PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 14 (28), 1972, pp. 39–48

ON CLASSES AND UNIVERSES

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(Communicated June 7, 1972)

1. Introduction.

In the mathematical development there is a permanent tendency of finding out an (axiomatized) framework within which one could develop all mathematics. The traditional idea is to do it within the (axiomatized) system of set theory. As it is well known there are many axiomatic approaches to this theory. However, in current use by mathematicians there are, in the main, two essentially different approaches, known as the ZFC and the NBG theories. Under the NBG theory we understand that one given in [6]. Both these theories are infinitely axiomatizable. However, the NBG theory does provide a foundation for mathematics which is free of the obvious paradoxes. In it, it is essentially true that any meaningful collection of mathematical objects exists and forms a class. Classes are subdivided into two types: those which are members and those which are not members of any class. The first are sets and the second ones are proper classes. This theory has given a good foundation for considerations of different mathematical objects and relations among them. Early in this century to these considerations had not been payed sufficient attention. Studies of isolated mathematical objects were predominant. However, since recently much more attention has been just payed to them. From such considerations, because of usefulness, in mathematics has arisen a new notion. That notion is a "category", which has developed into the separate theory. The basic characteristic of this theory is the consideration of a class of mathematical objects and relations among them without taking into considerations the nature of those objects. The development of category theory has posed problems of the set theoretic foundations of mathematics. For instance, the NBG foundation for category theory has not allowed the free formation of functor categories. In order to solve the noticed problems one had to drop the traditional idea of the foundation of mathematics and propose new ones. The formation of new foundations has always been done when in concrete cases one notices some disadvantages of existing foundations. Disadvantages pointed out by category theory caused dropping the NBG foundation for this theory. As new foundations different systems were proposed, obtained by a strengthening of the ZFC axioms. This was done, in the main, by requiring the existence of a large set (or many such sets [3]), called a "universe", such that each set is a member of it (or of a large set). Mac Lane [5] proposing his foundation started with the assumption that there is only one large set, elements of which are sets and classes are subsets of it. He proposed foundation: the ZFC axioms plus the axiom that there exists a universe. His universe is a transitive infinite set having certain closure properties. Feferman [2], using the so-called reflection principle, showed that category theory can be formulated set-theoretically without the axioms of universes. He proposed a system ZFC/s with a symbol s to which is applied a form of the reflection principle as one of its basic axion shemata. In such a system he considered the formulation of category theory.

Foundations mentioned in this paper, as well as others of the same type, not mentioned here, were sufficient for the formation of current category theory. However, a further and very quick development of this theory poses new problems for foundations. For instance, the mentioned foundations do not allow the formation of category of (all) categories. Because of that they also have to be dropped. Lawvere [4] tried to avoid the set theoretic foundation of mathematics and suggested it to be the category of all categories. He also proposed a system of axioms for such a foundation. Although one could put the question if such a foundation was sufficient and, also, if Lawvere's axioms were the right axioms for the purpose, one could freely say that his approach was very interesting.

In the present paper we tried to avoid noticed disadvantages of existing foundations and to propose an adequate foundation that will be our future framework. For this purpose we take the set theoretic approach to the problem, only that instead of one primitive notion we introduce the existence of a hierarchy of these notions. These notions we call "classes". If we denote a level in that hierarchy by i and all levels by $\mathcal{J}$ and introduce a letter c for the word "class" then we can write the hierarchy of classes as a family $\mathfrak{C}=\{c_i \mid i\in\mathcal{J}\}$. In such a way we distinguish among classes in the hierarchy by means of indices $i\in\mathcal{J}$. Then an entity c_i from $\mathfrak{C}$ denotes a class on the ith level in the hierarchy. The above family together with a family of bonding relations $\rho_{i(i+1)}$ between consecutive levels in $\mathfrak{C}$ we call a "spectar" and denote it by $\mathfrak{C}_\rho=\langle c_i;\ \rho_{i(i+1)}\rangle_{i\in\mathcal{J}}$. As bonding relations we have membership relations $\in_{i(i+1)}$. In that case the spectar is reduced to $\mathfrak{C}_\in$. Hence we have that a class on the level i is a member of a class on the level $i+1$. Moreover, we safeguard on each level $i\in\mathcal{J}$ the existence of an initial or limit class $\mathcal{U}_i$ and thus a family $\mathcal{U}=\{\mathcal{U}_i \mid i\in\mathcal{J}\}$ of these classes. Every class on a level i is a subclass of the limit class $\mathcal{U}_i$. These limit classes we call "universes". Investigating the properties of universes $\mathcal{U}_i$, $i\in\mathcal{J}$, we have obtained that they constitute an inductive spectar $\mathcal{U}_\prec=\langle\mathcal{U}_i;\ \prec_{i(i+1)}\rangle_{i\in\mathcal{J}}$ in which bonding relations are strict dominations $\prec_{i(i+1)}$. The inductive limit of the spectar $\mathcal{U}_\prec$ is the proper universe $\mathcal{U}$. It is also a proper class. Thus, according to the accepted approach the proper class exists as a limit class of all initial classes $\mathcal{U}_i$, $i\in\mathcal{J}$.

