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CHROMATIC NUMBER AND THE SPECTRUM OF A GRAPH

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We shall consider only finite, undirected graphs without loops or multiple edges.

The spectrum of a graph is defined as the spectrum of the adjacency matrix of the graph. Similarly, the characteristic polynomial of the adjacency matrix is called the characteristic polynomial of the graph. The spectrum contains the eigenvalues of the mentioned matrix so that each eigenvalue appears in the spectrum as many times as is its multiplicity. The spectrum of an undirected graph contains only real numbers. The greatest eigenvalue is called the index of the graph. If r denotes the index of a graph and the spectrum contains the eigenvalues $\lambda_1, \dots, \lambda_n$, then the relations $-r \leqslant \lambda_i \leqslant r$ $(i=1, \dots n)$ hold. For the graphs without loops we have $\lambda_1 + \cdots + \lambda_n = 0$.

In the last few years several papers, dealing with the connection between the chromatic number and the spectrum of a graph (for example [3], [10], [11], [13], [6], [7], [8]) have been published.

Mainly, some inequalities for the chromatic number of a graph on the basis of the spectrum have been obtained in the mentioned papers. In special cases the chromatic number can accurately be determined by means of the spectrum (for example, for bichromatic graphs [10] and for regular graphs of degree $n-3$, where n is the number of vertices [3]). The mentioned inequalities are not, in general, sufficiently sharp, but there are always some graphs for which they yield good estimations of the chromatic number. Different inequalities give good estimations for different classes of graphs. Therefore, all known estimations should be applied to the given graph, and then the best one should be chosen.

It seems possible to find much more inequalities for the chromatic number of a graph, based on the spectrum. We quote a simple possibility.

It is noticed in [2] that for the number of interior stability $\alpha(G)$ of the graph G the inequality

$$\alpha(G) \leqslant p_0 + \min(p_+, p_-)$$

holds, where p_-, p_0, p_+ represent respectively the number (having in view multiplicities) of negative, equal to 0 and positive eigenvalues in the spectrum of the graph G. If G has n vertices and the chromatic number $\gamma(G)$, then, on the basis of the well known inequality $\alpha(G)\gamma(G) \geqslant n$, we have

$$\gamma(G) \geqslant \frac{n}{p_0 + \min(p_+, p_-)}. \tag{1}$$

This inequality is sharp, for example, for complete graphs.

It is surprising that, on the basis of the spectrum some information on the chromatic number (a quantity which in general case cannot easily be determined) can be obtained. In connection with, this the following question can be raised: Can the chromatic number be accurately determined by means of the spectrum? Unfortunately, the answer is negative as we can see in the example from Fig. 1.

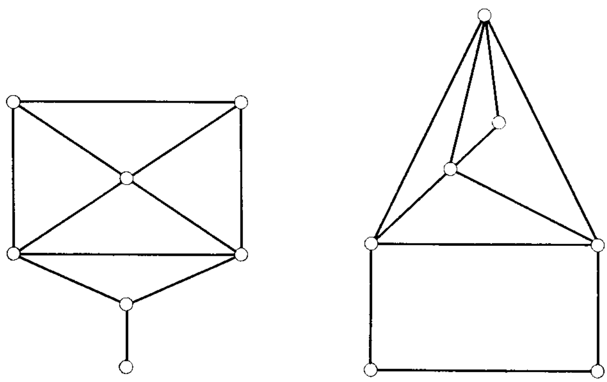

Fig. 1

Graphs from Fig. 1 have the same spectrum (according to [5], their common characteristic polynomial reads: $\lambda^7 - 11\lambda^5 - 10\lambda^4 + 16\lambda^3 + 16\lambda^2$) but different chromatic numbers (chromatic number of the first graph is equal to 3 and that of the second to 4).

