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SOME FIRST INTEGRALS OF DIFFERENTIAL EQUATIONS OF MOTION OF A MECHANICAL SYSTEM

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Abstract — In a general case, differential equations of the motion of a mechanical system in a configuration space represent a non-linear system of differential equations notwithstanding what the external forces are; the solutions of these equations are not known in general. In the present paper some general first integrals are dealt with for a non-linear system of differential equations where a mechanical system is being acted upon by one class of potential, gyroscopic and dissipative gyroscopic forces. The absolute integral of a tensor that is applied directly to the absolute differential of a tensor, or in this particular case, to the absolute derivative of the generalized velocities vector, was used in order to obtain this result.

1. Introduction

The motion of a holonomic, scleronomic system with n degrees of freedom can be described, as is well known, by n Lagrangian equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i}-\frac{\partial T}{\partial q^i}=Q_i \qquad (i=1, 2, \ldots, n) \tag{1}$$

where the usual notations are used.

Both left-and right-hand sides of these equations, in a general case, are non-linear. On the left-hand side we have the squares of the generalized velocities $\dot{q}^i=\frac{dq^i}{dt}$, since [3]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i}-\frac{\partial T}{\partial q^i}=a_{ij}\left(\frac{d^2 q^j}{dt^2}+\Gamma^j_{lk}\frac{d q^k}{dt}\frac{b q^l}{dt}\right)=Q_i.$$

where

$$a_{ij}=\sum_{\nu=1}^{N} m_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\cdot\frac{\partial \boldsymbol{r}_\nu}{\partial q^j}=a_{ji}(q^1, q^2, \ldots, q^n) \tag{2}$$

are the covariant coordinates of a metric tensor of the configuration space V_n. The coefficients of connexion Γ^k_{ij} are the Christoffel symbols. Both a_{ij} and Γ^k_{ij}, in a general case, are non-linear functions of the coordinates q^i.

For such a general system, it is well known that from the equation (1) it is possible to obtain one first integral of energy, provided the force is a conservative one, i. e. if $Q_i = -\dfrac{\partial \Pi}{\partial q^i}$, where $\Pi = \Pi(q^1, q^2, \ldots, q^n)$ is the potential energy. The formulation of all the first integrals of the system (1) was not possible in a final form, not even when all the coordinates of the generalized force were equal to nil, i. e. for $Q_i = 0$. In this paper, however, we have succeeded in finding all the first integrals even for a broader class of generalized forces.

Let us assume that the force has the generalized potential $V = \Pi_i \dot{q}^i + \Pi$ where $\Pi_i = \Pi_i(q^1, q^2, \ldots, q^n)$ and $\Pi = \Pi(q^1, q^2, \ldots, q^n)$ are functions of co-ordinates $q^1, q^2, \ldots, q^n$ the gradient coordinates $\dfrac{\partial \Pi_i}{\partial q^j}$ and $\dfrac{\partial \Pi}{\partial q^j}$ of which form covariantly constant tensors. In addition, let the system be acted upon by dissipative forces of the form $-b_{ij}\dot{q}^j$ where

$$b_{ij} = b_{ji}(q^1, q^2, \ldots, q^n) \tag{3}$$

are dissipative coefficients forming covariantly constant tensors.

With these limitations, the generalized force will have the following form

$$Q_i = \frac{d}{dt}\frac{\partial V}{\partial \dot{q}^i} - \frac{\partial V}{\partial q^i} - b_{ij}\dot{q}^j = G_{ij}\dot{q}^j - b_{ij}\dot{q}^j - f_i,$$

where

$$G_{ij} = \frac{\partial \Pi_i}{\partial q^j} - \frac{\partial \Pi_j}{\partial q^i} = -G_{ij}(q^1, q^2, \ldots, q^n) \tag{4}$$

are the gyroscopic anti-symmetric coefficients, while

$$f_i = \frac{\partial \Pi}{\partial q^i} = f_i(q^1, q^2, \ldots, q^n) \tag{5}$$

are covariant constants of the potential force.

Hence, we are going to consider the integration of a system of non-linear equations of motion of the following form:

$$a_{ij}\left(\frac{d^2 q^j}{dt^2} = \Gamma^j_{kl}\frac{d q^k}{dt}\frac{d q^l}{dt}\right) = (G_{ij} - b_{ij})\dot{q}^j - f_i(q^1, q^2, \ldots, q^n). \tag{6}$$

In view of the fact that for this integration we shall use a tensorial integral operator, we propose to rearrange the equations (6) in another way.

The left hand side of (6) can be expressed by means of the absolute derivative

$$\frac{D\dot{q}^j}{dt} = \frac{d\dot{q}^j}{dt} + \Gamma^j_{kl}\dot{q}^k\frac{d q^l}{dt},$$

i. e.

$$a_{ij}\left(\frac{d^2 q^j}{dt^2} + \Gamma^j_{kl}\frac{d q^k}{dt}\frac{d q^l}{dt}\right) = \frac{D p_i}{dt} \tag{7}$$

since the absolute derivative of a metric tensor is equal to zero, whereas, on the other hand, $p_i = a_{ij}\dot{q}^j$ is the generalized impulse of the scleronomic dynamic system.

