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ON THE STABILITY OF VARIOUS EQUILIBRIUM AND STATIONARY MOTION OF NONHOLONOMIC SYSTEMS

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There is a very small portion of the literature on stability of motion and the equilibrium state of mechanical systems dedicated to the nonholonomic systems. Together with the investigation of the state of equilibrium of nonholonomic systems there appeared different points of views and different results were obtained by various authors. Thus, for example, Whittaker considered that with the equations of perturbed motion one should use the linearized equations of nonholonomic constraints which can be then integrated, what makes disappear the difference between holonomic and nonholonomic systems. Bottema, however, proves that such an approach to this problem is wrong (for more details about this problem see [5]).

In this study we consider the equilibrium state and the stationary motion of nonholonomic systems, with closed holonomic sub-system (autonomous system Chaplygin's). We examine the stability by Liapunov using his direct method wherein the equations of perturbed motion and nonholonomic constraints are not linearized, and in this way all the uncertainess about this problem are avoided. Besides, following the works [1], [2], and [3], the Liapunov's function is chosen in the form of a sum of kinetic energy of the system and of one positive-definite function which depends only upon the position coordinates of the system and in this manner the practical part of the work about the investigation of stability is greatly minimized. Finally, the obtained expression, which helps us to judge about the stability of equilibrium's state and about stationary motion is rather simple and it does not request employing of the differential equations of system motion.

1. We consider mechanical system whose configuration is determined by n Lagrange coordinates q^i, which is subject to k nonholonomic constraints

$$\omega_{li}\,\dot{q}^i=0 \qquad (l=1,\ \dots,\ k;\quad i=1,\ \dots,\ n)$$

(we use Einstein's convention about summation by repeated indices). It is supposed that:

a) The system is scleronomic and the kinetic energy can be written in the form

$$T=T(q',\ \dots,\ q^m;\quad \dot{q}',\ \dots,\ \dot{q}^n) \qquad (m=n-k). \tag{1.1}$$

b) Nonholonomic constraints can be rewritten in the form

$$\dot{q}^h=b^h_\mu(q^1,\ \dots,\ q^m)\,\dot{q}^\mu \qquad (\mu=1,\ \dots,\ m;\quad h=m+1,\ \dots,\ n). \tag{1.2}$$

By substituting generalized velocities $\dot{q}^h$ with the relations (1.2) we obtain transformed expression for kinetic energy T which, in respect of hypothesis a) and b), represents homogenous quadratic form by $\dot{q}^\mu$ in which the coefficients depend only upon the first m coordinates

$$(1.3) \qquad 2\tilde{T} = a_{\mu\nu}(q', \ldots, q^m)\dot{q}^\mu \dot{q}^\nu \, (\mu, \nu = 1, \ldots, m).$$

In the following text it is considered that all indices have always the same values.

Let the system be subject to the generalized forces

$$Q_i = Q_i(q', \ldots, q^m; \dot{q}', \ldots, \dot{q}^n) \qquad (i = 1, \ldots, n).$$

For the investigation of motion of such a system we can use Chapligin's equations (special case Voronec's equations):

$$(1.4) \qquad \frac{d}{dt}\frac{\partial \tilde{T}}{\partial \dot{q}^\mu} + \frac{\partial \tilde{T}}{\partial q^\mu} - \left(\frac{\partial \tilde{T}}{\partial \dot{q}^h}\right)\gamma^h_{\mu\nu}\dot{q}^\nu - Q_\mu - Q_h b^h_\mu = 0$$

where

$$(1.5) \qquad \gamma^h_{\mu\nu} = \frac{\partial b^h_\mu}{\partial q^\nu} - \frac{\partial b^h_\nu}{\partial q^\mu} = -\gamma_{\nu\mu}^{\ h}$$

and the symbol above the bracket in the third member on the left side of equation denotes that in the expression which is in the bracket where was performed the substitution of all generalized velocities $\dot{q}^h$ with the (1.2). The equations (1.4) represent the complet system of m equations with respect to the coordinates q^μ from which we can obtain solutions $q^\mu = q^\mu(t)$. The other coordinates are obtained from (1.2) after expressing their right sides in the function of t. From (1.4) it follows that the equilibrium position is determined by equations