In order to handle a spectar of classes we have proposed an axiom system Σ_R. This system consists of a family of systems Σ_i, which are of NBG$^-$ type, where NBG$^-$ is the NBG system minus the axiom of choice AC and the axiom of regularity AR, and of a family of bonding rules $R_{i(i+1)}\colon\Sigma_i\to\Sigma_{i+1}$. Thus, the system Σ_R is a spectar $\langle\mathcal{U}_i;\ R_{i(i+1)}\rangle_{i\in\mathcal{J}}$. Let us see now what are models for such a system. To give a model for Σ_R we must have models for every Σ_i, $i\in\mathcal{J}$, and interpretations of bonding rules. A model for the system Σ_i is an interpretation in which are valid all axioms of Σ_i. Such a model we call an i-class. Interpretations for bonding rules are membership relations. If we consider a triple $(\Sigma_i,\ R_{i(i+1)},\ \Sigma_{i+1})$ then it is to be interpreted as follows: for every model t_i for Σ_i there is a model t_{i+1} for Σ_{i+1} and a membership relation $\in_{i(i+1)}$ such that $t_i \in_{i(i+1)} t_{i+1}$. Hence, a model for the system Σ_R is a spectar of classes $\mathfrak{C}_\in=\langle t_i;\ \in_{i(i+1)}\rangle_{i\in\mathcal{J}}$. In such a way we have a system by which we can handle spectars, respectively classes of different levels. It still remain the question of handling universes. The universe on a level, for instance i, satisfies all axioms of Σ_i except the power-class axiom. Namely we have $\mathcal{P}(\mathcal{U}_i)=\mathcal{U}_{i+1}$. Thus, the operation $\mathcal{P}$ transfers the i-universe $\mathcal{U}_i$ to the $(i+1)$-universe $\mathcal{U}_{i+1}$. Hence, the universe $\mathcal{U}_{i+1}$ strictly dominates $\mathcal{U}_i$ and thus, there is the spectar of universes which as bonding rules has strict dominations realized by means of $\mathcal{P}$. This spectar we have designated by $\mathcal{U}_{\prec}$. Between every two consecutive members in $\mathcal{U}_{\prec}$ does not exist any class. If we denote by GLH_i the statement that between $\mathcal{U}_i$ and $\mathcal{P}(\mathcal{U}_i)$ does not exist any class and by LH^i_{i+1} that $\mathcal{P}(\mathcal{U}_i)=\mathcal{U}_{i+1}$ then we have that both hypotheses hold in $\mathcal{U}_{\prec}$ and that $(\mathrm{LH}^i_{i+1}\ \&\ \mathrm{AC}_{i+1}) \Leftrightarrow \mathrm{GLH}_i$, where AC_{i+1} is the axiom of choice on the level $i+1$. This result is well known in the theory of transfinite cardinals-alephs. It is obvious that according to the properties our spectar corresponds to the sequence of initial ordinals. If initial ordinal numbers occur in their cardinal capacities then it corresponds to the sequence of alephs $\langle \aleph_i;\ <_{i(i+1)}\rangle_{i\in\mathcal{J}}$. The proper universe $\mathcal{U}$ corresponds to an inaccessible cardinal. Thus, if $\mathcal{U}_0$ is equipolent to ω then we can consider that the cardinality of the universe $\mathcal{U}_i$ is the aleph $\aleph_i$ and of the proper universe $\mathcal{U}$ an inaccessible cardinal.

