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S-BASES OF PROPOSITIONAL ALGEBRA

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1. The symmetry of propositional functions usualy simplifies the synthesis of switching circuits. Moreover, symmetric functions have algebraic properties which make it desirable to treat them as a separate class. The algebraic treatment of symmetric functions is derived from a general definition of the symmetric function and a number of theorems, first stated by C. E. Shannon in [17] and [18].

In this paper, the following problem is discussed and solved:

To find all subsets U of the set F_2 of all propositional functions which are S-bases for F_2 in the following sense:

1. U contains only symmetric functions.

2. Starting from elements of U, every element of F_2 can be obtained by composition.

3. No proper subset of U has the property 2. of U.

2. Definition 1. *An n-place propositional function $f(x_1, x_2, \ldots, x_n)$ is said to be symmetric if and only if the following equality is valid*:

$$f(x_1, x_2, \ldots, x_n)=f(y_1, y_2, \ldots, y_n),$$

where $(y_1, y_2, \ldots, y_n)$ is an arbitrary permutation of $(x_1, x_2, \ldots, x_n)$.

Since any permutation of variables may be obtained by successive interchanges of two variables, a necessary and sufficient condition that a function be symmetric is that any interchange of two variables leaves the function unaltered.

Starting from definition 1, the following property of symmetric functions can be easily proved ([14], p. 178.):

Lemma 1. *If the perfect disjunctive normal form (PDNF) of a n-place symmetric function contains a unit constituent with m, $(0 \leqslant m \leqslant n)$, unnegated variables, then that PDNF contains every unit constituent with the same property.*

Definition 2. *A propositional function is said to be basic symmetric function if and only if every unit constituent of its PDNF can be obtained by some permutation from an arbitrary unit constituent.*

A basic symmetric function is uniquely determined by two numbers — the number of independent variables n and so called a-number i. e. the number of unnegated variables k contained in an arbitrary unit constituent of *PDNF* of that function.

Let S_k^n denote n-place basic symmetric function with a-number k. Given n, there exist exactly $n+1$ basic symmetric functions: $S_0^n, S_1^n, \ldots, S_n^n$.

The following property of symmetric functions can be easily proved ([14], p. 178.):

L e m m a 2. *Every symmetric function can be uniquely represented as a disjunction of basic symmetric functions.*

This property enables a suitable way of notation for symmetric functions writing:

$$S_{k_1}^n \vee S_{k_2}^n \vee \cdots \vee S_{k_m}^n \overset{\text{Def.}}{=} S_{k_1, k_2, \ldots, k_m}^n, \qquad (n \geqslant 1).$$

The constants 0 and 1 are symmetric functions and we will denote them by $S_{\varnothing}^n$ and $S_{0,1,2,\ldots,n}^n$ respectively.

Let F_2 denote the set of all propositional functions, F_2^n the set of all n-place propositional functions and S the set of all symmetric propositional functions. It can be easily shown that the number of n-place symmetric functions is:

$$k\left(F_2^n \cap S\right) = 2^{n+1}.$$

For further investigations the following subsets of F_2 are important:

1. $T_0 = \{f \mid f \in F_2,\ f(0, 0, \ldots, 0) = 0\}$, i. e. the set of all functions preserving zero.

2. $T_1 = \{f \mid f \in T_2,\ f(1, 1, \ldots, 1) = 1\}$, i. e. the set of all functions preserving one.

3. $A = \{f \mid f \in F_2,\ f(x_1, x_2, \ldots, x_n) = \bar{f}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)\}$, i. e. the set of all self-dual functions.

4. $L = \{f \mid f \in F_2,\ f(x_1, x_2, \ldots, x_n) = a_0 + a_1 x_1 + \cdots + a_n x_n \pmod 2\}$, where $a_i \in \{0, 1\}$, $(i = 1, 2, \ldots, n)$, i. e. the set of all linear functions.

5. $M = \{f \mid f \in F_2,\ \forall (\hat{a}, \hat{b})\ (\hat{a} \leqslant \hat{b} \Rightarrow f(\hat{a}) \leqslant f(\hat{b}))\}$, where $\hat{a} = (a_1, a_2, \ldots, a_n)$, $\hat{b} = (b_1, b_2, \ldots, b_n)$; $a_i, b_i \in \{0, 1\}$, $(i = 1, 2, \ldots, n)$, and
$\hat{a} \leqslant \hat{b} \overset{\text{Def.}}{\Leftrightarrow} a_i \leqslant b_i$, $(i = 1, 2, \ldots, n)$, i. e. the set of all monotonic functions.

