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SOME REMARKS ON CHOICE TOPOLOGY

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The purpose of this paper is to give some remarks as well as some improvements on the choice topology introduced by Dacić [1].

Let $\mathcal{D}=\{D_\alpha \mid \alpha \in A\}$ be a decomposition of a topological space X, i. e. $\mathcal{D}$ is a collection of non-empty disjoint subsets of X whose union is X. Any mapping $\varphi : \mathcal{D} \to X$ such that $\varphi(D_\alpha) \in D_\alpha$ for every $\alpha \in A$ is called a choice function. If Z is the set of all choice functions for a given decomposition $\mathcal{D}$, then the choice topology τ_Z on $\mathcal{D}$ is defined to be the coarsest topology on $\mathcal{D}$ for which all choice functions are continuous.

Recall [1] that a decomposition $\mathcal{D}$ is called I-non-void iff for each $D_\alpha \in \mathcal{D}$ there exists some $x \in D_\alpha$ and a neighborhood V of x such that $D_\beta \setminus V \neq \varnothing$ for all $\beta \neq \alpha$.

It was shown in [1; Theorem 1.2] that τ_Z is discrete iff the decomposition $\mathcal{D}$ is I-non-void. However, there is a gap in the proof that if τ_Z is discrete then $\mathcal{D}$ is I-non-void. In fact, this statement is false in general. An example of a decomposition $\mathcal{D}$ which is not I-non-void but τ_Z is discrete can be easily constructed. For example, let $X=\{1, 2, 3, 4\}$ with $\{1, 4\}$ and $\{2, 3\}$ as the only proper open sets of X. It is easy to check directly that the decomposition $\mathcal{D}=\{\{1, 2\}, \{3\}, \{4\}\}$ of X is not I-non-void but $(\mathcal{D}, \tau_Z)$ is a discrete space. Note that the space X here is not Hausdorff. Such examples for Hausdorff spaces can also be constructed without difficulty. (See example following Proposition 4)

P r o p o s i t i o n 1. *Let X be a Hausdorff space and $\mathcal{D}=\{D_\alpha \mid \alpha \in A\}$ a decomposition of X and $\alpha_0 \in A$. If D_{α_0} has more than one point, then $\{D_{\alpha_0}\} \in \tau_Z$.*

P r o o f. Let $p, q \in D_{\alpha_0}$, $p \neq q$. Since X is Hausdorff, there exist disjoint open sets U and V of X containing p and q respectively. Let $\varphi_1 : \mathcal{D} \to X$ be any choice function such that $\varphi_1(D_\alpha) \in D_\alpha \cap U$ whenever $D_\alpha \cap U \neq \varnothing$. Define a choice function $\varphi_2 : \mathcal{D} \to X$ by putting $\varphi_2(D_{\alpha_0}) = q$ and $\varphi_2(D_\alpha) = \varphi_1(D_\alpha)$ if $\alpha \neq \alpha_0$. Observing that $q \in D_{\alpha_0} \cap V$ we have $\{D_{\alpha_0}\} = \varphi_1^{-1}(U) \cap \varphi_2^{-1}(V)$. Hence $\{D_{\alpha_0}\} \in \tau_Z$.

P r o p o s i t i o n 2. *If X is a Hausdorff space and $\mathcal{D}$ is a decomposition of X satisfying any one of the following conditions:*

(a) *$\mathcal{D}$ contains no singleton sets*

(b) *$\mathcal{D}$ is a locally finite family*

then τ_Z is discrete.

Proof. Let $\mathcal{D}=\{D_\alpha \mid \alpha\in A\}$. If (*a*) is satisfied, then each $D_\alpha\in\mathcal{D}$ has more than one point. Hence by Proposition 1, $\{D_\alpha\}\in\tau_Z$ for all $\alpha\in A$, i. e. τ_Z is discrete.

