PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 14 (28), 1972, pp. 15–18

FIXED POINT THEOREMS FOR A SEQUENCE OF MAPPINGS WITH CONTRACTIVE ITERATES

S. K. Chatterjea

(Received March 30, 1972)

1. Introduction: Let (X, d) be a metric space, $T: X \to X$ a mapping and $0 < \alpha < 1$. Then T is said to be α-contractive endomorphism if

$$d(Tx, Ty) \leqslant \alpha\, d(x, y)$$

for all $x, y \in X$. It is well-known [1] that for X complete, each continuous endomorphism T satisfying the condition that T^n is α-contractive for some n, has a unique fixed point. The object of this paper is to consider some fixed point theorems for a sequence of mappings with contractive iterates and to show that particular cases of our theorems are those discussed by B. Ray [2]. Our theorems are stated in § 2 — § 4.

2. First we shall prove the following fixed point theorem for a sequence of mappings with contractive iterates:

Theorem I. *If there exists a sequence of continuous mappings $\{T_n\}$ of a complete metric space (X, d) into itself such that for some m and $0 < \alpha < 1$*

(*i*) *for any two mappings T_i and T_j, we have*

$$d(T_i^m x,\ T_j^m y) \leqslant \alpha\, d(x, y), \quad x, y \in X,$$

(*ii*) *T_i commutes with T_j, $i \neq j$,*

then $\{T_n\}$ has a unique common fixed point.

Proof: Let x_0 be any point X, and

$$x_1 = T_1^m x_0, \quad x_2 = T_2^m x_1, \ldots, x_n = T_n^m x_{n-1}, \ldots .$$

Now

$$d(x_1, x_2) = d(T_1^m x_0,\ T_2^m x_1) \leqslant \alpha\, d(x_0, x_1)$$

$$d(x_2, x_3) = d(T_2^m x_1,\ T_3^m x_2) \leqslant \alpha\, d(x_1, x_2) \leqslant \alpha^2 d(x_0, x_1)$$

and so on. Generally we have

$$d(x_n, x_{n+1}) \leqslant \alpha^n d(x_0, x_1).$$

For $p>0$, we have

$$d(x_n, x_{n+p}) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})$$
$$\leqslant (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{n+p-1})\, d(x_0, x_1)$$
$$< \frac{\alpha^n}{1-\alpha} d(x_0, x_1).$$

Since $0<\alpha<1$, $d(x_n, x_{n+p}) \to 0$, as $n \to \infty$. Thus $\{x_n\}$ is fundamental in X. Again (X, d) being a complete metric space, $\lim\limits_{n\to\infty} x_n = \bar{x} \in X$.

Now for any fixed k, we shall show that $T_k \bar{x} = \bar{x}$.

If possible, suppose $T_k \bar{x} \neq \bar{x}$. Then there exists a pair of disjoint close dneighbourhoods U and V such that $\bar{x} \in U$, $T_k \bar{x} \in V$ and also

$$(*) \qquad \rho = \inf\{d(x, y) : x \in U, y \in V\} > 0.$$

Since T_k is continuous, $x_n \in U$ and $T_k x_n \in V$ for sufficiently large n.

However we notice that

$$d(T_k x_n, x_n) = d(T_n^m T_k x_{n-1}, T_n^m x_{n-1})$$
$$\leqslant \alpha\, d(T_k x_{n-1}, x_{n-1})$$
$$\leqslant \cdots\cdots\cdots$$
$$\leqslant \alpha^n d(T_k x_0, x_0) \to 0, \quad as\ n \to \infty,$$

which contradicts $(*)$. Thus $T_k \bar{x} = \bar{x}$.

Lastly we show that $\bar{x}$ is unique. If possible, let $\bar{y}$ be a fixed point common to T_n $(n = 1, 2, \cdots)$ such that $\bar{y} \neq \bar{x}$.

Then we have

$$d(\bar{x}, \bar{y}) = d(T_i^m \bar{x}, T_j^m \bar{y})$$
$$\leqslant \alpha\, d(\bar{x}, \bar{y}),$$

which is impossible, since $0<\alpha<1$. Hence $\bar{x} = \bar{y}$.

It may be noted that if $T_1 = T_2 = \cdots = T$ *(say)*, then theorem I reduces to the well-known theorem as mentioned in the introduction.