$$(1.6) \qquad Q_\mu + Q_h b^h_\mu = 0$$

These m equations determine in n dimension space of V_n coordinates $q^1, q^2, \ldots, q^n$ equilibrium variety O_p whose dimension is $p \geqslant n - m = k$ (symbol $\geqslant$ is present because all equations of the system need not be independent). Now, we investigate the stability of equilibrium variety in the Liapunov sense by using the direct method of Liapunov. Let us choose the point $M \in O_p$ with coordinates $q^1_0, q^2_0, \ldots, q^n_0$. Without loss of generality of investigation we can choose the system of coordinates so that $q^1_0 = q^2_0 = \cdots = q^n_0 = 0$. In this manner the equations of perturbated equilibrium state of the system will have the form (1.4) and the initial values of the perturbation must be in accordance with the equations (1.2). The Liapunov's function we use in the form of the sum of kinetic energy (which is strictly positive definite function) and the function W which is strictly positive definite in the region H is continuous together with its partial derivative $\frac{\partial W}{\partial q^\mu}$ in the same region:

$$(1.7) \qquad V = \tilde{T} + W; \qquad W = W(q^1, \ldots, q^m) \in C^{(0,1)}(H)$$

The rate of time of this function is formed in the sense of equations of disturbed equilibrium state (1.4):

$$\dot{V} = \frac{dV}{dt} = \frac{d\tilde{T}}{dt} + \frac{dW}{dt} = \left(\frac{d}{dt}\frac{\partial \tilde{T}}{\partial \dot{q}^\mu} - \frac{\partial \tilde{T}}{\partial q^\mu}\right)\dot{q}^\mu + \frac{\partial W}{\partial q^\mu}\dot{q}^\mu =$$

$$= \widetilde{\left(\frac{\partial T}{\partial \dot{q}^h}\right)}\gamma^h_{\mu\nu}\dot{q}^\mu \dot{q}^\nu + \left(Q_\mu + Q_h b^h_\mu + \frac{\partial W}{\partial q^\mu}\right)\dot{q}^\mu.$$

Taking into consideration the asymetric of coefficients $\gamma^h_{\mu\nu}$ we obtain at last:

$$\dot{V} = \left(Q_\mu + Q_h b^h_\mu + \frac{\partial W}{\partial q^\mu}\right)\dot{q}^\mu \tag{1.8}$$

From this follows the following conclusion:

If there exists strictly positive definite function $W \in C^{(0,1)}(H)$ so that for nonholonomic Chaplygin's system in the region H

$$\left(Q_\mu + Q_h b^h_\mu + \frac{\partial W}{\partial q^\mu}\right)\dot{q}^\mu \leqslant 0$$

the point $M \in O_p$ is stable equilibrium position related to coordinates q^μ.

If at the same time $\dot{V}$ is strictly negative definite function, the position of equilibrium is asymptotic stable.

From this, however, we can not come to any conclusion about coordinates q^h. This, our case concerns the stability with respect to the part of coordinates q^μ. With the help of equations (1.2) we can come although to an interesting conclusion. Let us suppose that the observed point $M \in O_p$ is asymptotic stable equilibrium position of the nonholonomic system. Then, it follows that

$$\lim_{t\to\infty} q^\mu(t) = 0, \qquad \lim_{t\to\infty} \dot{q}^\mu(t) = 0$$

and from (1.2) we obtain

$$\lim_{t\to\infty} \dot{q}^h(t) = 0; \qquad \lim_{t\to\infty} q^h(t) = \text{const.} \neq q^h_0$$

That means that the system may not oscillate round one fixed point in the V space. This point can displace over hypersurface which is represented by equilibrium various. In general case some equilibrium various O_p will have the points in which the equilibrium is stable, and the points in which the equilibrium is unstable. If we mark the set of the first with O^s_p and the set of the others with O^n_p then we can speak about the stable $O^s_p \subset O_p$ and unstable $O^n_p \subset O_p$ region of equilibrium various.

