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THE SERIES OF MIKUSIŃSKI OPERATORS OF THE SPECIAL FORM

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Introduction

We are going to investigate the convergence of the series of operators in the field $\mathcal{K}$ of Mikusiński in which general terms are the rational functions of the differential operator s. We shall also investigate the analytic form for the operators represented by the series of such form.

At first, we are going to investigate the convergence of that form of operator:

$$\sum_{n=0}^{\infty} a_n \frac{1}{s+\beta_n} \tag{1}$$

where s is differential operator, $a_n \in \mathcal{K}$ and

$$0<\beta_0<\beta_1<\cdots, \quad \beta_n \to \infty, \quad n\to\infty.$$

The problem of convergence of the series of the operator (1) we shall connect with the coresponding Stieltjes integral of operational function.

Then using the isomorphism of the field $L \subset \mathcal{K}$ of operators and the set constructed by Laplace transformation, we will give the representation in the field $\mathcal{K}$ for convergent series:

$$\sum_{n=1}^{\infty} \frac{1}{s^2+4n^2\pi^2}. \tag{2}$$

At last, using the properties of E. M. Wright's function we shall prove that for $0<\nu<1$, implies:

$$\sum_{n=1}^{\infty} \frac{1}{s^{2\nu}+4n^2\pi^2} = \frac{e^{-s^\nu}}{1-e^{-s^\nu}} - \frac{1}{s^\nu} + \frac{1}{2}, \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{16\pi^2 n^2}{(s^{2\nu}+4n^2\pi^2)^2} = \frac{e^{-s^\nu}}{s^\nu(1-e^{-s^\nu})} - \frac{e^{-s^\nu}}{(1-e^{-s^\nu})^2} + \frac{1}{2s^\nu}. \tag{6}$$

The convergence of the series (1) of operators

Proposition 1, which concerns the simplest series (1), shows the difficulties by the usual investigation of the convergence of such series.

Proposition 1. *If $\beta_1, \beta_2, \ldots$ is a sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} < \infty,$$

then the series of operators:

$$\sum_{n=1}^{\infty} \frac{1}{s+\beta_n}$$

diverges in the field $\mathcal{K}$.

Proof. — Suppose the opposite of it; namely that the given series is convergent, then an $f \in \mathcal{C}$, $f \neq 0$ should exist, so that $f \sum_{n=1}^{\infty} \frac{1}{s+\beta_n}$ represents the series of continuous functions, uniformly convergent with respect to t on each finite interval $[0, T]$, $T > 0$. We know that:

$$\frac{1}{s+\beta_n} = \frac{1}{\beta_n} - \frac{s}{\beta_n(s+\beta_n)}$$

then

$$f \sum_{n=1}^{\infty} \frac{1}{s+\beta_n} = f \sum_{n=1}^{\infty} \frac{1}{\beta_n} - f \sum_{n=1}^{\infty} \frac{s}{\beta_n(s+\beta_n)}.$$

The series of operators: $\sum_{n=1}^{\infty} \frac{s}{\beta_n(s+\beta_n)}$ converges in $\mathcal{K}$ (it is enough to take $f = l^2$).

According to the supposition of the proposition that $\sum \frac{1}{\beta_n} = \infty$, it follows that $\sum_{n=1}^{\infty} \frac{1}{s+\beta_n}$ diverges in $\mathcal{K}$.

The problem of the convergence of the series of operators (1) we shall connect with the coresponding Stieltjes integral of operator function which was defined by E. Gesztelyi [2].