The universe $\mathcal{U}$ is otherwise a model for the limit system of Σ_R. This system is the NBG system, even the NBG + GCH, where the GCH is only a statement concerned with the crossing from a universe level to higher one in $\mathcal{U}$. Thus, the NBG system is a limit system of the spectar Σ_R. It is obviously unique up to an equivalence.

Since we have presupposed the existence of classes of different levels we have to find a confirmation for such a presupposition. We find it in the Universum **U**. We shall consider **U** as a real foundation. Every collection of elements in it is contained in a larger collection. If we start from the basic undivisible elements known as protons, neutrons and electrons, then we come to the collections known as atoms. Atoms are elements of higher level collections known as molecules. These ones are further elements of real objects. Real objects generate Planets. Those are elements of the Solar system. This is further element of the Galaxy. Clearly, this process can be continued indefinitely. Thus, for every collection of natural elements in **U** there is a larger collection containing it. Hence, we have that the Universum **U** as a frame which contains all collections of natural elements is inductive.

As a conclusion we have that in a real foundation indeed there exist collections of objects of different levels. Thus, if we want to consider the organisation of **U**, we have to consider it on particular levels and among levels. To a more detailed consideration of **U** we shall dedicate an another paper.

2. Class axioms. Universes.

In the introduction we have confirmed the presupposition of an inductive process for introducing classes according to which each class is a member of a larger class. If we introduce a binary relation $\in$ to designate the belongness of a class to larger one, then we have a spectar $\mathfrak{C}_{\in}=\langle c_i;\ \in_{i(i+1)}\rangle_{i\in\mathcal{J}}$. Now we shall design an axiom system to handle spectars. We propose it to be an axiom spectar Σ_R consisting of axiom system Σ_i and bonding rules $R_{i(i+1)}$ i.e. $\Sigma_R=\langle\Sigma_i;\ R_{i(i+1)}\rangle_{i\in\mathcal{J}}$. In what follows we shall describe this system. We start its description with undefined notions. As undefined we choose the following notions:

1) Starting objects, and
2) T_i — unary predicates, and $\in_{i(i+1)}$ — binary relations, for every $i\in\mathcal{J}$.

We designate variables by small letters of Latin alphabet. If t is a variable then $T_i(t)$ means "t is an i-class". We now introduce relative unary predicates T_i^{i-k}, $k=0, 1, 2$ to designate $(i-2)$-, $(i-1)$-, and i-class in relation to an i-class. The statement $T_i^{i-2}(t)$ means "t is an $(i-2)$-class", $T_i^{i-1}(t)$ that "t is an $(i-1)$-class" and $T_i^i(t)$ that "t is i-class" all in relation to an i-class. The first class we call an i-atom, the second an i-set and the third one a proper i-class. We shall define the notions of proper i-class and i-set and deduce of i-atom. We define an i-set as an i-class which is an element of an i-class. Those i-classes which are not elements of i-classes are proper i-classes.

Definition.

$$T_i^{i-1}(t) =_{\text{Def}} \{T_i(t) \ \& \ (\exists s)(T_i(s) \ \& \ t \in_{ii} s)\},$$

$$T_i^i(t) =_{\text{Def}} \{T_i(t) \ \& \ (\forall s)(T_i(s) \Rightarrow t \notin_{ii} s)\}.$$

In such a way we distinguish on a level between classes which are elements and those which are not. If an i-class is an element of an i-class then it is an $(i-1)$--class, respectively an i-set. The proper i-class is not an element of any i-class. Hence, in any spectar $\langle t_i; \in_{i(i+1)} \rangle_{i \in \mathcal{J}}$ does not exist the identity relation $\in_{ii}$ for any $i \in \mathcal{J}$. However, if an $(i-1)$-class is the set in relation to an $(i-1)$-class and this one in relation to an i-class then it is an i-atom. Hence, we have $T_i^{i-1}(T_{i-1}^{i-2}(t)) = T_i^{i-2}(t)$. Thus, there is a composition of relations joining consecutive levels in a spectar, i.e. $\in_{i-1}^{i-2} \cdot \in_i^{i-1} = \in_i^{i-2}$.