The index of a graph, i.e. the greatest eigenvalue in the spectrum, is, of course, a very important quantity, which appears in investigation of several properties of graphs. The chromatic number of a graph can also be estimated on the basis of the index of the graph. H. S. Wilf [13] has proved that for the connected graph G with index r and chromatic number $\gamma(G)$ the inequality

$$\gamma(G) \leqslant r + 1$$

holds, where equality holds if and only if G is a complete graph or a cycle of an odd length.

On the basis of the index, a lower bound for the chromatic number can be found, too. It is expressed in the following theorem, which, as a matter of fact, represents the main result of this paper.

Theorem. *If G is a graph with n vertices, whose index is equal to r and chromatic number equal to $\gamma(G)$, then the following inequality holds*

$$\gamma(G) \geqslant \frac{n}{n-r}. \tag{3}$$

P r o o f. In the proof we shall use the concept of the k-complete graph. A graph is said to be k-complete if the set of its vertices can be partitioned into k nonempty subsets, so that no pair of vertices from the same subset is joined by an edge and that each pair of vertices from different subsets is joined by an egde. If the mentioned subsets contain $n_1, \ldots, n_k$ vertices respectively, then the k-complete gıaph is denoted by $K_{n_1,\ldots,n_k}$.

The characteristic polynomial of the k-complete graph was determined in [2] and [5]*). It can be represented in the following two forms:

$$p_{K_{n_1,\ldots,n_k}}(\lambda)=\lambda^{n-k}\left(1-\sum_{i=1}^{k}\frac{n_i}{\lambda+n_i}\right)\prod_{i=1}^{k}(\lambda+n_i), \tag{4}$$

$$p_{K_{n_1,\ldots,n_k}}(\lambda)=\sum_{i=0}^{k}(1-i)\,S_i\lambda^{n-i}, \tag{5}$$

where S_i $(i=1, \ldots, k)$ is the elementary symmetric function of order i of variables $n_1, \ldots, n_k$, $S_0=1$ and $n_1+\cdots+n_k=n$. Equality of (4) and (5) can easily be proved.

The polynomial (4) has only one positive root [12]. It is necessary to find the lower bound of this root for the given n and k.

It is natural to conjecture that the maximum value of the root is achieved when all n_i's are mutually equal and, if n is not divisible by k, when n_i's are mutually as close as possible. Assume, for a moment, that n_i need not be integers and put $n_1=n_2=\cdots=n_k=\frac{n}{k}$ in (4). We obtain then $\lambda=\frac{k-1}{k}n$ for the positive root of (4). So we suppose that for the given n and k the positive root λ of the polynomial (4), i.e. (5), satisfies the inequality $\lambda\leqslant\frac{k-1}{k}n$.

We prove this conjecture by proving a lemma. Let $\alpha_1, \ldots, \alpha_k$ $(k\geqslant 2)$ be positive numbers satisfying relation $\alpha_1+\cdots+\alpha_k=\alpha$, where α is a given (positive) number. Let S_i be the elementary symmetric function of the order i of variables $\alpha_1, \ldots, \alpha_k$.

L e m m a. *The positive root* λ *of the equation*

$$\lambda^k=S_2\lambda^{k-2}+2\,S_3\lambda^{k-3}+\cdots+(k-1)\,S_k \tag{6}$$

satisfies the inequality

$$\lambda\leqslant\frac{k-1}{k}\alpha. \tag{7}$$

P r o o f of the Lemma. We shall use a result from [9], p. 223, which reads:

Let $a_1, \ldots, a_k$ be real nonnegative numbers that $\sum_{i=1}^{k} a_i>0$. Let λ be a positive root of the equation

$$\lambda^k=a_1\lambda^{k-1}+a_2\lambda^{k-2}+\cdots+a_k$$

and let $t_1, \ldots, t_{k-1}$ be arbitrary positive numbers. Then

$$\lambda\leqslant\max\left(t_1, \ldots, t_{k-1},\ a_1+\frac{a_2}{t_1}+\cdots+\frac{a_k}{t_{k-1}^{k-1}}\right). \tag{8}$$

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*) In [5], in formula (4), λ^{n-t} stands instead of λ^{n-r}.