Definition 1. *The step function $A(x)$ of the sequence $a_n \in \mathcal{K}$ with jumps at $x = \beta_n$, $0 < \beta_1 < \beta_2 < \cdots$, $\beta_n \to \infty$, $n \to \infty$, is defined as follows:*

$$A(x) = \sum_{\beta_n \leqslant x} a_n, \quad A(x) = \begin{cases} 0 & x = \beta_1 \\ \sum_{k=1}^{n} a_k & \beta_n \leqslant x < \beta_{n+1}. \end{cases}$$

According to Gesztelyi's theorem [2] it is easy to prove:

Proposition 2. *Suppose that:*

i. *$f(x)$ is continuous operational function on interval $[0, \infty)$ i. e.*

$$f(x) \in C[0, \infty)\mathcal{M}.$$

ii. *In the field $\mathcal{K}$ exists the limit* $\lim\limits_{x\to\infty} \int_0^x f(u)\, dA(u)$ *and equals:*

$$\lim_{x\to\infty} \int_0^x f(u)\, dA(u) = \int_0^\infty f(u)\, dA(u).$$

Then the series of operators $\sum\limits_{k\geqslant 1} f(\beta_k)\, a_k$ *converges in the field $\mathcal{K}$ and*

$$\int_0^\infty f(u)\, dA(u) = \sum_{k=1}^\infty a_k f(\beta_k).$$

P r o o f. — If we denote by

$$q_k(u) = \begin{cases} 0 & 0 \leqslant u < \beta_k \\ 1 & \beta_k \leqslant u < +\infty \end{cases} \qquad k = 1, 2, \ldots$$

then the step function $A(u)$ of a sequence of operators a_n can be written in the form:

$$\begin{aligned} A(u) &= a_1 q_1(u) + \cdots + a_n q_n(u) + \cdots \\ &= a_1 \int_0^u dq_1(v) + \cdots + a_n \int_0^u dq_n(v) + \cdots \\ &= \sum_{\beta_n \leqslant u} a_n \int_0^u dq_n(v). \end{aligned}$$

As $A(u)$, according to the theorem in [2], is the operational function of bounded variation in $[0, x]$, and $f(x)$ is the continuous operational function in each interval $[0, x] \subset [0, \infty)$, $\int_0^x f(u)\, dA(u)$ exists and it equals:

$$\int_0^x f(u)\, dA(u) = \sum_{\beta_n \leqslant x} \int_0^x f(u)\, a_n\, dq_n(u) = \sum_{\beta_n \leqslant x} a_n \left\{ \int_0^x f(u)\, dq_n(u) \right\}.$$

According to the supposition **i.**

$$f(u) = p\{f_1(u, t)\}; \quad p \in \mathcal{K},\ f_1(u, t) \in C[0, x]\mathcal{C},$$

$$\begin{aligned} \sum_{\beta_n \leqslant x} a_n \left\{ \int_0^x f(u)\, dq_n(u) \right\} &= \sum_{\beta_n \leqslant x} a_n p \left\{ \int_0^x f_1(u, t)\, dq_n(u) \right\} \\ &= \sum_{\beta_n \leqslant x} a_n p \{f_1(\beta_n, t)\} \\ &= \sum_{\beta_n \leqslant x} a_n f(\beta_n). \end{aligned}$$

From the supposition **ii.** and from that relation follows the convergence of the series of operators: $\sum\limits_{n=1}^\infty a_n f(\beta_n)$ and the relation:

$$\int_0^\infty f(u)\, dA(u) = \sum_{n=1}^\infty a_n f(\beta_n).$$

Corollary 1. $f(x)=\frac{1}{s+x}=\{e^{-xt}\}$ *is continuous operational function, namely* $f(x)=f(x, t)\in C[0, \infty)\mathscr{C}$. *If for all* $t\geqslant 0$ *exists*

$$\lim_{x\to\infty}\int_0^x e^{-ut}\,dA(u)$$

where $A(u)$ *is the step function of the operator sequence* a_n *defined by Definition* 1, *then the series of operators* (1) *converges and implies:*

$$\sum_{n=1}^{\infty} a_n\frac{1}{s+\beta_n}=\lim_{x\to\infty}\int_0^x e^{-xt}\,dA(x).$$

Corollary 2. *The series of operators:*

$$\sum_{n=0}^{\infty}\frac{1}{s+\beta_n}=\left\{\sum_{n=0}^{\infty}e^{-\beta_n t}\right\},$$

where $0<\beta_0<\beta_1<\cdots$, $\beta_n\to\infty$, $n\to\infty$ *does not converge in the field* $\mathscr{K}$.