Now we shall start with a description of undefined notions. That we shall do by means of axioms and axiom schemata. The first axiom is concerned with starting objects. In the process of introducing classes we start from a level. We take it to be 0. Clearly, a class on that level contains certain elements. These elements we call starting objects and denote their level by -1. They are atoms in relation to a class on the level 1, namely they are 1-atoms. The first axiom in our list of axioms will characterize these objects.

A1: Characterization of Starting Objects

(SO) $$T_i^{-1}(a) \Rightarrow (\neg T_i(a) \ \& \ (\forall t)(t \notin a)).$$

A starting object is not an i-class and has no elements. Thus, our starting objects are undivisible elements i.e. they are those objects which contain no elements but which are elements. We shall denote by $\mathcal{U}_0$ the class of all such objects. Otherwise, the level 0 is only a set level and not a proper class level. This one starts from 1.

By means of next axioms we shall describe the properties of the second type of undefined notions. Let us consider a level i and state the first axiom for this purpose.

A2: Axiom of Extensionality

$$T_i(t) \ \& \ T_i(t') \Rightarrow ((\forall s)(s \in_{(i-1)i} t \Leftrightarrow s \in_{(i-1)i} t') \Rightarrow t = t').$$

Hence, if two i-classes have the same elements they are identical, and conversely. Thus, an i-class is determined by its elements.

Using the binary relations $\in_{(i-1)i}$, we can define what it means for one i-class to be a subclass of another i-class. As a digression in formation of the list of axioms, what is our main purpose, we give the definition of the i-subclass.

Definition.

$$T_i(t) \ \& \ T_i(t') \Rightarrow (t \subseteq t' =_{\mathrm{Def}} (\forall s)\ (s \in_{(i-1)i} t \Rightarrow s \in_{(i-1)i} t')),$$

$$T_i(t) \ \& \ T_i(t') \Rightarrow (t \subset t' =_{\mathrm{Def}} (t \subseteq t \ \& \ t \neq t')).$$

Let us continue with the list of axioms. Now we give the essential axiom, namely a general rule for the existence of classes on the considered level i. According to it, given a property $\mathbf{F}_i$, there is a class whose elements are precisely those $(i-1)$-classes having property $\mathbf{F}_i$. Its formulation is as follows.

A3: If $\mathbf{F}_i$ as wff in which t is not a free variable, on the level i, then the following statement is an axiom

$$(\exists t)\ (T_i(t) \ \& \ (\forall t')\ (t' \in_{(i-1)i} t \Leftrightarrow \mathbf{F}_i)).$$

The formulated axiom is really an axiom schema, namely a rule for producing axioms. Together with Al it gives the existence of a unique i-class t. From A3 follows the existence of different kinds of i-classes. For us of particular interest will be the universal i-class. The universal i-class is such an i-class t that any $(i-1)$--class is its element. Thus, it is the class of all $(i-1)$-classes. Hence, every i-class is its subclass, that means that it is an initial or limit class on the level i. According to A2 it is unique. Such a class we call an i-universe. The above axiom also gives the existence of an i-class which has no $(i-1)$-class as its element. It is the empty i-class, and it exists for every i. Thus, the empty class is a class for every i. Because of that we can denote it by $\varnothing$ only, without designating the level. According to A2 it is also unique. We omit the consideration of other kinds of classes and proceed to complete the list of axioms.

By means of previous two axioms we ensured on a level i the existence of different classes and their uniqueness. Now we shall introduce some axioms to ensure certain useful properties of these classes. Before all we shall ensure the possibility for relating elements in them. That we do by postulating the existence of sets which have at most two elements.

A4: Pairing Axiom

$$(\exists t)\ (T_i^{i-1}(t) \ \& \ (\forall s)\ (s \in_{(i-1)(i-1)} t \Leftrightarrow (s = u \vee s = v)).$$

The above unique set t is called unordered pair. By means of it we can define rules for relating elements in a class and also ensure in it the existence of sets with some finite number of sets as their elements.