The contravariant coordinates $\dot{q}^j$ of the velocity that appear on the right-hand side of (6) can be expressed by means of the absolute derivative. In [6] a proof is given that in the configuration space $V_n \ni q^1, q^2, \ldots, q^n$ the relationship

$$\dot{q}^i = \frac{D\rho^i}{dt} = \frac{d\rho^i}{dt} + \Gamma^i_{jk}\rho^j \frac{dq^k}{dt} \tag{8}$$

is valid, in which $\rho^i = a^{ij}\rho_j$, and $\rho_j = \sum_{\nu=1}^{N} m_\nu \boldsymbol{r}_\nu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^j}$

Hence, by substituting (8) and (7) into (6), we obtain the following system of differential equations of motion

$$\frac{Dp_i}{dt} = (G_{ij} - b_{ij})\frac{D\rho^j}{dt} + f_i. \tag{9}$$

It is possible thus to apply this system of equations, and the equations (1) directly both to the systems with a finite number of degrees of freedom, and to the motion of a rigid body.

Prior to evaluating the first integrals of the equations (9), we propose to outline a theoretical approach to the integration of an absolute differential of a tensor or a vector.

2. The Integral of the Absolute Differential of a Vector

Two years ago in the paper [5] the concept of an absolute integral of a tensor was first introduced. It was proved that this integral can be applied very successfully in Mechanics. The equations (9) can be integrated very easily by means of this integral. Therefore, let us enumerate those properties of the absolute integral that we need here.

If we have a covariant vector v_i the absolute differential Dv_i of which is known with respect to the absolute integral of a tensor, then we have

$$\hat{\int} D\,v_i = v_i - {}_c\mathcal{A}_i \tag{10}$$

where ${}_c\mathcal{A}_i$ is the covariantly constant (or antiparallel) vector. The vector ${}_c\mathcal{A}_i$ is not constant, but is, in a general case, a function of the coordinates of the vector $v_i(t_0)$ at the point $q^i_0 = q^i(t_0)$, i. e. at the instant $t = t_0$, and of the coordinates q^i of any point at the instant t. The vector ${}_c\mathcal{A}_i$ is in fact the vector $v_i(t_0)$ at the point $t = t_0$, the latter vector having been displaced parallel to itself along the trajectory to a point. A parallel displacement of the vector $v_i(t_0)$ to any other point is possible by means of a bipunctual tensor which is to be found in papers [2] and [4] where they are called-shifter. For the sake of our considerations within the configuration space, the tensor a^K_i is determined by means of the covariant bipunctual fundamental tensor a_{iK}, the latter being defined in the same way as the metric tensor (3); but is calculated in two points, i. e.

$$a_{iK} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} \cdot \left(\frac{\partial \boldsymbol{r}_\nu}{\partial q^K}\right)_{q^K = q^K(t_0) = q^K}. \tag{11}$$

Thus, the coordinates of the bipunctual tensor occur as functions of the coordinates $q_0^1, \ldots, q_0^n$ of the initial point, and of the coordinates $q^1, q^2, \ldots, q^n$ of a point at the instant t. A composition of the tensor (11) and the contravariant metric tensor a^{ij} will yield

$$a_K^j = a_{iK} a^{ij} = a_K^j \left(\overset{1}{q_0}, \overset{2}{q_0}, \ldots, \overset{n}{q_0};\; q^1, q^2, \ldots, q^n \right) \tag{12}$$

where we use, similarly to the double tensor fields [2], capital letters in the index in order to denote the coordinates of a vector or a tensor at a fixed point. In a parallel displacement of a vector, for instance v_i, from the point $t = t_0$ to the point t, we have [2]

$$\mathcal{A}_i = v_K(t_0)\, a_i^K \qquad (i, K = 1, 2, \ldots, n) \tag{13}$$

where $a_i^K = a_{iL} a^{KL}$. Hence, the integral (10) becomes

$$\hat{\int} D v_i = v_i - v_K(t_0)\, a_i^K. \tag{14}$$

The accuracy of the relationships (10) or (14) is readily proved by the absolute differentiation

$$D \hat{\int} D v_i = D v_i$$

since $D\mathcal{A}_i = 0$, for the antiparallel vector, or $D v_K(t_0) = 0$ for the constant, and $D a_i^K = 0$, as proved in [1].

We propose to show other properties of the absolute integral, which we may need later on, in integrating the non-linear differential equations of motion (9).