The following relations, which can be found in [1], are valid:

L e m m a 3. $T_0 \cap T_1 \cap L \cap F_2^n \subseteq A \cap F_2^n$,

L e m m a 4. $T_0 \cap T_1 \cap A \cap L \cap M \cap F_2^n = \{x_1, x_2, \ldots, x_n\}$.

3. D e f i n i t i o n 3. *S-basis of propositional algebra is a basis which contains only symmetrical functions.*

First, we shall give some lemmas.

L e m m a 5. *The only degenerated n-place symmetric functions* (*i. e. symmetric functions which actually do not depend on all n variables*), $(n \geqslant 1)$, *are the constants* 0 *and* 1.

C o r o l l a r y. *There are exactly* $2^{n+1} - 2$ *nondegenerated n-place symmetric functions.*

L e m m a 6. *The following equalities are valid*:

$$k\left((S \cap F_2^n) \setminus (T_0 \cup T_1)\right) = k\left((S \cap T_0 \cap F_2^n) \setminus T_1\right) =$$

$$= k\left((S \cap T_1 \cap F_2^n) \setminus T_0\right) = k\left(S \cap T_0 \cap T_1 \cap F_2^n\right) = 2^{n-1}.$$

L e m m a 7. *If n is an even number, then every n-place symmetric function is not self-dual.*

P r o o f. If $n=2m$, $(m\in N)$, then PDNF of the function

$$f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{2m})$$

either contains the basic symmetric function S_m^{2m} or not.

In the first case, PDNF of the function f contains all unit constituents with m negated variables, so we can find two opposite valuations of the variables $x_1, x_2, \ldots, x_{2m}$, e. g. $(\underbrace{0, \ldots, 0}_{m}, \underbrace{1, \ldots, 1}_{m})$ and $(\underbrace{1, \ldots, 1}_{m}, \underbrace{0, \ldots, 0}_{m})$, such that

$$f(0, \ldots, 0, 1, \ldots, 1)=f(1, \ldots, 1, 0, \ldots, 0)=1$$

holds.

In the second case, PDNF of the function f does not contain such unit constituents, so it is

$$f(0, \ldots, 0, 1, \ldots, 1)=f(1, \ldots, 1, 0, \ldots, 0)=0.$$

In both cases, the function f is not self-dual, for a self-dual function has opposite values on the opposite valuations of variables.

L e m m a 8. *There are exactly $n+2$ n-place monotonic symmetric functions: $S^n_{0,1,2,\ldots,n}=1$; $S^n_{1,2,\ldots,n}$; $S^n_{2,\ldots,n}$; $\ldots$; S^n_n; $S^n_\varnothing=0$. If n is even, none of them is self-dual; if n is odd, only one among them is self-dual, that is the function $S^{2m+1}_{m+1,m+2,\ldots,2m,2m+1}$, $(n=2m+1)$.*

P r o o f. The first part of proposition is a consequence of the obvious fact that PDNF of a symmetric monotonic function containing S^n_i, must contain all S^n_j, for $i\leqslant j\leqslant n$.

According to lemma 7, if n is even, none of monotonic n-place symmetric functions is self-dual.

If n is odd, i. e. $n=2m+1$, then $S^{2m+1}_{m+1,m+2,\ldots,2m,2m+1}$ is self-dual function, for it has opposite values on the opposite valuations of variables. The function $S^{2m+1}_{i,i+1,\ldots,2m+1}$, $i\neq m+1$, cannot be self-dual, for the numbers of 0's and 1's in the truth table of that function are different.

L e m m a 9. *The only nondegenerated n-place linear symmetric functions are:*

$$s_1(x_1, x_2, \ldots, x_n)=\sum_{i=1}^{n} x_i \pmod 2$$

and

$$s_2(x_1, x_2, \ldots, x_n)=1+\sum_{i=1}^{n} x_i \pmod 2.$$

P r o o f. The symmetry of the functions s_1 and s_2 is obvious. On the other hand, for every linear function $f(x_1, x_2, \ldots, x_n)=a_0+a_1x_1+a_2x_2+\cdots+a_nx_n$ (mod 2), different from s_1 and s_2, must be $a_i=0$ for at least one $i\in\{1, 2, \ldots, n\}$, so f is degenerated, since, it cannot be symmetric.