Applying this result to (6) we shall put $t_1=t_2=\cdots=t_{k-1}=\frac{k-1}{k}\alpha$. Lemma will be proved if we prove the inequality

$$a\overset{\text{def}}{=}a_1+\frac{a_2}{t_1}+\cdots+\frac{a_k}{t_{k-1}^{k-1}}\leqslant\frac{k-1}{k}\alpha.$$

We have

$$a=\sum_{i=1}^{k}\frac{(i-1)S_i}{\left(\frac{k-1}{k}\alpha\right)^{i-1}}=\frac{k-1}{k}\alpha\sum_{i=1}^{k}(i-1)\frac{S_i k^i}{(k-1)^i\alpha^i}.$$

Introduce new variables α_j' by $\alpha_j'=\frac{\alpha_j k}{\alpha(k-1)}$ $(j=1,\ldots,k)$. Let S_i' $(i=1,\ldots,k)$ be elementary symmetric functions of quantities α_j'. Then we have

$$a=\frac{k-1}{k}\alpha\sum_{i=1}^{k}(i-1)S_i'=\frac{k-1}{k}\alpha\left[1-\left(1-\sum_{i=1}^{k}\frac{\alpha_i'}{1+\alpha_i'}\right)\prod_{i=1}^{k}(1+\alpha_i')\right].$$

The expression in square brackets is, naturally, positive but we must prove that it is not greater than 1. For this it is sufficient to prove that the expression $b\overset{\text{def}}{=}1-\sum_{i=1}^{k}\frac{\alpha_i'}{1+\alpha_i'}$, naturally under condition $\alpha_i'+\cdots+\alpha_k'=\frac{k}{k-1}$, is nonnegative. Introducing quantities $x_i=1+\alpha_i'$ $(i=1,\ldots,k)$ we have

$$b=\sum_{i=1}^{k}\frac{1}{x_i}-k+1\quad\left(x_1+\cdots+x_k=\frac{k^2}{k-1}\right).$$

Since on the basis of the relation between harmonic and arithmetic mean we have

$$\left(\frac{1}{k}\sum_{i=1}^{k}\frac{1}{x_i}\right)^{-1}\leqslant\frac{1}{k}\sum_{i=1}^{k}x_i=\frac{k}{k-1},$$

i. e.

$$\sum_{i=1}^{k}\frac{1}{x_i}\geqslant k-1,$$

we obtain $b\geqslant 0$.

This completes the proof of the Lemma.

Thus, the index of $K_{n_1,\ldots,n_k}$, where $n_1+\cdots+n_k=n$, is not greater than $\frac{k-1}{k}n$.

In Appendix the spectra of some k-complete graphs are given.

We continue with the proof of the Theorem.

If $\gamma(G)=k$, the set of vertices of G can be partitioned into k non-empty subsets, so that the subgraph induced by any of these subsets contains no edges. If the mentioned subsets contain $n_1,\ldots,n_k$ $(n_1+\cdots+n_k=n)$ vertices respectively, then by adding new edges to G we can obtain $K_{n_1,\ldots,n_k}$. It is known (see, for example, [1], that the index of a graph does not decrease when adding new edges to the graph. Therefore, the index of G is not greater than the index of $K_{n_1,\ldots,n_k}$. According to this and foregoing, we can write $r \leqslant \frac{k-1}{k} n$, which implies $k \geqslant \frac{n}{n-r}$.

This completes the proof of the Theorem.

We shall now compare this result with results known in literature.

In [4] the inequality $\gamma(G) \geqslant \frac{n^2}{n^2-2m}$ was given, where n is the number of vertices and m the number of edges of the graph G. Since $\frac{2m}{n} \leqslant r$ ([1]) and since the above inequality can be written in the form $\gamma(G) \geqslant \frac{n}{n-\frac{2m}{n}}$, we see that (3) is an improvement of this inequality.