If (*b*) is satisfied, let $D_{\alpha_0}\in\mathcal{D}$ be arbitrary; we shall show that $\{D_{\alpha_0}\}\in\tau_Z$. By virtue of Proposition 1, we may assume $D_{\alpha_0}=\{x\}$ for some $x\in X$. Since $\mathcal{D}$ is locally finite, there is an open neighborhood W of x such that $W\cap D_\alpha=\varnothing$ for all but at most a finite number of α's in A. Let A_0 be the finite subset of A such that $W\cap D_\alpha\neq\varnothing$ iff $\alpha\in A_0$. If $A_0=\{\alpha_0\}$, let $\varphi:\mathcal{D}\to X$ be any choice function such that $\varphi(D_{\alpha_0})=x$; then $\varphi^{-1}(W)=\{D_{\alpha_0}\}$ so that $\{D_{\alpha_0}\}\in\tau_Z$. If $A_0=\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ for some $\alpha_1, \ldots, \alpha_n\in A$, let $q_i\in W\cap D_{\alpha i}$ for $i=1, 2, \ldots, n$. Since X is Hausdorff, there is an open set $W_0\subset W$ such that $x\in W_0$ and $W_0\cap\{q_1, \ldots, q_n\}=\varnothing$. Let $\varphi:\mathcal{D}\to X$ be any choice function such that $\varphi(D_{\alpha_0})=x$ and $\varphi(D_{\alpha i})=q_i$ for $i=1, 2, \ldots, n$. Then $\varphi^{-1}(W_0)=\{D_{\alpha_0}\}$. Hence $\{D_{\alpha_0}\}\in\tau_Z$.

Corollary (*a*) *If X is a discrete space then τ_Z is discrete.*

(*b*) *If X is Hausdorff and $\mathcal{D}$ is a finite decomposition, then τ_Z is discrete.*

Proposition 3. *Let $\mathcal{D}$ be a decomposition of a space X. For each $D_{\alpha_0}\in\mathcal{D}$ such that D_{α_0} has only one point, we have $\{D_{\alpha_0}\}\in\tau_Z$ iff $\{D_{\alpha_0}\}=\varphi^{-1}(U)$ for some choice function $\varphi:\mathcal{D}\to X$ and some open set U of X.*

Proof. The sufficiency follows from the definition of τ_Z.

Suppose $\{D_{\alpha_0}\}\in\tau_Z$, then by definition of τ_Z, there are choice functions $\varphi_1, \ldots, \varphi_n:\mathcal{D}\to X$ and open sets $U_1, \ldots, U_n$ of X such that $\{D_{\alpha_0}\}=\varphi_1^{-1}(U_1)\cap\cdots\cap\varphi_n^{-1}(U_n)$. Now $D_{\alpha_0}=\{p\}$ for some $p\in X$ by hypothesis, so $p=\varphi_i(D_{\alpha_0})\in U_i$ for all $i=1, 2, \ldots, n$. Let $U=U_1\cap\cdots\cap U_n$, then U is open in X. Now, for each $D_\alpha\in\mathcal{D}$, $\alpha\neq\alpha_0$, there exists i_0, $1\leqslant i_0\leqslant n$ such that $\varphi_{i_0}(D_\alpha)\notin U_{i_0}$. Hence $D_\alpha\setminus U_{i_0}\neq\varnothing$. Since $U\subseteq U_{i_0}$, we have $D_\alpha\setminus U\neq\varnothing$. Define a choice function $\varphi:\mathcal{D}\to X$ by putting $\varphi(D_{\alpha_0})=p$ and $\varphi(D_\alpha)\in D_\alpha\setminus U$ for $\alpha\neq\alpha_0$. We have $\{D_{\alpha_0}\}=\varphi^{-1}(U)\in\tau_Z$.