The convergence of the series of operators and the field $\hat{L}$

Let $\hat{L}$ be the subset of the field $\mathscr{K}$, elements of which have the property: In the class of equivalence which defines the element $a\in\mathscr{K}$, exist such elements f and $g\neq 0$ in $\mathscr{C}$, $a=\frac{f}{g}$, which have the absolute convergent Laplace transformations in the halfplane $Re\,z\geqslant x_0$, where $x_0\geqslant 0$, and x_0 changes according to f and g. With the induced operations from $\mathscr{K}$, L becomes field, and it is known that $L\neq\mathscr{K}$ [1].

If we denote Laplace transformation of the function f as $\mathscr{L}\{f\}$, then

$$a=\frac{f}{g}\in L,\quad \mathscr{L}\{a\}=\frac{\mathscr{L}\{F\}}{\mathscr{L}\{g\}}.$$

Let:

$$\hat{L}=\{\mathscr{L}\{a\}:a\in L\},$$

then $(\hat{L}, +, .)$ is also a field.

Convergence in $\hat{L}$: A sequence $\frac{F_n(z)}{G(z)}\in\hat{L}$ converges in $\hat{L}$ iff there exists a right halfplane $Re\,z\geqslant x_0$, x_0 is independent of n, so that $F_n(z)$ converges in that halfplane.

Theorem A: *There exists an algebraic isomorphism between L and* $\hat{L}$ [1]:

$$L \underset{\mathscr{L}^{-1}}{\overset{\mathscr{L}}{\rightleftarrows}} \hat{L}$$

Examples of this isomorphism

$a \in L$	$\mathcal{L}\{a\} \in \hat{L}$
l^{α}	$z^{-\alpha}$
$e^{-\lambda s}$, $\lambda > 0$	$e^{-\lambda z}$
$\frac{1}{s^2+k^2}$	$\frac{1}{z^2+k^2}$

We can define relative topology for L induced by the topology $\mathcal{G}$ of the field $\mathcal{K}$. Beside this one we can also define a finer topology τ.

Definition 2. *Let $a_n \in L$ for $n=1, 2, \ldots$ and let $a \in L$. Suppose that*

i. *there is a non zero b in $L \cap \mathcal{C}$ such that $a_n b = c_n \in \mathcal{C}$*

ii. *$c_n \to ab \in \mathcal{C}$ almost uniformly*

iii. *there exist $n_0 \in \mathcal{N}$ and $\alpha \in \mathcal{R}$ such that:*

$$|c_n(t)| < e^{\alpha t}, \quad n \geq n_0.$$

Then we say $\lim_{n \to \infty} a_n = a$ *in the topology τ, and we shall call b a regularization factor.*

Lemma 1. *Let $a_n \in L$ for $n=1, 2, \ldots$ and $a_n \to a \in L$, in the topology τ. Then $A_n(z)$ converges to $A(z)$ in L, where $A_n(z) = \mathcal{L}\{a_n\}$ and $A(z) = \mathcal{L}\{a\}$.*

Proof. — $a_n \to a \in L$ in the topology τ, then there exists $b \in L \cap \mathcal{C}$, $b \neq 0$, such that $ba_n = c_n \in \mathcal{C}$, and $c_n \to ab \in \mathcal{C}$. By the Theorem A, for each $a_n \in L$ there exists $A_n(z) \in \hat{L}$ so that:

$$A_n(z) = \frac{C_n(z)}{B(z)}.$$

We will show that $A_n(z)$ converges to $\frac{C(z)}{B(z)} \in \hat{L}$ in $\hat{L}$, where $C(z) = \mathcal{L}\{c(t)\}$:

$$\lim_{n \to \infty} C_n(z) = \lim_{n \to \infty} \mathcal{L}\{c_n(t)\} = \mathcal{L}\{\lim_{n \to \infty} c_n(t)\} = \mathcal{L}\{c(t)\}$$

because

$$|c_n(t)| < e^{\alpha t}, \quad n \geq n_0.$$

Then

$$\lim_{n \to \infty} A_n(z) = \lim_{n \to \infty} \frac{C_n(z)}{B(z)} = \frac{\mathcal{L}\{a \cdot b\}}{\mathcal{L}\{b\}} = \mathcal{L}\{a\}.$$