2. Let us consider some special cases.

A. Let us suppose that the nonholonomic system is subjected to generalized forces only with the potential which is presented by the analytic function $\Pi = \Pi(q^1, \ldots, q^n)$ Expending Π on a power series round the point $M(0, \ldots, 0) \in O^s_p$ we obtain

$$\Pi = c_0 + c_i q^i + \frac{1}{2} c_{ij} q^i q^j + \Pi^* \qquad (i, j = 1, \ldots, n) \tag{2.1}$$

where

$$c_0 = \Pi(0, \ldots, 0): \quad c_i = \left(\frac{\partial \Pi}{\partial q^i}\right)_0; \quad c_{ij} = \left(\frac{\partial^2 \Pi}{\partial q^i \partial q^j}\right)_0$$

and Π^* is the rest of our series. In respect to $q^i_0 = 0$ then we obtain

$$c_\mu + c_h b^h_{\mu 0} = 0 \qquad (b^h_{\mu 0} = b^h_\mu(0, \ldots, 0)). \tag{2.2}$$

From (2.2) it is clear, that in the equilibrium position of nonholonomic system the generalised forces $Q_i = -\dfrac{\partial \Pi}{\partial q^i}$, differing from holonomic systems may not be equal to zero. From this follows that the series (2.1) will have also linear terms, which is not the case with the holonomic system. Besides we can always choose $c_0 = 0$. If the potential energy Π possesses the strict minimum at the point $q^i = 0$ in respect to coordinates q^μ, then considering that $W = \Pi$ from (1.9) it follows that this position of the equilibrium is stable. Therefore, as a generalization of Lagrange's theorem for holonomic conservative systems we can obtain for the nonholonomic system the following conclusion.

Equilibrium position of nonholonomic systems of Chapligin's type wherein the potential energy has the strict minimum related to the part of variables q^μ is the position of the stable equilibrium.

B. Let the observed system, besides potential forces $Q_i^{(1)}$ be also subjected to the forces, which are linear function of the generalised velocities:

$$Q_i^{(2)} = -f_{ij}\dot{q}^j$$

By separating the matrix coefficients f_{ij} in the symmetric and screwsymmetric part and by introducing Rayleigh's function we can obtain

$$Q_i = Q_i^{(1)} + Q_i^{(2)} = -\frac{\partial \Pi}{\partial q^i} - f_{(ij)}\dot{q}^j - f_{[ij]}\dot{q}^j = -\frac{\partial \Pi}{\partial q^i} - \frac{\partial \Phi}{\partial \dot{q}^i} - \gamma_{[ij]}\dot{q}^j (\gamma_{[ij]} = -\gamma_{[ji]}).$$

Introducing this expression for generalized forces into (1.9) we obtain

$$\left(Q_\mu + Q_h b_\mu^h + \frac{\partial W}{\partial q^\mu}\right)\dot{q}^\mu = \left(-\frac{\partial \Pi}{\partial q^\mu} - \frac{\partial \Pi}{\partial q^h} b_\mu^h\right)\dot{q}^\mu - 2\Phi.$$

From this expression results the direct conclusion which is exposed in [8]: Dissipative and gyroscope forces do not disturb the stability of equilibrium position of nonholonomic system. Dissipative forces can fix the stable equilibrium position to asymptotic stability.

3. Within the system, which is considered under the 1. and 2. it is obvious that the coordinates q^h are cyclical and the coordinates q^μ noncyclical. Such systems, in some cases and for definite values q_0^μ, $\dot{q}_0^\mu = 0$, $\dot{q}_0^h$ can perform stationary motion, i. e., motion in which all position coordinates and the cyclical velocites are constant during the entire time of motion and the cyclical coordinates are linear functions of time:

$$\text{(3.1)} \qquad q^\mu = \text{const.} = q_0^\mu; \quad \dot{q}^h = \text{const.} = \dot{q}_0^h; \quad q^h = \dot{q}_0^h t + q_0^h.$$

The constants q_0^μ can be determined by with the chosen values $\dot{q}_0^h$ from the equation of motion (1.4), taking that $\dot{q}^\mu = 0$

$$\text{(3.2)} \qquad Q_\nu(q_0^\mu, \dot{q}_0^h) + O_r(q_0^\mu, \dot{q}_0^h)\, b_\nu^r(q_0^\mu) = 0 \qquad (r = m+1, \ldots, n)$$

System (3.2) determines in the space V_n of coordinates q^i various O_s of the stationary motions by the given values $\dot{q}_0^h$, dimensions $s \geqslant n - m$ by hypothesis that the solution of equations (3.2) is

$$q_0^\mu = f^\mu(\dot{q}_0^{m+1}, \ldots, \dot{q}_0^n)$$

uniformy defined, meaning that there is no points of bifurcation.