Proposition 4. *A decomposition $\mathcal{D}=\{D_\alpha \mid \alpha\in A\}$ is* I-*non-void iff for every $D_\alpha\in\mathcal{D}$ there exist a choice function $\varphi_\alpha:\mathcal{D}\to X$ and an open set V_α of X such that $\varphi_\alpha^{-1}(V_\alpha)=\{D_\alpha\}$.*

Proof. If $\mathcal{D}$ is I-non-void, then for every $D_\alpha\in\mathcal{D}$, there exist $x\in D_\alpha$ and open neighborhood V_α of x such that $D_\beta\setminus V_\alpha\neq\varnothing$ for all $\beta\neq\alpha$. Define $\varphi_\alpha:\mathcal{D}\to X$ by $\varphi_\alpha(D_\beta)\in D_\beta\setminus V$ for all $\beta\neq\alpha$ and $\varphi_\alpha(D_\alpha)=x$. Then $\varphi_\alpha^{-1}(V_\alpha)=\{D_\alpha\}$.

Conversely, suppose that the condition is satisfied and $D_\alpha\in\mathcal{D}$. Then there exists choice function $\varphi_\alpha:\mathcal{D}\to X$ and open set V_α of X with $\varphi_\alpha^{-1}(V_\alpha)=\{D_\alpha\}$. Let $x=\varphi_\alpha(D_\alpha)\in V_\alpha$, then V_α is a neighborhood of x and $D_\beta\setminus V_\alpha\neq\varnothing$ for all $\beta\neq\alpha$, $\beta\in A$. Hence $\mathcal{D}$ is I-non-void.

Example. Let $X=\left\{0, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}\cup\left\{1, 1\frac{1}{2}, 1\frac{1}{3}, \ldots, 1\frac{1}{n}, \ldots\right\}$ with the relative topology of the real line. Then X is Hausdorff. Let

$$\mathcal{D}=\left\{\{0, 1\}, \left\{\frac{1}{2}\right\}, \left\{\frac{1}{3}\right\}, \ldots, \left\{\frac{1}{n}\right\}, \ldots, \left\{1\frac{1}{2}\right\}, \left\{1\frac{1}{3}\right\}, \ldots, \left\{1\frac{1}{n}\right\}, \ldots\right\}$$

be a decomposition of X. It is clear that the set Z of all choice functions from $\mathcal{D}$ to X has exactly two elements, namely φ_1, φ_2 where $\varphi_1(\{0, 1\})=0$ and $\varphi_2(\{0, 1\})=1$, (φ_1, φ_2 map the other elements in the obvious unique way). Since

$\left\{\frac{1}{n}\right\}$, $\left\{1\frac{1}{n}\right\}$ are open in X for all $n=2, 3, \ldots$; we have $\left\{\left\{\frac{1}{n}\right\}\right\}$, $\left\{\left\{1\frac{1}{n}\right\}\right\}\in\tau_Z$ for all $n=2, 3, \ldots$; $\{\{0, 1\}\}\in\tau_Z$ follows from Proposition 1. Hence τ_Z is discrete. On the other hand, $\varphi^{-1}(V)=\{\{0, 1\}\}$ cannot hold for all choice of φ in Z and all choice of V open in X. Hence by proposition 4, $\mathcal{D}$ is not I-non-void.

Remarks (1) In the above example, X is compact but $(\mathcal{D}, \tau_Z)$ is not compact. Hence in general, the compactness of X does not imply the compactness of $(\mathcal{D}, \tau_Z)$.

(2) Let X be a Hausdorff space and $\mathcal{D}$ a decomposition of X. Let

$\mathcal{D}_1=\{D\in\mathcal{D}\mid D$ contains more than one point$\}$

$\mathcal{D}_2=\{D\in\mathcal{D}\mid D$ is a singleton set such that $\{D\}=\varphi^{-1}(U)$ for some choice function $\varphi:\mathcal{D}\to X$ and some open set U of $X\}$

$\mathcal{D}_3=\mathcal{D}\setminus(\mathcal{D}_1\cup\mathcal{D}_2)$.

Then it is easy to see from Proposition 1 and the definition of choice topology of $\mathcal{D}$ that $\mathcal{D}_1\cup\mathcal{D}_2$ is an open discrete subspace of $(\mathcal{D}, \tau_Z)$ and $\mathcal{D}_3$, $\mathcal{D}_2\cup\mathcal{D}_3$ as subspaces of $(\mathcal{D}, \tau_Z)$ are homeomorphic with the subspaces $\bigcup\mathcal{D}_3$ and $\bigcup(\mathcal{D}_2\cup\mathcal{D}_3)$ of X respectively.