Now we shall safeguard some good properties of classes with respect to certain operations in them. Certainly we can do some operations on elements in a class. But we do not know what their levels will be after an operation. We would expect them to be sets. However, this does not follow from any of the axioms which have been stated previously, so if we want it to be true we have to postulate it. For this purpose we introduce two axioms. Before we state them we shall do some considerations concerning operations that we shall regard. Let us neglect for the moment the explicit referring to the class levels and consider which are these operations. All operations on a class t we shall devide into two types. To the first type belong operations of domination, namely those operations on a class the result of which dominates the class itself. If $\mathrm{Func}(F)$ & $F: t \xrightarrow{1-1} s$, then one says that s dominates t, or t is dominated by s. If besides $\neg F: t \xrightarrow[\text{onto}]{1-1} s$ then one says that the domination is strict. We shall designate by $t \leqslant_d s$ the statement that t is dominated by s. The strict domination will be designated by $<_d$. The instances of the above type of operations are the union $\cup$ and power-class operation $\mathcal{P}$. The operation $\cup$ is such an operation on a class t that $t \leqslant_d \cup(t)$, where $\cup(t)$ is the union of all classes in the class t being of the same level as t itself. However, for the power-class operation, $\mathcal{P}(t)$ strictly dominates t, i.e. $t <_d \mathcal{P}(t)$. The second type are operations of codomination, namely those operations on a class the result of which codominates the class itself. If $\mathrm{Func}(G)$ & $G: t \xrightarrow{\text{onto}} s$ then one says that s codominates t. The codomination we shall designate by ${}_c\geqslant$. The statement $t\,{}_c\geqslant s$ means that s codominates t. The instances of this type of operations are the image under a function F and hence the subclass of a class. Now we add to the existing list of axioms two basic types of axioms.

A5: Axiom of Domination

$$T_i^{i-1}(t) \Rightarrow T_i^{i-1}(\Delta(t)),$$

where Δ denotes operations of domination, namely those operations on the class t such that $t \leqslant_d \Delta(t)$ or $t <_d \Delta(t)$.

A6: Axiom of Codomination

$$T_i^{i-1}(t) \Rightarrow T_i^{i-1}(\nabla(t)),$$

where ∇ denotes operations of codomination, namely those operations on the class t such that $t\,{}_c\geqslant \nabla(t)$.

To the above types of axioms correspond different particular cases. The power-class and union axiom are of the type A5, and the axiom of replacement of the type A6. The above axioms assert that all reasonable operations applied to sets in a class give again sets. Then we say the class is closed with respect to operations on its elements. From the axioms A4—A6 one can further deduce another closure properties of a class. For instance a class is also closed with respect to direct products and exponentiations.

By means of proposed axioms A1—A6 we have described the properties of starting objects and have ensured on a level i the existence of unique classes and their closure properties. The system of axioms A2—A6 we shall designate by Σ_i, where i designates the class-level. Thus, on our list of axioms we have so far the axioms $A1+\Sigma_i$.

As we have said in the Introduction the system Σ_i is the NBG^- type system on the level i, namely the NBG system without AC and AR. If we exclude the axioms A5 and A6 from Σ_i then we obtain the system of axioms of the type theory given by Tarski and Church [1].

Let us consider now a class t on the level i denoted by t_i. By means of the system Σ_i we have ensured that the class t_i has certain closure properties with respect to the operations on its elements. However we can also do the same operations, entering the Σ_i, on the class t_i itself. The question arises how to ensure the closeness of t_i with respect to the operations on itself. If there exists a class s such that $t_i \in s$ and if s satisfies the same system of axioms as the class t_i itself, then s will have the same closure properties as t_i and thus t_i will be closed with respect to the operations on itself. Clearly, the class s must be of the level $i+1$. Thus, in order to ensure the closeness of a class t_i with respect to the operations on itself we must ensure the existence of a class of the $(i+1)$-level which will have the same properties as the class t_i and of which t_i will be an element. That we do by means of a new axiom. By means of it we shall ensure that, if Σ_i is the axiom system then the Σ_{i+1} is also the axiom system.

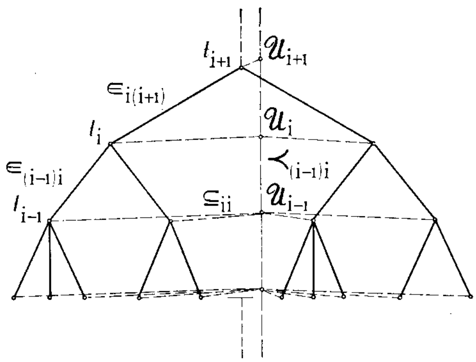
A7: Existence of Higher Level Classes

$$(\forall i)\ (\forall s)\ (i \in \mathcal{J}\ \&\ T_i(s)) \Rightarrow (\exists t)\ (T_i(t)\ \&\ s \in_{ii} t).$$