In [6] the following lower bound for the chromatic number was given:

$$\gamma(G) \geqslant \frac{r}{|q|} + 1, \tag{9}$$

where q denotes the lowest eigenvalue from the spectrum of G. In order to compare (3) with (9), write (3) in the from $\gamma(G) \geqslant 1 + \frac{r}{n-r}$. We see that (3) is better than (9), if $n-r < |q|$. Such graphs exist (for example, the graph obtained from the complete graph with 10 vertices by deleting an edge). Of course, there are also graphs for which (9) is better than (3).

APPENDIX*

In the following table the characteristic polynomials and spectra of $K_{n_1,\ldots,n_k}$ $n_1+\cdots+n_k=n$) for $k=1,\ldots,10$ and $n=k, k+1, \ldots, 10$ are given. The characteristic polynomial (4) or (5) can be written in the form

$$P_{K_{n_1,\ldots,n_k}}(\lambda) = \lambda^{n-k}\left(\lambda^k - \sum_{i=2}^{k} b_i \lambda^{k-i}\right).$$

In the table before every group of graphs, for which k and n are constant, the values of k and n are indicated. A graph is determined by the partition of n into

* The table of this Appendix has been computed by Slobodan K. Simić.

$n_1, \ldots, n_k$ (for example, $3^2\,1^3$ denotes the partition 3, 3, 1, 1, 1). After this, the coefficients b_i $(i=2, \ldots, k)$ and finally the approximative values of roots of

$$\lambda^k - \sum_{i=2}^{k} b_i \lambda^{k-i} = 0$$

in the decreasing order are given.

TABLE

$k=2$	$n=2$		
1^2		1	
		1	−1
$k=2$	$n=3$		
2 1		2	
		1,41	−1,41
$k=2$	$n=4$		
3 1		3	
		1,73	−1,73
2^2		4	
		2	−2
$k=2$	$n=5$		
4 1		4	
		2	−2
3 2		6	
		2,45	−2,45
$k=2$	$n=6$		
5 1		5	
		2,24	−2,24
4 2		8	
		2,83	−2.83
3^2		9	
		3	−3

$k=2$	$n=7$		
6 1		6	
		2,45	−2,45
5 2		10	
		3,16	−3,16
4 3		12	
		3,46	−3,46
$k=2$	$n=8$		
7 1		7	
		2,65	−2,65
6 2		12	
		3,46	−3,46
5 3		15	
		3,87	−3,87
4^2		16	
		4	−4
$k=2$	$n=9$		
8 1		8	
		2,83	−2,83
7 2		14	
		3,74	−3,74
6 3		18	
		4,24	−4,24
5 4		20	
		4,47	−4,47
$k=2$	$n=10$		
9 1		9	
		3	−3
8 2		16	
		4	−4
7 3		21	
		4,58	−4,58
6 4		24	
		4,90	−4,90
5^2		25	
		5	−5

$k=3$	$n=3$			
1^3		3	2	
		2	−1	−1
$k=3$	$n=4$			
$2\ 1^2$		5	4	
		2,56	−1	−1,56
$k=3$	$n=5$			
$3\ 1^2$		7	6	
		3	−1	−2
$2^2\ 1$		8	8	
		3,24	−1,24	−2
$k=3$	$n=6$			
$4\ 1^2$		9	8	
		3,37	−1	−2,37
3 2 1		11	12	
		3,77	−1,28	−2,48
2^3		12	16	
		4	−2	−2
$k=3$	$n=7$			
$5\ 1^2$		11	10	
		3,70	−1	−2,70
4 2 1		14	16	
		4,22	−1,30	−2,92
$3^2\ 1$		15	18	
		4,37	−1,37	−3
$3\ 2^2$		16	24	
		4,61	−2	−2,61
$k=3$	$n=8$			
$6\ 1^2$		13	12	
		4	−1	−3
5 2 1		17	20	
		4,62	−1,31	−3,31
4 3 1		19	24	
		4,89	−1,41	−3,48
$4\ 2^2$		20	32	
		5,12	−2	−3,12
$3^2\ 2$		21	36	
		5,27	−2,27	−3