It was shown in [1; Theorem 3.1] that if X is regular (respectively normal) then τ_Z is regular (respectively normal). However, each of the proofs provided there contains a gap. We are able to provide an elementary proof for the case of regularity and an example to show that the case of normality does not hold in general.

Proposition 5. *If X is regular, then $(\mathcal{D}, \tau_Z)$ is regular.*

Proof. Let $\mathcal{G}$ be an open subset of $\mathcal{D}=\{D_\alpha\mid\alpha\in A\}$ and $D_{\alpha_0}\in\mathcal{G}$. We shall show that there is $\mathcal{W}$ open in $\mathcal{D}$ such that $D_{\alpha_0}\in\mathcal{W}\subset\overline{\mathcal{W}}\subset\mathcal{G}$. Since $\mathcal{G}$ is open, there are choice functions $\varphi_1, \ldots, \varphi_n$ and open sets $U_1, \ldots, U_n$ of X such that

$$D_{\alpha_0}\in\varphi_1^{-1}(U_1)\cap\cdots\cap\varphi_n^{-1}(U_n)\subset\mathcal{G}.$$

Now $\varphi_i(D_{\alpha_0})\in U_i$ for each $i=1, 2, \ldots, n$. By regularity of X, there are open sets $V_1, \ldots, V_n$ of X such that

$$\varphi_i(D_{\alpha_0})\in V_i\subset\overline{V_i}\subset U_i \quad \text{for all} \quad i=1, 2, \ldots, n.$$

Then

$$\begin{aligned} D_{\alpha_0}&\in\varphi_1^{-1}(V_1)\cap\cdots\cap\varphi_n^{-1}(V_n)\\ &\subset\overline{\varphi_1^{-1}(V_1)\cap\cdots\cap\varphi_n^{-1}(V_n)}\\ &\subset\overline{\varphi_1^{-1}(V_1)}\cap\cdots\cap\overline{\varphi_n^{-1}(V_n)}\\ &\subset\varphi_1^{-1}(\overline{V_1})\cap\cdots\cap\varphi_n^{-1}(\overline{V_n}) \end{aligned}$$

The last inclusion sign follows since φ_i are continuous. But

$$\varphi_1^{-1}(\overline{V_1})\cap\cdots\cap\varphi_n^{-1}(\overline{V_n})\subset\varphi_1^{-1}(U_1)\cap\cdots\cap\varphi_n^{-1}(U_n)\subset\mathcal{G}.$$

Hence it suffices to let $\mathcal{W}=\varphi_1^{-1}(V_1)\cap\cdots\cap\varphi_n^{-1}(V_n)$.

The following example shows that the normality of X does not imply the normality of $(\mathcal{D}, \tau_Z)$ in general.

Example. Let α be a fixed non-limit ordinal and $\alpha > \Omega$, the first uncountable ordinal. The set $W(\alpha)$ of all ordinals less than α with the order topology is a compact Hausdorff space. Let $X = W(\alpha) \times W(\alpha)$. Then X is compact Hausdorff and hence normal. Let $T = \{(x, y): x \leqslant \omega, \ y \leqslant \Omega\} \setminus \{(\omega, \Omega)\} \subset X$, where ω is the first infinite ordinal. Then T as a subspace of X is not normal ([2] p. 132). Let $\mathcal{D} = \{X \setminus T\} \cup \{\{p\}: p \in T\}$. By Proposition 1, $\{X \setminus T\}$ is open in $(\mathcal{D}, \tau_Z)$, so $A = \{\{p\}: p \in T\}$ is a closed subspace of $(\mathcal{D}, \tau_Z)$ and A is homeomorphic with the subspace T of X. Thus A is not normal. Therefore $(\mathcal{D}, \tau_Z)$ cannot be normal, because the normality of $(\mathcal{D}, \tau_Z)$ would imply the normality of the closed subspace A.