Lemma 2: *Let*

i. *$a_n \in L$ for $n=1, 2, \ldots$ and the series of operators Σa_n converges to a in the topology τ with regularization factor b.*

ii. *There exists* $x_0 \geqslant 0$, *so that for* $\mathcal{L}\{a_n\} = A_n(z)$ *the series* $\Sigma A_n(z)$ *converges to* $A(z)$ *in the halfplane* $Re\, z \geqslant x_0$.

iii. $$\lim_{T\to\infty} \sum_{n=0}^{\infty} \int_T^{\infty} e^{-zt}(a_n(t) * b(t))\, dt = 0, \quad Re\, z \geqslant x_0,$$

then $A(z) \in \hat{L}$ *and* $\mathcal{L}\{a\} = A(z)$.

P r o o f. — According to **i.** and as the series $\sum_{n=0}^{\infty} (a_n(t) * b(t))$ converges uniformly with respect to t on each finite interval $[0, T]$, $T > 0$ we have:

$$\int_0^T \left(\sum_{n=0}^{\infty} a_n(t) * b(t)\right) e^{-zt} dt = \sum_{n=0}^{\infty} \int_0^T (a_n(t) * b(t))\, e^{-zt} dt =$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} e^{-zt} (a_n(t) * b(t))\, dt - \sum_{n=0}^{\infty} \int_T^{\infty} e^{-zt} (a_n(t) * b(t))\, dt =$$

$$= B(z) \sum_{n=0}^{\infty} A_n(z) - \sum_{n=0}^{\infty} \int_T^{\infty} e^{-zt} (a_n(t) * b(t))\, dt.$$

Hence by supposition **iii.** we get the proposition of lemma 2.

Using lemma 1 and lemma 2 we shall find the representation in the field $\mathcal{K}$ for the series of operators of the special form.

L e m m a 3: *If there exists* $n_0 \in \mathcal{N}$ *such that:*

$$(n \geqslant n_0) \Rightarrow (\beta_n > n^{1/2+\varepsilon}), \quad \varepsilon > 0,$$

then the series of operators of the following form:

$$\sum_{n=1}^{\infty} \frac{1}{s^2 + \beta_n^2}$$

converges in the topology τ *as well as in the field* $\mathcal{K}$.

In the proof we shall only use the following representation:

$$l \sum_{n=1}^{\infty} \frac{1}{s^2 + \beta_n^2} = \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} \{1 - \cos \beta_n t\}.$$

P r o p o s i t i o n 3: *The operator* $\frac{e^{-s}}{1 - e^{-s}}$ *can be written in the form of a convergent series of operators:*

$$\frac{e^{-s}}{1 - e^{-s}} = l - \frac{1}{2} + 2s \sum_{n=1}^{\infty} \frac{1}{s^2 + 4n^2\pi^2} \tag{2}$$

P r o o f. — Let $\beta_n = 2n\pi$ in the series of lemma 3; in the theory of complex function it is known that:

$$\frac{e^{-z}}{1 - e^{-z}} = \frac{1}{z} - \frac{1}{2} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 + 4n^2\pi^2}; \quad z^2 + 4n^2\pi^2 \neq 0.$$

The suppositions **i.** and **ii.** of lemma 2 are satisfied for $x_0>0$, and as:

$$l\sum_{n=1}^{\infty}\frac{1}{s^2+4n^2\pi^2}=\sum_{n=1}^{\infty}\frac{1}{4n^2\pi^2}\{1-\cos 2n\pi t\},$$

$$\left|\sum_{n=1}^{\infty}\int_T^{\infty}e^{-zt}\frac{1}{4n^2\pi^2}(1-\cos 2n\pi t)\,dt\right|\leqslant\sum_{n=1}^{\infty}\frac{1}{2n^2\pi^2}\int_T^{\infty}e^{-x_0t}\,dt=$$

$$=\frac{e^{-x_0T}}{x_0}\sum_{n=1}^{\infty}\frac{1}{2n^2\pi^2}\to 0,\quad T\to\infty$$

so is the supposition **iii.** of lemma 2 satisfied too and it completes the proof.