Let us consider the stability by Liapunov of a certain stationary motion which belongs to O_s. Whithout reducing the generality it can be considered that for the chosen fixed values $\dot{q}_0^h$ all position coordinates are $q_0^\mu = 0$. It can be considered that to the stationary motion in the space V_n corresponds equilibrium position in the various O_s. Let us report very little initial disturbances to the observed system. We mention that they can not be arbitrary but they have to be in accordance with the equations of nonholonomic constraints (1.2). For disturbed motion it is

$$q^\mu = \xi^\mu, \quad \dot{q}^\mu = \eta^\mu, \quad q^h = q_0^h + \xi^h, \quad \dot{q}^h = \dot{q}_0^h + \eta^h \tag{3.3}$$

where ξ^i and η^i are disturbances.

As the cyclic coordinates in the linear way depend on time, the system in respect to them is certainly unstable. Therefore we omit them in further investigation. Introducing (3.3) into (3.2) we obtain

$$[Q_\nu (q^\mu, \dot{q}_0^h + \eta^h) + Q_r (q^\mu, \dot{q}^h + \eta^h) b_\nu^r q^\mu)]_{q_\mu = 0} \neq 0.$$

That is to say the equations (3.2) need not be exact even after the substitution with assistance of (3.3). Therefore the stationary motion will be subject to costant disturbances. We will not consider this case here, and to avoid it, we shall put tha. all η^h are equal to zero. In this way we will investigate the stability in respect to the position of coordinates and their generalized velocities

$$q^\mu = \xi^\mu, \quad \dot{q}^\mu = \eta^\mu$$

that represents the conditional stability by Liapunov.

The expression for kinetic energy of disturbed motion is

$$2\, T^* = a_{\mu\nu} (\xi^1, \ldots, \xi^m)\, \eta^\mu \eta^\nu. \tag{3.4}$$

The equation of nonholonomic constraints in variations is

$$\dot{q}_0^h = b_\mu^h (\xi^1, \ldots, \xi^m)\, \eta^\mu \tag{3.5}$$

and the Chapligin's equations in variations are

$$\frac{d}{dt} \frac{\partial \tilde{T}^*}{\partial \eta^\mu} - \frac{\partial \tilde{T}^*}{\partial \xi^\mu} - \left(\widetilde{\frac{\partial T^*}{\partial \dot{q}^h}} \right) \gamma_{\mu\nu}^{*h} \eta^\nu - Q_\mu^* - Q_h^* b_\mu^h = 0 \tag{3.6}$$

where $*$ denotes that the corresponding terms are given by the help of disturbances ξ^μ and η^μ.

The stability of observed stationary motion we investigate with the help of direct method by Liapunov. As $\tilde{T}^*$ is obviously positive definite quadratic form in respect to η^μ, we can choose the function of Liapunov in the form

$$V = \tilde{T}^* + W^* \tag{3.7}$$

where

$$W^* = W^* (\xi^1, \ldots, \xi^m) \in C^{(0,1)} (H), \quad W^* \geqslant 0, \quad W^* = 0 \Leftrightarrow \xi^1 = \cdots = \xi^m = 0.$$

Analogously to the calculus carried out in 1. we obtain

$$\dot{V} = \left(Q_\mu^* + Q_h^* b_\mu^h + \frac{\partial W^*}{\partial \xi^\mu} \right) \eta^\mu. \tag{3.8}$$

From the previous expression here results the following conclusion:

If for the stationary motion $q^{\mu}=\text{const.}$, $\dot{q}^{h}=\text{const.}$ of nonholonomic system exists positive definite function $W \in C^{(0,1)}(H)$ for which it is

$$\left(Q_{\mu}^{*}+Q_{h}^{*} b_{\mu}^{h}+\frac{\partial W}{\partial \xi^{\mu}}\right) \eta^{\mu} \leqslant 0$$

that motion is stable. Besides, if (3.8) is strictly negative definite function, then the motion is asymptoticly stable.

To show that the dissipative forces do not disturb the stability of stable stationary motion it is easy to prove it in the way analogous to that in the item 1.

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