$k=3$	$n=9$			
7 1^2		15	14	
		4,27	—1	—3,27
6 2 1		20	24	
		4,98	—1,31	—3,66
5 3 1		23	30	
		5,35	—1,43	—3,92
5 2^2		24	40	
		5,58	—2	—3,58
4^2 1		24	32	
		5,46	—1,46	—4
4 3 2		26	48	
		5,85	—2,34	—3,51
3^3		27	54	
		6	—3	—3

$k=3$	$n=10$			
8 1^2		17	16	
		4,53	—2	—3,53
7 2 1		23	28	
		5,32	—1,32	—4
6 3 1		27	36	
		5,77	—1,45	—4,32
6 2^2		28	48	
		6	—2	—4
5 4 1		29	40	
		5,97	—1,49	—4,48
5 3 2		31	60	
		6,36	—2,36	—4
4^2 2		32	64	
		6,47	—2,47	—4
4 3^2		33	72	
		6,62	—3	—3,62

$k=4$	$n=4$				
1^4		6	8	3	
		3	—1	—1	—1

$k=4$	$n=5$				
2 1^3		9	14	6	
		3,65	—1	—1	—1,65

$k=4$ $n=6$				
$3\ 1^3$	12	20	9	
	4,16	−1	−1	−2,16
$2^2\ 1^2$	13	24	12	
	4,37	−1	−1,37	−2
$k=4$ $n=7$				
$4\ 1^3$	15	26	12	
	4,61	−1	−1	−2,61
$3\ 2\ 1^2$	17	34	18	
	4,96	−1	−1,44	−2,52
$2^3\ 1$	18	40	24	
	5,16	−1,16	−2	−2
$k=4$ $n=8$				
$5\ 1^3$	18	32	15	
	5	−1	−1	−3
$4\ 2\ 1^2$	21	44	24	
	5,46	−1	−1,46	−3
$3^2\ 1^2$	22	48	27	
	5,61	−1	−1,61	−3
$3\ 2^2\ 1$	23	56	36	
	5,81	−1,18	−2	−2,63
2^4	24	64	48	
	6	−2	−2	−2
$k=4$ $n=9$				
$6\ 1^3$	21	38	18	
	5,36	−1	−1	−3,36
$5\ 2\ 1^2$	25	54	30	
	5,91	−1	−1,47	−3,44
$4\ 3\ 1^2$	27	62	36	
	6,16	−1	−1,67	−3,49
$4\ 2^2\ 1$	28	72	48	
	6,36	−1,19	−2	−3,18
$3^2\ 2\ 1$	29	78	54	
	6,50	−1,21	−2,30	−3
$3\ 2^3$	30	88	72	
	6,69	−2	−2	−2,69