Note the following properties of s_1 and s_2:

The function $s_1(x_1, x_2, \ldots, x_n)$ is: nonmonotonic, if $n>1$; self-dual, if n is odd; not self-dual, if n is even; preserving zero; preserving one, if n is odd; not preserving one, if n is even.

The function $s_2(x_1, x_2, \ldots, x_n)$ is: nonmonotonic; self-dual, if n is odd; not self-dual, if n is even; not preserving zero; preserving one, if n is even; not preserving one, if n is odd.

4. According to [1], the function f belonging to the set T_1 is said to have the property T_1 e. c. The function having e. g. the properties T_0, T_1, L and not having the properties A, M is said to be a $|T_0, T_1, L|$-function. Krnič in [1], p. 27, proved that the number of different types of functions is 15. Every function belongs to one and only one type. We are going to prove that every of 15 types contains symmetric functions. More precisely, if $k\,|T_0, T_1, A|_s(n)$ denote the number of n-place symmetric $|T_0, T_1, A|$-functions e. c., then the following proposition is valid:

Theorem 1.

1. $$k\,|\varnothing|_s(n)=\begin{cases}2^{n-1}, & \text{if } n \text{ is even}\\ 2^{n-1}-2^{\frac{n-1}{2}}, & \text{if } n \text{ is odd.}\end{cases}$$

The functions of this type are: $S^n_{0,k_1,k_2,\ldots,k_l}$, where $\{k_1, k_2, \ldots, k_l\}\subseteq H=\{1, 2, \ldots, n-1\}$, and if n is odd, the following condition is satisfied: there exists at least one j such that j and $n+j$ either both belong or both do not belong to the set $\{k_1, k_2, \ldots, k_l\}$.

2. $$k\,|A|_s(n)=\begin{cases}0, & \text{if } n \text{ is even}\\ 2^{\frac{n-1}{2}}-1, & \text{if } n \text{ is odd.}\end{cases}$$

If n is odd, the functions of this type are: $S^n_{0,k_1,k_2,\ldots,k_l}$, where $\{k_1, k_2, \ldots, k_l\}\subseteq H$, and for every $j\in H$, one and only one among j and $n-j$ belongs to the set $\{k_1, k_2, \ldots, k_l\}$. The function $s_2(x_1, x_2, \ldots, x_n)=S^n_{0,2,4,\ldots,n-1}$ is excluded.

3. $$k\,|T_0|_s(n)=\begin{cases}2^{n-1}-2, & \text{if } n \text{ is even}\\ 2^{n-1}-1, & \text{if } n \text{ is odd.}\end{cases}$$

The functions of this type are: $S^n_{k_1,k_2,\ldots,k_l}$, where $\{k_1, k_2, \ldots, k_l\}\subseteq H$; the functions $S^n_\varnothing=0$ and $s_1(x_1, x_2, \ldots, x_n)=S^n_{1,3,5,\ldots,n-1}$ are excluded if n is even, and only $S^n_\varnothing=0$ if n is odd.

4. $$k\,|T_1|_s(n)=\begin{cases}2^{n-1}-2, & \text{if } n \text{ is even}\\ 2^{n-1}-1, & \text{if } n \text{ is odd.}\end{cases}$$

The functions of this type are: $S^n_{0,k_1,k_2,\ldots,k_l,n}$, where $\{k_1, k_2, \ldots, k_l\}\subseteq H$; the functions $S^n_{0,1,2,\ldots,n}=1$ and $s_2(x_1, x_2, \ldots, x_n)=S^n_{0,2,4,\ldots,n}$ are excluded if n is even, and only $S^n_{0,1,2,\ldots,n}=1$ if n is odd.

5. $$k\,|A, L|_s(n)=\begin{cases}0, & \text{if } n \text{ is even}\\ 1, & \text{if } n \text{ is odd.}\end{cases}$$

If n is odd, the only function of this type is $s_2(x_1, x_2, \ldots, x_n)=S^n_{0,2,4,\ldots,n-1}$.