Hence it follows that $T_{i+1}(t)$. Thus, the system Σ_i has a lifting to the level $i+1$, i.e. Σ_{i+1} is also the axiom system. As a conclusion we have that Σ_i is the axiom system for every $i \in \mathcal{J}$. According to that we have a family $\Sigma = \{\Sigma_i \mid i \in \mathcal{J}\}$ of the axiom systems Σ_i, i.e. an axiom system with different levels. As every model for Σ_i is contained in a model for Σ_{i+1} then we can consider that in the family Σ is defined a family $\mathcal{R}$ of bonding rules $R_{i(i+1)}: \Sigma_i \to \Sigma_{i+1}$. These rules are to be interpreted as follows: For every model t_i for Σ_i there is a model t_{i+1} for Σ_{i+1} and the membership relation $\in_{i(i+1)}$ such that $t_i \in_{i(i+1)} t_{i+1}$. The family Σ together with the family $\mathcal{R}$ we call an axiom spectar. This spectar is obviously inductive. Its models are different spectars of classes. We denote it by Σ_R.

With the axiom A7 we have completed our list of axioms. Thus, our axiom system is $\mathrm{SO} + \Sigma_R$. By means of SO we have described the starting objects and by means of Σ_R have ensured, on every level $i \in \mathcal{J}$, the existence of different classes, their uniqueness and closeness with respect to certain operations on elements. As we have seen without the axioms SO, A5 and A6 our axiom system is to be reduced to the system of type theory, namely our theory is then the type theory given by Tarski and Church.

Now we shall give a sketch of a spectar $\mathfrak{C}_{\in} = \langle t_i : \in_{i(i+1)} \rangle_{i \in \mathcal{J}}$. Let us consider the level i and the class t_i. This class contains $(i-1)$-classes as its elements and is contained as an element in the class t_{i+1}. Moreover, each other i-class t_i having a property $\mathbf{F}_i$ as the class t_i is also an element of t_{i+1}. Hence, a part of $\mathfrak{C}_{\in}$ can be sketched as follows.



Thus, the spectar $\mathfrak{C}_{\in}$ is only a representative for a more complex situation in a class hierarchy.

As we have seen models for the axiom spectar Σ_R are spectars of classes. However, there is still one special spectar which is not a model for the Σ_R. This spectar is very important. Because of that we shall describe it. According to the axiom A2 we have that on every level $i \in \mathcal{J}$ there exists a universe $\mathcal{U}_i$ and thus a family $\mathcal{U} = \{\mathcal{U}_i \mid i \in \mathcal{J}\}$. As we have seen the universe $\mathcal{U}_i$ is an initial class on the level i, every i-class is its subclass. Hence $\mathcal{U}$ is a family of initial classes. In the sequel we shall investigate the properties of these classes, determine bonding rules between them and thus constitute a spectar. Let $\mathcal{U}_i$ and $\mathcal{U}_{i+1}$ be two consecutive elements in $\mathcal{U}$. We show first that there exists no class between them. The assumption that there exists an i-class will contradict the universality of $\mathcal{U}_i$. Namely if $\mathcal{U}'_i$ is between $\mathcal{U}_i$ and $\mathcal{U}_{i+1}$ then it will, in a manner, dominate $\mathcal{U}_i$. Hence, there will exist (i—1)-classes which are not in $\mathcal{U}_i$ contradicting the universality of $\mathcal{U}_i$. If between $\mathcal{U}_i$ and $\mathcal{U}_{i+1}$ there exists an (i+1)-class $\mathcal{U}'_{i+1}$ then $\mathcal{U}_{i+1}$ will dominate it. Hence, the universe $\mathcal{U}_{i+1}$ will contain i-classes which are out of $\mathcal{U}'_{i+1}$. This further implies that elements of these i-classes are out of $\mathcal{U}_i$ contradicting the universality of $\mathcal{U}_i$. Thus, it follows that between any two consecutive elements of $\mathcal{U}$ there exist no class. Starting from this truth we conclude that for all operations of domination except the strict one, i.e. $\mathcal{P}$ we have $\Delta(\mathcal{U}_i)=\mathcal{U}_i$ for every $i\in\mathcal{J}$. For the case of the strict domination $\mathcal{P}$ the only possibility is $\mathcal{P}(\mathcal{U}_i)=\mathcal{U}_{i+1}$. Thus, the operation $\mathcal{P}$ associates to each $\mathcal{U}_i\in\mathcal{U}$ the universe $\mathcal{U}_{i+1}$ and a relation $\prec_{i(i+1)}$ such that $\mathcal{U}_i\prec_{i(i+1)}\mathcal{U}_{i+1}$. For every $i\in\mathcal{J}$ we have that the universe $\mathcal{U}_i$ statisfies all axioms of the Σ_i except the power-class axiom. However, every subclass of $\mathcal{U}_i$ satisfies all axioms of the Σ_i. Hence, every model for Σ_i is a subclass of the universe $\mathcal{U}_i$, respectively an element of the universe $\mathcal{U}_{i+1}$. The family of universes $\mathcal{U}$ together with the family of strict dominations form a spectar $\mathcal{U}_\prec=\langle\mathcal{U}_i;\ \prec_{i(i+1)}\rangle_{i\in\mathcal{J}}$. We call this spectar initial or limit spectar. The first element in $\mathcal{U}_\prec$ is the universe $\mathcal{U}_0$ containing all starting objects. If the universe $\mathcal{U}_0$ is empty, namely if there is no starting object, then the spectar $\mathcal{U}_\prec$ is the sequence of natural numbers.