Let us investigate the convergence of the operator series:

$$\sum_{n=1}^{\infty}\frac{1}{s^{2\nu}+\beta_n^2},\qquad 0<\nu<1.$$

Proposition 4: *If there exists* $n_0\in\mathcal{N}$ *such that:*

$$(n\geqslant n_0)\ \Rightarrow\ (\beta_n>n^{\frac{1}{2}+\varepsilon});\qquad \varepsilon>0$$

then the series of operators of the following form:

$$\sum_{n=1}^{\infty}\frac{1}{s^{2\nu}+\beta_n^2},\qquad 0<\nu<1,$$

converges in the topology τ.

Proof. — Using the following known relation [5]: If $\mathcal{L}\{f\}=\varphi(z)$ then:

$$\mathcal{L}\left\{\int_0^{\infty}\Phi(0,\ -\nu;\ -tx^{-\nu})f(t)\frac{dt}{\nu t}\right\}=z^{\nu-1}\varphi(z^{\nu}),\quad 0<\nu<1. \tag{3}$$

Let us consider the function:

$$\varphi(z)=\frac{1}{z(z^2+\beta_n^2)}=\mathcal{L}\left\{\frac{1-\cos\beta_n t}{\beta_n^2}\right\}.$$

According to (3), for $0<\nu<1$, values:

$$\mathcal{L}\left\{\int_0^{\infty}\Phi(0,\ -\nu;\ -xt^{-\nu})\frac{1-\cos\beta_n x}{\beta_n^2}\frac{dx}{\nu x}\right\}=\frac{z^{\nu-1}}{z^{\nu}(z^{2\nu}+\beta_n^2)}.$$

The function $\psi_n(t)$, where

$$\psi_n(t)=\int_0^{\infty}\Phi(0,\ -\nu;\ -xt^{-\nu})\frac{1-\cos\beta_n x}{\beta_n^2}\frac{dx}{\nu x}$$

has the property that for $t>0$, $0<\nu<1$, values:

(4) $$|\psi_n(t)| \leqslant \frac{2}{\beta_n^2} \int_0^{+\infty} \Phi(0, -\nu; -xt^{-\nu}) \frac{dx}{\nu x} = \frac{2}{\beta_n^2}$$

$\left[\text{see [5] p. 95, 97, 99 where } \frac{1}{\nu x} F_\nu(\sigma x^{-1/\nu}) = \frac{1}{\nu x} \Phi(0, -\nu; -x\sigma^{-\nu})\right]$.

Besides this property we have for $t \to 0$:

$$0 \leqslant \liminf_{t\to 0} \psi_n(t) \leqslant \limsup_{t\to 0} \psi_n(t) \leqslant \frac{2}{\beta_n^2}$$

and $|\psi_n(t)| \leqslant \frac{2}{\beta_n^2}$ values for all $t \geqslant 0$.

Using (3) we have that:

$$\sum_{n=1}^{\infty} \frac{1}{s^{2\nu}+\beta_n^2} = s \sum_{n=1}^{\infty} \frac{s^{\nu-1}}{s^\nu (s^{2\nu}+\beta_n^2)} = s \sum_{n=1}^{\infty} \{\psi_n(t)\}.$$

The series of continuous function $\sum_{n=1}^{\infty} \psi_n(t)$ converges uniformly with respect to t on each finite interval $[0, T]$, and for all $t \geqslant 0$ is:

$$\left|\sum_{n=1}^{\infty} \psi_n(t)\right| \leqslant \sum_{n=1}^{\infty} |\psi_n(t)| \leqslant \sum_{n=1}^{\infty} \frac{2}{\beta_n^2}.$$

That shows that the series:

$$\sum_{n=1}^{\infty} \frac{1}{s^{2\nu}+\beta_n^2}$$

converges in the topology τ.