$k=4$ $n=10$					
$7\ 1^3$	24	44	21		
	5,69	−1	−1	−3,69	
$6\ 2\ 1^2$	29	64	36		
	6,33	−1	−1,48	−3,85	
$5\ 3\ 1^2$	32	76	45		
	6,66	−1	−1,71	−3,96	
$5\ 2^2\ 1$	33	88	60		
	6,86	−1,19	−2	−3,66	
$4^2\ 1^2$	33	80	48		
	6,77	−1	−1,77	−4	
43 2 1	35	100	72		
	7,11	−1,21	−2,37	−3,53	
$4\ 2^3$	36	112	96		
	7,29	−2	−2	−3,29	
$3^3\ 1$	36	108	81		
	7,24	−1,24	−3	−3	
$3^2\ 2^2$	37	120	108		
	7,42	−2	−2,42	−3	
$k=5$ $n=5$					
1^5	10	20	15	4	
	4	−1	−1	−1	−1
$k=5$ $n=6$					
$2\ 1^4$	14	32	27	8	
	4,70	−1	−1	−1	−1,70
$k=5$ $n=7$					
$3\ 1^4$	18	44	39	12	
	5,27	−1	−1	−1	−2,27
$2^2\ 1^3$	19	50	48	16	
	5,46	−1	−1	−1,46	−2
$k=5$ $n=8$					
$4\ 1^4$	22	56	51	16	
	5,77	−1	−1	−1	−2,77
$3\ 2\ 1^3$	24	68	69	24	
	6,09	−1	−1	−1,55	−2,55
$2^3\ 1^2$	25	76	84	32	
	6,27	−1	−1,27	−2	−2

$k=5$ $n=9$						
$5\ 1^4$	26	68	63	20		
	6,22	−1	−1	−1	−3,22	
$4\ 2\ 1^3$	29	86	90	32		
	6,64	−1	−1	−1,57	−3,07	
$3^2\ 1^3$	30	92	99	36		
	6,77	−1	−1	−1,77	−3	
$3\ 2^2\ 1^2$	31	102	120	48		
	6,95	−1	−1,31	−2	−2,64	
$2^4\ 1$	32	112	144	64		
	7,12	−1,12	−2	−2	−2	
$k=5$ $n=10$						
$6\ 1^4$	30	80	75	24		
	6,62	−1	−1	−1	−3,62	
$5\ 2\ 1^3$	34	104	111	40		
	7,13	−1	−1	−1,58	−3,56	
$4\ 3\ 1^3$	36	116	129	48		
	7,37	−1	−1	−1,86	−3,51	
$4\ 2^2\ 1^2$	37	128	156	64		
	7,54	−1	−1,32	−2	−3,23	
$3^2\ 2\ 1^2$	38	136	171	72		
	7,67	−1	−1,35	−2,32	−3	
$3\ 2^3\ 1$	39	148	204	96		
	7,84	−1,13	−2	−2	−2,70	
2^5	40	160	240	128		
	8	−2	−2	−2	−2	
$k=6$ $n=6$						
1^6	15	40	45	24	5	
	5	−1	−1	−1	−1	−1
$k=6$ $n=7$						
$2\ 1^5$	20	60	75	44	10	
	5,74	−1	−1	−1	−1	−1,74
$k=6$ $n=8$						
$3\ 1^5$	25	80	105	64	15	
	6,36	−1	−1	−1	−1	−2,36
$2^2\ 1^4$	26	88	123	80	20	
	6,53	−1	−1	−1	−1,53	−2

$k=6$ $n=9$							
4 1^5	30	100	135	84	20		
	6,90	−1	−1	−1	−1	−2,90	
3 2 1^4	32	116	171	116	30		
	7,19	−1	−1	−1	−1,62	−2,57	
2^3 1^3	33	126	198	144	40		
	7,36	−1	−1	−1,36	−2	−2	
$k=6$ $n=10$							
5 1^5	35	120	165	104	25		
	7,39	−1	−1	−1	−1	−3,39	
4 2 1^4	38	144	219	152	40		
	7,78	−1	−1	−1	−1,64	−3,14	
3^2 1^4	39	152	237	168	45		
	7,90	−1	−1	−1	−1,90	−3	
3 2^2 1^3	40	164	273	208	60		
	8,06	−1	−1	−1,40	−2	−2,66	
2^4 1^2	41	176	312	256	80		
	8,22	−1	−1,22	−2	−2	−2	
$k=7$ $n=7$							
1^7	21	70	105	84	35	6	
	6	−1	−1	−1	−1	−1	−1
$k=7$ $n=8$							
2 1^6	27	100	165	144	65	12	
	6,77	−1	−1	−1	−1	−1	−1,77
$k=7$ $n=9$							
3 1^6	33	130	225	204	95	18	
	7,42	−1	−1	−1	−1	−1	−2,42
2^2 1^5	34	140	225	244	120	24	
	7,58	−1	−1	−1	−1	−1,58	−2
$k=7$ $n=10$							
4 1^6	39	160	285	264	125	24	
	8	−1	−1	−1	−1	−1	−3
3 2 1^5	41	180	345	344	175	36	
	8,27	−1	−1	−1	−1	−1,67	−2,60
2^3 1^4	42	192	387	408	220	48	
	8,42	−1	−1	−1	−1,42	−2	−2