Compare now our spectar of universes $\mathcal{U}_\prec$ with the sequence of transfinite cardinals. In the theory of transfinite cardinals are well known the aleph hypothesis AH, which in our case we shall designate by LH^i_{i+1}, indicating its relation to the levels i and $i+1$, and the generalized continuum hypothesis GCH, that we shall designate by GLH_i, indicating its relation to the level i. The definition of these notions relating to our spectar of universes will be

Definition.

$$LH^i_{i+1}=_{\mathrm{Def}}(\forall i)\ (\mathcal{P}(\mathcal{U}_i)=\mathcal{U}_{i+1}),$$

$$GLH_i=_{\mathrm{Def}}(\forall\mathcal{U}_i)\ (\mathrm{Univ}(\mathcal{U}_i)\Rightarrow\neg(\exists\mathcal{V})(\mathcal{U}_i\prec\mathcal{V}\prec\mathcal{P}(\mathcal{U}_i))).$$

Obviously in our spectar of universes $\mathcal{U}_\prec$ these hypotheses hold, what follows from our preceding discussion. Now we prove some propositions concerning these hypotheses. They are well known for the case of transfinite cardinals.

Proposition. $GLH_i\Rightarrow LH^i_{i+1}$.

Proof. According to hypothesis, $\mathcal{P}(\mathcal{U}_i)$ is a unique immediate successor of $\mathcal{U}_i$. If there exists another successor $\mathcal{U}_{i+1}$ of $\mathcal{U}_i$, then $\mathcal{U}_{i+1}=\mathcal{P}(\mathcal{U}_i)$. ∎

Proposition. $GLH_i\Rightarrow AC_{i+1}$.

Proof. According to hypothesis we have that $\mathcal{U}_i$ and $\mathcal{P}(\mathcal{U}_i)$ form a pair of consecutive universes such that $g:\mathcal{U}_i\xrightarrow{1-1}\mathcal{P}(\mathcal{U}_i)$. Let $\mathcal{P}(\mathcal{U}_i)$ not contain the empty class and consider a surjection $f:\mathcal{P}(\mathcal{U}_i)\to\mathcal{U}_i$. Because of uniqueness of universes we have that g is a section of f namely that $f\circ g=1_{\mathcal{U}_i}$. ∎

Proposition. $(AC_{i+1}$ & $LH^i_{i+1}) \Leftrightarrow GLH_i$.