6. $k \mid T_1, L\mid_s(n) = \begin{cases} 1, \text{ if } n \text{ is even} \\ 0, \text{ if } n \text{ is odd.} \end{cases}$

If n is even, the only function of this type is $s_2(x_1, x_2, \ldots, x_n) = S^n_{0, 2, 4, \ldots, n}$.

7. $k \mid T_0, L\mid_s(n) = \begin{cases} 1, \text{ if } n \text{ is even} \\ 0, \text{ if } n \text{ is odd.} \end{cases}$

If n is even, the only function of this type is $s_1(x_1, x_2, \ldots, x_n) = S^n_{1, 3, 5, \ldots, n-1}$.

8. $k \mid T_0, T_1\mid_s(n) = \begin{cases} 2^{n-1} - n, \text{ if } n \text{ is even} \\ 2^{n-1} - 2^{\frac{n-1}{2}} - n + 1, \text{ if } n \text{ is odd.} \end{cases}$

The functions of this type are: $S^n_{k_1, k_2, \ldots, k_l, n}$, where $\{k_1, k_2, \ldots, k_l\} \subseteq H$ and, if n is odd, there exists at least one j such that j and $n-j$ either both belong or both do not belong to the set $\{k_1, k_2, \ldots, k_l\}$. In both cases (if n is even and if n is odd), the functions $S^n_{j, j+1, \ldots, n}$, where $j = 1, 2, \ldots, n$, are excluded.

9. $k \mid T_1, L, M\mid_s(n) = 1$.

The only function of this type is the constant $1 = S^n_{0, 1, \ldots, n}$.

10. $k \mid T_0, L, M\mid_s(n) = 1$.

The only function of this type is the constant $0 = S^n_{\emptyset}$.

11. $k \mid T_0, T_1, M\mid_s(n) = \begin{cases} n, \text{ if } n \text{ is even} \\ n-1, \text{ if } n \text{ is odd.} \end{cases}$

The functions of this type are: $S^n_{j, j+1, \ldots, n-1, n}$, where $j \in \{1, 2, \ldots, n\}$, if n is even and $j \in \left\{1, 2, \ldots, \frac{n-1}{2}, \frac{n+3}{2}, \ldots, n\right\}$, if n is odd.

12. $k \mid T_0, T_1, A\mid_s(n) = \begin{cases} 0, \text{ if } n \text{ is even or } n = 1 \\ 2^{\frac{n-1}{2}} - 2, \text{ if } n \text{ is odd} \geqslant 3. \end{cases}$

If n is odd $\geqslant 3$, the functions of this type are: $S^n_{k_1, k_2, \ldots, k_l, n}$, where $\{k_1, k_2, \ldots, k_l\} \subseteq H$, and for every $j \in H$, one and only one among j and $n-j$ belongs to the set $\{k_1, k_2, \ldots, k_l\}$. The functions $S^n_{\frac{n+1}{2}, \frac{n+3}{2}, \ldots, n}$ and $s_1(x_1, x_2, \ldots, x_n) = S^n_{1, 3, 5, \ldots, n}$ are excluded.

13. $k \mid T_0, T_1, A, M\mid_s(n) = \begin{cases} 0, \text{ if } n \text{ is even or } n = 1 \\ 1, \text{ if } n \text{ is odd } \geqslant 3. \end{cases}$

If n is odd, the only function of this type is $S^n_{\frac{n+1}{2}, \frac{n+3}{2}, \ldots, n}$.

14. $k \mid T_0, T_1, A, L \mid_s (n) = \begin{cases} 0, & \text{if } n \text{ is even or } n=1 \\ 1, & \text{if } n \text{ is odd } \geqslant 3. \end{cases}$

If n is odd, the only function of this type is $s_1(x_1, x_2, \ldots, x_n) = S^n_{1,3,5,\ldots,n}$.

15. $k \mid T_0, T_1, A, L, M \mid_s (n) = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{if } n>1. \end{cases}$

If $n=1$, the only function of this type is $f(x_1)=x_1$.

Remark. $|\varnothing|$-functions i. e. the functions which have not any of properties T_0, T_1, A, L, M are so called Sheffer functions. Each of them considered separately represents a basis.

Proof.