3. In this section we investigate a theorem similar to theorem I, without assuming the continuity of the sequence of maps $\{T_n\}$. Actually we prove the following theorem:

Theorem II. If there exists a sequence of mappings $\{T_n\}$ of a complete metric space (X, d) into itself such that

(i) for any two mappings T_i, T_j, we have

$$1)\ d(T_i^m x, T_j^m y) \leqslant \alpha\, d(x, y)$$
$$2)\ d(T_i^m x, T_j y) \leqslant \alpha\, d(x, y)$$

for some m and $0<\alpha<1$; $x, y \in X$,

then $\{T_n\}$ has a unique common fixed point.

P r o o f : Taking an arbitrary point $x_0 \in X$, we can construct the same sequence $\{x_n\}$ as in theorem I. It follows therefore that $\{x_n\}$ is fundamental in the complete metric space (X, d). Thus $\lim\limits_{n\to\infty} x_n = \bar{x} \in X$.

Now for a fixed k, we shall show that $T_k \bar{x} = \bar{x}$

Indeed, we have

$$\begin{aligned} d(\bar{x}, T_k \bar{x}) &\leqslant d(\bar{x}, x_n) + d(x_n, T_k \bar{x}) \\ &= d(\bar{x}, x_n) + d(T_n^m x_{n-1}, T_k \bar{x}) \\ &\leqslant d(\bar{x}, x_n) + \alpha\, d(x_{n-1}, \bar{x}). \end{aligned}$$

Since $x_n \to \bar{x}$, it follows that $\bar{x} = T_k \bar{x}$.

The uniqueness of $\bar{x}$ follows similarly as in theorem I.

It is interesting to note that theorem II reduces to theorem 1.1 of Ray [2, p. 7], when $m = 1$.

4. Lastly we consider another extension of a theorem of Ray. Let us prove the following theorem:

T h e o r e m III. If there exists a sequence of mappings $\{T_n\}$, each mapping a complete metric space (X, d) into itself such that

(i) for any two mappings T_i, T_j, we have

$$1)\ d(T_i^m x, T_j^m y) \leqslant \alpha_{i,j}\, d(x, y)$$

$$2)\ d(T_i^m x, T_j y) \leqslant \alpha_{i,j}\, d(x, y)$$

for some m and $0 < \alpha_{i,j} < 1$; $x, y \in X$, and $\sum\limits_{i=1}^{\infty} \alpha_{i,i+1}$ is $(C, 1)$ — summable, then $\{T_n\}$ has a unique common fixed point.

P r o o f : As in theorem II, we construct the sequence $\{x_n\}$. Now we shall show that $\{x_n\}$ is also fundamental in the complete metric space (X, d).

By routine calculation we have

$$d(x_n, x_{n+1}) \leqslant \left(\prod_{i=1}^{n} \alpha_{i,i+1}\right) d(x_0, x_1).$$

Thus for $p > 0$, we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leqslant \left(\sum_{k=n}^{n+p-1} \prod_{i=1}^{k} \alpha_{i,i+1}\right) d(x_0, x_1) \\ &\leqslant \sum_{k=n}^{n+p-1} \left(\frac{\sum\limits_{i=1}^{k} \alpha_{i,i+1}}{k}\right)^k d(x_0, x_1) \end{aligned}$$

Now by $(C, 1)$ — summability of $\sum\limits_{i=1}^{\infty} \alpha_{i,i+1}$, we have $\sum\limits_{k=1}^{\infty} S_k < +\infty$, where

$$S_k = \left(\sum_{\nu=1}^{k} s_\nu\right) \Big/ k \quad \text{and} \quad s_k = \sum_{i=1}^{k} \alpha_{i,i+1}.$$

Since $0<\alpha_{i,j}<1$, $(s_k/k)^k \leqslant \frac{s_k}{k} \leqslant S_k$. Thus the series $\sum_{k=1}^{\infty} \frac{s_k}{k}$ is convergent.

$$\therefore\ d(x_n, x_{n+p}) \to 0, \quad as\ n \to \infty.$$

Hence $\{x_n\}$ is fundamental. Thus $\lim_{n\to\infty} x_n = \bar{x} \in X$.

The existence of the common fixed point $\bar{x}$ and its uniqueness follow in exactly the same way as in theorem II.

It may be noted that theorem III reduces to theorem 3.1 of Ray [2. p. 9], when $m=1$.

The author wishes to record his deep gratitude to Dr. M. Dutta for his valuable suggestions.

REFERENCES

[1] A. N. Kolmogorov and S. V. Fomin, *Elements of the theory of functions and functional analysis*, vol. I, Graylock Press (1957), N. Y., p. 50.

[2] Baidyanath Ray, *On a paper of Kannan*, Bull. Cal. Math. Soc., vol. 63(1971), pp. 7-10.

Dept. of Pure Mathematics,
Calcutta University, Calcutta, India.