$k=8$ $n=8$										
1^8	28	112	210	224	140	48	7			
	7	−1	−1	−1	−1	−1	−1	−1		
$k=8$ $n=9$										
$2\ 1^7$	35	154	315	364	245	90	14			
	7,80	−1	−1	−1	−1	−1	−1	−1,80		
$k=8$ $n=10$										
$3\ 1^7$	42	196	420	504	350	132	21			
	8,48	−1	−1	−1	−1	−1	−1	−2,48		
$2^2\ 1^6$	43	208	465	585	425	168	28			
	8,62	−1	−1	−1	−1	−1	−1,62	−2		
$k=9$ $n=9$										
1^9	36	168	378	504	420	216	63	8		
	8	−1	−1	−1	−1	−1	−1	−1	−1	
$k=9$ $n=10$										
$2\ 1^8$	44	224	546	784	700	384	119	16		
	8,82	−1	−1	−1	−1	−1	−1	−1	−1,82	
$k=10$ $n=10$										
1^{10}	45	240	630	1008	1050	720	315	80	9	
	9	−1	−1	−1	−1	−1	−1	−1	−1	−1

REFERENCES

[1] L. Collatz, U. Sinogowitz, *Spektren endlicher Graphen*, Abh. Math. Sem. Univ. Hamburg 21 (1957), 63—77.

[2] D. M. Cvetković, *Graphs and their spectra*, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 354—No. 356 (1971), 1—50.

[3] H. J. Finck, *Vollständiges Produkt, chromatische Zahl und charakteristisches Polynom regulärer Graphen II*, Wiss. Z. Techn. Hochsch. Ilmenau 11 (1969), 81—87.

[4] D. P. Geller, E. F. Schmeichel, *Chromatic number of a simple graph*, Problem 5713, Amer. Math. Monthly 78 (1971), 205—206.

[5] F. Harary, C. King, A. Mowshowitz, R. C. Read, *Cospectral graphs and digraphs*, Bull. London Math. Soc. 3 (1971), 321—328.

[6] A. J. Hoffman, *On eigenvalues and colouring of graphs I*, Graph theory and its applications (edited by B. Harris), New York—London 1970, 79—91.

[7] A. J. Hoffman, L. Howes, *On eigenvalues and colouring of graphs II*, Ann. New York, Acad. Sci. 175 (1970), No. 1, 238—242.

[8] L. L. Kraus, D. M. Cvetković, *Evaluation of a lower bound for the chromatic number of the complete product of graphs*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357 — No. 380, 63—68.

[9] D. S. Mitrinović, *Analytic Inequalities*, New York—Heidelberg—Berlin 1970.

[10] H. Sachs, *Über Teiler, Faktoren und characteristische Polynome von Graphen I*, Wiss. Z. Techn. Hochsch. Ilmenau 12 (1966), 7—12.

[11] H. Sachs, *Über Teiler, Faktoren und characteristische Polynome von Graphen II*, Ibid. 13 (1967) 405—412.

[12] J. H. Smith, *Some properties of the spectrum of a graph*, Combinatorial structures and their applications, New York—London—Paris 1970, 403—406.

[13] H. S. Wilf, *The eigenvalues of a graph and its chromatic number*, J. London Math. Soc. 42 (1967), 330—332.