If X is T_1 (respectively Hausdorff) then $(\mathcal{D}, \tau_Z)$ is T_1 (respectively Hausdorff) [1; Theorem 3.1]. It is not difficult to see that if X is T_0 then $(\mathcal{D}, \tau_Z)$ is also T_0. We shall show that if X is completely regular then $(\mathcal{D}, \tau_Z)$ is completely regular. We begin by,

Let E be any topological space. Recall [3] that a space S is called E-completely regular iff S is homeomorphic to a subspace of some topological powers of E, (i. e., to a subspace of some product of spaces each being E). In [3; 3.9] it was shown that a T_0-space S is E-completely regular iff for every net $\{x_\delta \mid \delta \in A\}$ of points of S, the condition "$x_\delta \to x$ iff $f(x_\delta) \to f(x)$ for every continuous function $f: S \to E$" holds.

Proposition 6. *Let X be a T_0-space, then $(\mathcal{D}, \tau_Z)$ is X-completely regular.*

Proof. Since X is T_0 implies that $(\mathcal{D}, \tau_Z)$ is also T_0, by the quoted result [3; 3.9], it suffices to show for every net $\{D_\delta \mid \delta \in A\}$ of points of $\mathcal{D}$, the condition "$D_\delta \to D$ iff $f(D_\delta) \to f(D)$ for every continuous function $f: \mathcal{D} \to X$" holds.

$D_\delta \to D$ implies $f(D_\delta) \to f(D)$ is obvious. Suppose $f(D_\delta) \to f(D)$ for every continuous function $f: \mathcal{D} \to X$; we shall show $D_\delta \to D$. Let $\mathcal{G}$ be a neighborhood of D in $\mathcal{D}$. Then since $\mathcal{G}$ is open, there are choice functions $\varphi_1, \ldots, \varphi_n$ and open sets $U_1, \ldots, U_n$ of X such that

$$D \in \varphi_1^{-1}(U_1) \cap \cdots \cap \varphi_n^{-1}(U_n) \subset \mathcal{G}.$$

For all $i = 1, 2, \ldots, n$, $\varphi_i(D) \in U_i$. But φ_i is continuous, so

$$\varphi_i(D_\delta) \to \varphi_i(D) \in U_i.$$

There exist $\delta_i \in A$ such that $\varphi_i(D_\delta) \in U_i$ for $\delta \geqslant \delta_i$. Let $\delta_0 \geqslant \delta_1, \ldots, \delta_n$; $\delta_0 \in A$ then $\varphi_i(D_\delta) \in U_i$ for all $\delta \geqslant \delta_0$ and all $i = 1, 2, \ldots, n$. Hence for all $\delta \geqslant \delta_0$,

$$D_\delta \in \varphi_1^{-1}(U_1) \cap \cdots \cap \varphi_n^{-1}(U_n) \subset \mathcal{G} \text{ i. e. } \qquad D_\delta \to D.$$

Corollary. *If X is T_0, completely regular then $(\mathcal{D}, \tau_Z)$ is completely regular.*

Proof. If X is completely regular, then every X-completely regular space is completely regular.

Remark. Regularity, E-complete regularity are productive and hereditary properties. Therefore Propositions 5 and 6 also follow from the fact that the set Z of all choice functions from $\mathcal{D}$ into X separates the points of $\mathcal{D}$, so the evaluation map e from $(\mathcal{D}, \tau_Z)$ into the product X^Z defined by $e(D)(\varphi)=\varphi(D)$ for each $D\in\mathcal{D}$ and $\varphi\in Z$ is a homeomorphism [4, Theorem 8.12]. Moreover, T_0-ness can be omitted in Proposition 6.

Next, we shall investigate the relations among τ_Z and some other topologies of a decomposition $\mathcal{D}=\{D_\alpha \mid \alpha\in A\}$ of a topological space (X, τ). For each $U\in\tau$, let $U^*=\{D\in\mathcal{D} \mid D\cap U\neq\varnothing\}$, $U^+=\{D\in\mathcal{D} \mid D\subset U\}$. Then $\{U^* \mid U\in\tau\}$ and $\{U^+ \mid U\in\tau\}$ are subbases for some topologies τ^* and τ^+ for $\mathcal{D}$ respectively.

