{i+1}(s'_{i+1}) \vdash_{i+1} q_{i+1}(s_{i+1})$. The class of all these rules is *fc* over $f'_{i+2}(p_{i+1})$ with the vertex $q_{i+1}(s_{i+1})$.∎

In the next proposition we give a characterization of this *l*-semigroupoid.

P r o p o s i t i o n 12. *The l-semigroupoid* $f_{i+2}(p_{i+1})$ *is an atomic l-semigroupoid.*

P r o o f. Let f_{i+1} be a filter in $f_{i+2}(p_{i+1})$ such that $f_{i+1} \neq 0_{i+1}$. Let us consider the class of all object $g_{i+1} \in f_{i+2}(p_{i+1})$ through which the rule $\vdash_{i+1} : 0_{i+1} \to$

$\rightarrow f_{i+1}$ is reducible. This class we denote by $\mathbf{G}_{i+2}$. Certainly it has *lcc* in $f_{i+2}(p_{i+1})$. The covertex of *lcc* over it we denote by k_{i+1}. Since the rule $\vdash_{i+1}$ is also reducible through k_{i+1}, then k_{i+1} belongs to $\mathbf{G}_{i+2}$. The object k_{i+1} is an atom because the condition $k_{i+1}=0_{i+1}$ would lead to a contradiction, namely that $0_{i+1} \in \mathbf{G}_{i+2}$.∎

Atoms of the *l*-semigroupoid $f_{i+2}(p_{i+1})$ we shall call *ultrafilters.* The usefulness of these filters we shall see in the next paper.

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