1. If n is even, then, according to lemma 6, there are 2^{n-1} n-place symmetric functions not preserving any of two constants. Lemma 7 implies that all those functions are not self-dual. According to a well known statement ([5], p. 44.), then, they are nonlinear and nonmonotonic too.

If n is odd, i. e. if $n=2m+1$, $(m=0, 1, \ldots)$, then, according to lemma 6, there are $2^{n-1}=2^{2m}$ n-place symmetric functions not preserving any of two constants. Let us calculate the number of self-dual functions among them.

A self-dual $(2m+1)$-place symmetric function not preserving any of two constants is determined if one knows which of the basic symmetric functions among $S_1^{2m+1}, S_2^{2m+1}, \ldots, S_m^{2m+1}$, are included in its PDNF, for PDNF of a self-dual function contains one and only one of S_i^{2m+1} and S_{2m+1-i}^{2m+1}, $(i=1,2, \ldots, m)$. There are, however, $\binom{m}{0}+\binom{m}{1}+\cdots+\binom{m}{m}=2^m=2^{\frac{n-1}{2}}$ ways to include some of S_i^{2m+1}, $(i=1, 2, \ldots, m)$, in PDNF, hence, there are the same number of self-dual symmetric functions not preserving any of two constants. Now, the number of n-place symmetric functions not preserving any of two constants, which are not self-dual, is equal to $2^{n-1}-2^{\frac{n-1}{2}}$. They are nonlinear and nonmonotonic too.

2. If n is even, the statement is a consequence of lemma 7.

If n is odd, then, there are $2^{\frac{n-1}{2}}$ n-place symmetric self-dual functions not preserving any of two constants, as we proved. Clearly, they are all nonmonotonic. The only linear function $s_2(x_1, x_2, \ldots, x_n)$ must be excluded.

3. According to lemma 6, there are 2^{n-1} n-place symmetric functions preserving zero and not preserving one. Obviously, none of them is self-dual. They are all nonlinear — with the exception of $s_1(x_1, x_2, \ldots, x_n)$ if n is even, and nonmonotonic — with the exception of the constant 0.

4. The proof is analogous to the previous. Here, we must exclude the functions 1 and $s_2(x_1, x_2, \ldots, x_n)$ if n is even and only the constant 1 if n is odd.

5, 6, 7. The statements are direct consequences of lemma 9 and the properties of the functions s_1 and s_2.

8. If n is even, the statement is a consequence of lemmas 3, 6,7 and 8.

If n is odd, then, similarly as in the first part of this proof, it can be proved that the number of n-place symmetric functions preserving both constants, which are not self-dual, is equal $2^{n-1} - 2^{\frac{n-1}{1}}$. According to lemma 3, they are all nonlinear. Yet, we must exclude $n-1$ monotonic functions which are not self-dual (lemma 8).

9, 10. It can be easily verified.

11. According to lemma 8.

12. It is a consequence of lemma 7 if n is even, and of lemmas 6 and 8 and the properties of the function s_1 if n is odd.

13. It is a consequence of lemma 7 if n is even, and of lemma 8 if n is odd.

14. According to lemma 9 and the properties of the functions s_1 and s_2.

15. According to lemmas 4 and 5.

Let $k\,|\,T_0, T_1, A\,|_s(\leqslant n)$ denote the number of symmetric $|\,T_0, T_1, A\,|$-functions depending on at most n variables e. c., then starting from theorem 1 and using lemma 5, the following equalities can be proved:

Theorem 2.

1. $k\,|\,\varnothing\,|_s(\leqslant n) = 2^n - 2^{\left[\frac{n+1}{2}\right]}$.

2. $k\,|\,A\,|_s(\leqslant n) = 2^{\left[\frac{n+1}{2}\right]} - \left[\frac{n+3}{2}\right]$.

3, 4. $k\,|\,T_0\,|_s(\leqslant n) = k\,|\,T_1\,|_s(\leqslant n) = 2^n - \left[\frac{3n+2}{2}\right]$.

5. $k\,|\,A, L\,|_s(\leqslant n) = \left[\frac{n+1}{2}\right]$.

6, 7. $k\,|\,T_0, L\,|_s(\leqslant n) = k\,|\,T_1, L\,|_s(\leqslant n) = \left[\frac{n}{2}\right]$.

8. $k\,|\,T_0, T_1\,|_s(\leqslant n) = 2^n - 2^{\left[\frac{n+1}{2}\right]} - \left[\frac{n^2}{2}\right]$.