Proposition 5. *The operator* $\frac{e^{-s^\nu}}{1-e^{-s^\nu}}$ *can be written in the form of convergent series of operators:*

$$\frac{e^{-s^\nu}}{1-e^{-s^\nu}} = \frac{1}{s^\nu} - \frac{1}{2} + 2s^\nu \sum_{n=1}^{\infty} \frac{1}{s^{2\nu}+4\pi^2\pi^2}; \quad 0<\nu<1.$$

Proof. — From the proved proposition 3 and 4 follows that the suppositions **i.** and **ii.** of lemma 3 are satisfied. For the supposition **iii.** we have for $Re\, z \geqslant x_0 > 0$

$$\left|\sum_{n=1}^{\infty} \int_T^{\infty} e^{-zt} \psi_n(t)\, dt\right| \leqslant \sum_{n=1}^{\infty} \frac{1}{2n^2\pi^2} \int_T^{\infty} e^{-x_0 t} dt = \frac{e^{-x_0 T}}{x_0} \sum_{n=1}^{\infty} \frac{1}{2n^2\pi^2} \to 0, \ T \to \infty.$$

This completes the proof.

Proposition 6: *If there exists* $n_0 \in \mathcal{N}$ *such that:*

$$(n \geqslant n_0) \Rightarrow (\beta_n \geqslant n^{\frac{1}{2}+\varepsilon}); \quad \varepsilon > 0$$

and if D is algebraic derivate, then for each $k \in \mathcal{N}$ values:

$$D^k \sum_{n=1}^{\infty} \frac{s}{s^2+\beta_n^2} = \sum_{n=1}^{\infty} D^k \frac{s}{s^2+\beta_n^2},$$

and also

$$D^k \sum_{n=1}^{\infty} \frac{1}{s^{2\nu}+\beta_n^2} = \sum_{n=1}^{\infty} D^k \frac{1}{s^{2\nu}+\beta_n^2}; \quad 0<\nu<1.$$

these series of operators converge in the field $\mathcal{K}$.

The proof follows from continuity and linearity of the operator D^k [3] [4], and the fact that it is defined for each $a \in \mathcal{K}$. Namely:

$$D^k \sum_{i=1}^{n} \frac{s}{s^2+\beta_i^2} = \sum_{i=1}^{n} D^k \frac{s}{s^2+\beta_i^2}$$

$$D^k \sum_{i=1}^{n} \frac{1}{s^{2\nu}+\beta_i^2} = \sum_{i=1}^{n} D^k \frac{1}{s^{2\nu}+\beta_i^2}; \quad 0<\nu<1.$$

Using lemma 3, proposition 4 and the limit we have:

$$D^k \sum_{i=1}^{\infty} \frac{s}{s^2+\beta_i^2} = \sum_{i=1}^{\infty} D^k \frac{s}{s^2+\beta_i^2}$$

$$D^k \sum_{i=1}^{\infty} \frac{1}{s^{2\nu}+\beta_i^2} = \sum_{i=1}^{\infty} D^k \frac{1}{s^{2\nu}+\beta_i^2}; \quad 0<\nu<1,$$

and these series of operators converge in the field $\mathcal{K}$.

Using the representation of operator (5) and according to proposition 6, we have for example:

$$\frac{e^{-s^\nu}}{s^\nu(1-e^{-s^\nu})} - \frac{e^{-s^\nu}}{(1-e^{-s^\nu})^2} + \frac{1}{2s^\nu} = 16\pi^2 \sum_{n=1}^{\infty} \frac{n^2}{(s^{2\nu}+4n^2\pi^2)^2}; \quad 0<\nu<1. \tag{6}$$

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