An element U in τ is said to be $\mathcal{D}$-saturated if $U=\cup\{D_\alpha \mid \alpha\in I\}$ for some $I\subset A$. If U is $\mathcal{D}$-saturated then $U^*=U^+$. Let S^* be the topology of $\mathcal{D}$ with $\{U^* \mid U\in\tau$ and U is $\mathcal{D}$-saturated$\}$ as subbase. Then $S^*\subset\tau^*\cap\tau^+$. Let p be the projection map from X into $\mathcal{D}$ and τ_q be the quotient topology of $\mathcal{D}$, we shall show that $S^*=\tau_q$. If $U\in\tau$ and U is $\mathcal{D}$-saturated then $p^{-1}(U^*)=p^{-1}(p(U))=U$. This shows that p is a continuous mapping from (X, τ) into $(\mathcal{D}, S^*)$. For each G in τ_q, $p^{-1}(G)\in\tau$ and $p^{-1}(G)$ is $\mathcal{D}$-saturated, so $p^{-1}(G)^*=p(p^{-1}(G))=G\in S^*$. Hence $\tau_q\subset S^*$. But τ_q is the largest topology of $\mathcal{D}$ such that p is continuous, thus $\tau_q=S^*$.

Next, for any $U\in\tau$, let φ_1, φ_2 be choice functions from $\mathcal{D}$ into X such that $\varphi_1(D_\alpha)\in D_\alpha\cap U$ if $D_\alpha\cap U\neq\varnothing$, $\varphi_2(D_\alpha)\in D_\alpha$ if $D_\alpha\subset U$ and $\varphi_2(D_\alpha)\in D_\alpha\setminus U$ if $D_\alpha\setminus U\neq\varnothing$. Then $U^*=\varphi_1^{-1}(U)$ and $U^+=\varphi_2^{-1}(U)$. Hence $\tau^*\subset\tau_Z$ and $\tau^+\subset\tau_Z$.

We have shown that:

Proposition 7. (a) $\tau_q=S^*$

(b) $\tau_q\subset\tau^*\subset\tau_Z$

(c) $\tau_q\subset\tau^+\subset\tau_Z$.

In concluding, we shall give examples to show that the inclusions in (*b*), (*c*) may be proper and that no general containment holds between τ^* and τ^+.

Examples. (1) In this example, $\tau_q\neq\tau^*=\tau^+$. Let $X=[0, \infty)$ with the usual topology and $\mathcal{D}=\{[n-1, n) \mid n=1, 2, 3, \ldots\}$. Then τ_q is not discrete and τ^*, τ^+ are discrete.

(2) In this example, $\tau^*\subset\tau^+$ and $\tau^*\neq\tau^+$. Let (X, τ) be the real line with usual topology and $\mathcal{D}=\{\{n\}, (n, n+1) \mid n$ is an integer$\}$. Then τ^+ is discrete but τ^* is not discrete.

(3) In this example, $\tau^+\subset\tau^*$ and $\tau^*\neq\tau^+$. Let $X=\{1, 2, 3\}$, $\tau=\{\varnothing, \{3\}, X\}$ and $\mathcal{D}=\{\{1\}, \{2, 3\}\}$. Then τ^+ is indiscrete and τ^* is not indiscrete since $\{\{2, 3\}\}\in\tau^*$.

REFERENCES

[1] R. Dacić, *Choice Topology*, Publ. Inst. Math., Beograd, 8 (22) 1968, 93—96.

[2] J. L. Kelly, *General Topology*, Van Nostrad, Princeton 1955.

[3] S. Mrówka, *Further results on E-compact spaces I*, Acta Math., 120 (1968) 161—185.

[4] S. Willard, *General Topology*, Addison-Wesley, Ontario 1970.

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