Proof. Let $\mathcal{U}_{i+1}$ be the $(i+1)$-universe. Because of AC_{i+1}, it is well ordered and contains the initial element. The initial element in it is the inductive limit taken over well-ordered carrier of all elements from $\mathcal{U}_{i+1}$. It is easy to see on the base of properties of universes that the inductive limit over all elements from $\mathcal{U}_{i+1}$ is the i-universe $\mathcal{U}_i$. From LH^i_{i+1} i.e. from $\mathcal{U}_{i+1} = \mathcal{P}(\mathcal{U}_i)$ we have GLH_i.▮

Thus, we have the following properties of the spectar $\mathcal{U}_{\prec}$. Between any two consecutive elements in $\mathcal{U}_{\prec}$, for instance $\mathcal{U}_i$ and $\mathcal{U}_{i+1}$, do not exist any class; $\mathcal{U}_i$ is the initial element in $\mathcal{U}_{i+1}$; then, for every $i \in \mathcal{J}$ the universe $\mathcal{U}_i$ is stable under all operations which enter Σ_i except the operation $\mathcal{P}$. We have for it $\mathcal{P}(\mathcal{U}_i) = \mathcal{U}_{i+1}$. Further, we have that for every $i \in \mathcal{J}$ the universe $\mathcal{U}_{i+1}$ is well-ordered. As $\mathcal{U}_{i+1}$ is well-ordered then it can be represented as a well-ordered inductive spectar $\langle t_i^\alpha; \leqslant_{ii}\rangle_{\alpha<\lambda_{i+1}}$ where $\leqslant_{ii}$ is a well-ordering on $\mathcal{U}_{i+1}$ and λ_{i+1} is an ordinal which corresponds to the universe $\mathcal{U}_{i+1}$. The inductive limit of this spectar is the universe $\mathcal{U}_i$. Thus $\mathcal{U}_i = \underset{\longrightarrow}{\text{Ind}} \langle t_i^\alpha; \leqslant_{ii}\rangle_{\alpha<\lambda_{i+1}}$. In that way the spectar is completely investigated.

As we have seen the universe $\mathcal{U}_i$, for every $i \in \mathcal{J}$, is not a model for Σ_i because of the power-class axiom. If we take the inductive limit of the spectar $\mathcal{U}_{\prec}$, i.e. its initial element, then we obtain a universe $\mathcal{U}$ which is also stable under the operation $\mathcal{P}$ in the sense that $\mathcal{U}_i \prec_i \mathcal{U} \Rightarrow \mathcal{P}(\mathcal{U}_i) \prec_{i+1} \mathcal{U}$. As a limit of $\mathcal{U}_{\prec}$ it is unique. For every $i \in \mathcal{J}$ it i-dominates the universe $\mathcal{U}_i$. Otherwise, each class t of any level $i \in \mathcal{J}$ is an element of $\mathcal{U}$, i.e. $(\forall t)(\forall i)(t_i \in_i \mathcal{U})$. Hence it follows that $\mathcal{U}$ is the proper universe. The system of which $\mathcal{U}$ is a model we shall call the limit system and denote it by $\Sigma_{\rightarrow}$. This system is unique. If we analyse it we shall see that it is the NBG system [6], even the NBG+GCH. The GCH here is only a statement concerned with the crossing from a level universe to the higher one in $\mathcal{U}$. Thus, the NBG system is to be obtained as a limit of an inductive spectar of NBG^- systems.

If under the proper class we understand a class which cannot be an element of any other class then with respect to the hierarchy $\mathcal{J}$ we have the following:

Proposition. *$\mathcal{U}$ is the proper class.*▮

As we have said $\mathcal{U}$ is also the proper universe. We shall call it, in future, simply the universe. If axioms A 1, A 5 and A 6 are excluded then we shall call it the preuniverse. The universe $\mathcal{U}$ is sufficient for our future intentions. It will be our future framework. In $\mathcal{U}$ one can speak of all objects on a level and then of all objects on all levels. That is a quite natural matter. In several forthcoming papers we shall show the adequacy of such an assumption. For that purpose we shall consider some fundamental structures in $\mathcal{U}$, namely structures on a relevant domain in $\mathcal{U}$. This domain will be an i-class t_i of some $(i-1)$th mathematical objects. Structures on a class in $\mathcal{U}$ we shall call horizontal structures in $\mathcal{U}$. For instance an i-category of (i—1)th mathematical objects is a horizontal structure in $\mathcal{U}$ that we shall call a fundamental semigroupoid. There are also in $\mathcal{U}$ structures among mathematical objects on consecutive levels. Such structures we shall call vertical structures in $\mathcal{U}$. Topological spaces are instances of such structures.

At the end of this paper we say that a formalisation of these investigations will be our final step.

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