We shall add here that the covariant coordinates of the vector v_i can be expressed in the well known manner by means of the corresponding contravariant coordinates v^j, i. e. $v_i = a_{ij} v^j$. In view of the fact that the absolute differential of the metric tensor is equal to zero, $D a_{ij} = 0$, the integral (10) can also be written in the following form

$$\hat{\int} a_{ij} D v^j = a_{ij} v^j - \mathcal{A}_i. \tag{15}$$

This property of the absolute differential is valid not only for the metric tensor a_{ij}, but also for all covariantly constant tensor, because if a tensor $U_i^j = U_i^j(q^1, q^2, \ldots, q^n)$ is a covariantly constant tensor then it is always possible to write $D\left(U_i^j v_k^l\right) = U_i^j D v_k^l$.

3. First Integrals of the differential Equations of Motion of Systems

Since the absolute differential of a scalar is equal to the common differential, i. e. $Dt = dt$ we can apply the absolute integral (10) to the equation (9),

$$\hat{\int} \{D p_i + (b_{ij} - G_{ij}) D \rho^j - f_i Dt\} = 0.$$

But, as the absolute integral of a summation is equal to the summation of absolute integrals [5], we shall have

$$\hat{\int} D p_i + \hat{\int} (b_{ij} - G_{ij}) D \rho^j - \hat{\int} f_i D t = 0.$$

Remembering that (3), (4) and (5) are covariantly constant tensors, on the grounds of (10) and (15) we obtain

$$p_i + (b_{ij} - G_{ij})\, \rho^j - t f_i = \mathcal{A}_i. \tag{16}$$

The coordinates of an autoparallel vector $\mathcal{A}_i$ are determined from the initial conditions

$$p_K(t_0) = a_{KL}(q_0^1, q_0^2, \ldots, q_0^n)\, \dot q_0^L, \quad b_{KL} = b_{KL}(q_0^1, q_0^2, \ldots, q_0^n)$$

$$G_{KL} = G_{KL}(q_0^1, q_0^2, \ldots, q_0^n), \quad \rho_0^L, \quad f_K = f_K(q_0^1, q_0^2, \ldots, q_0^n).$$

and transposed thereupon to the point under consideration at the instant t. This is performed as in the relationship (13) by means of the bipunctual tensor (12); thus we have

$$\mathcal{A}_i = [a_{KL}\dot q_0^L + (b_{KL} - G_{KL})\, \rho_0^L]\, a_i^K$$

or

$$\mathcal{A}_i = a_{iL}\dot q_0^L + a_i^K (b_{KL} - G_{KL})\, \rho_0^L,$$

since $a_{KL}\, a_i^K = a_{iL}$, [2].

Thus we obtain n first absolute integrals of the nonlinear differential equations of motion (6) in the form of

$$a_{ij}\dot q^j + (b_{ij} - G_{ij})\, \rho^j - t f_i = a_{iL}\dot q_0^L + (b_{KL} - G_{KL})\, \rho_0^L\, a_i^K \tag{17}$$

or by means of (8)

$$a_{ij}\frac{D\rho^j}{dt} + (b_{ij} - G_{ij})\, \rho^j - t f_i = a_{iL}\dot q_0^L + (b_{KL} - G_{KL})\, \rho_0^L\, a_i^K. \tag{18}$$

If a verification of the accuracy of the above procedures is required, then it can be simply made by a repeated absolute differentiation which brings us to the initial equations (9).

From (18) there follow simpler integrals for other special types of forces. Thus if the generalized force potential V is equal to the potential Π, $V = \Pi$, and $\frac{\partial V}{\partial q_i} = f_i(q^1, \ldots, q^n)$ or if $\frac{\partial \Pi_i}{\partial q^j} = \frac{\partial \Pi_j}{\partial q^i}$, then the gyroscopic forces do not act, since $G_{ij} = 0$, then from (18) we have

$$a_{ij}\frac{D\rho^j}{dt} + b_{ij}\rho^j - t f_i = a_{iL}\dot q_0^L + b_{KL}\rho_0^L\, a_i^K. \tag{19}$$

The dissipative forces $-b_{ij}\dot q^j$ will be equal to zero provided $b_{ij} = 0$. Therefore, if neither dissipative nor gyroscopic forces are acting, then from (17) therefore follows

$$a_{ij}\dot q^j = t f_i + a_{iL}\dot q_0^L \tag{20}$$

If there is also a condition that the potential covariantly constant forces be equal to zero, then we obtain first integrals of the equations (1) for motion under the influence of inertia in the following form:

$$a_{ij}\dot q^j = a_{iL}\dot q_0^L.$$

The class of generalized forces for which the relationships (17) or (18) are valid, or if there exist the equations (19) and (20), occurs very frequently in problems of Mechanics. Therefore, the general integrals that were obtained above can be directly used, without setting up the equations (1), for simpler examples of the motion of a mechanic holonomic scleronomic system with n degrees of freedom. This is also valid, of course, as a consequence, for systems with one degree of freedom only. Since the equations (1) or (6) can be applied to consider the motion of a rigid body, the integrals obtained in our foregoing discussions are valid for the motion of a rigid body.

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