9, 10. $k\,|\,T_0, L, M\,|_s(\leqslant n) = k\,|\,T_1, L, M\,|_s(\leqslant n) = 1$.

11. $k\,|\,T_0, T_1, M\,|_s(\leqslant n) = \left[\frac{n^2}{2}\right]$.

12. $k\,|\,T_0, T_1, A\,|_s(\leqslant n) = \begin{cases} 2^{\frac{n}{2}} - n, & \text{if } n \text{ is even} \\ 2^{\frac{n+1}{2}} - n - 1, & \text{if } n \text{ is odd.} \end{cases}$

13. $k\,|\,T_0, T_1, A, M\,|_s(\leqslant n) = \left[\frac{n-1}{2}\right]$.

14. $k\,|\,T_0, T_1, A, L\,|_s(\leqslant n) = \left[\frac{n-1}{2}\right]$.

15. $k\,|\,T_0, T_1, A, L, M\,|_s(\leqslant n) = 1$.

According to theorem 1, the symmetric functions of a given type can be more precisely specified. For example, all symmetric $|\,A, L\,|$-functions are the functions $s_2(x_1, x_2, \ldots, x_n) = S^n_{0, 2, 4, \ldots, n-1}$, where n is an odd number. All symmetric $|\,T_0\,|$ -functions are the functions $S^n_{k_1, k_2, \ldots, k_l}$, where $n \in N$, $\{k_1, k_2, \ldots, k_l\} \subseteq H = \{1, 2, \ldots, n-1\}$; the functions $S^n_{1, 3, 5, \ldots, n-1}$ if n is even and $S^n_{\varnothing} = 0$ being excluded.

Given n, then using theorem 1 and the catalogue of the types of bases of propositional algebra ([1], pp. 29—30.), all S-bases consisting from n-place functions (constants may be included) can be constructed. For example, all S-bases consisting from four n-place functions, n odd $\geqslant 3$, are the bases of the form:

$$\left\{0, \quad 1, \quad S^n_{1, 3, 5, \ldots, n} = \sum_{i=1}^{n} x_i \,(\mathrm{mod}\ 2), \quad S^n_{j, j+1, \ldots, n-1, n}\right\}, \quad \text{where } j \in \{1, 2, \ldots, n\}.$$

Similarly, all S-bases of propositional algebra can be constructed. For example, all S-bases containing four functions are the bases of the form:

$$\left\{0, \quad 1, \quad S^{2m+1}_{1, 3, 5, \ldots, 2m+1} = \sum_{i=1}^{2m+1} x_i \,(\mathrm{mod}\ 2), \quad S^r_{j, j+1, \ldots, r-1, r}\right\},$$

where $m \geqslant 1$, $r \geqslant 2$, $j \in \{1, 2, \ldots, r\}$.

Let N^n_i and $N^{(\leq n)}_i$, $(i = 1, 2, 3, 4)$, denote the numbers of S-bases consisting from i n-place functions and functions depending on at most n variables respectively, then, starting from theorems 1 and 2 and according to the catalogue of the types of bases ([1], pp. 29—30.), the following equalities can be proved:

Corollary 1.

1. $N^n_1 = \begin{cases} 2^{n-1}, \text{ if } n \text{ is even} \\ 2^{n-1} - 2^{\frac{n-1}{2}}, \text{ if } n \text{ is odd.} \end{cases}$

2. $N^n_2 = \begin{cases} 4\,(4^{n-2} - 1), \text{ if } n \text{ is even} \\ 2^{n-1}\,(2^{n-1} - 1) + 3\left(2^{\frac{3(n-1)}{2}} - 1\right), \text{ if } n \text{ is odd.} \end{cases}$

3. $N^n_3 = \begin{cases} 0, \text{ if } n = 1 \\ 2^{n-1} - n, \text{ if } n \text{ is even} \\ 2^{n-1} + 2^{\frac{n+1}{2}} - n - 3, \text{ if } n \text{ is odd } \geqslant 3. \end{cases}$

4. $N^n_4 = \begin{cases} 0, \text{ if } n \text{ is even or } n = 1 \\ n, \text{ if } n \text{ is odd } \geqslant 3. \end{cases}$

Corollary 2.

1. $N_1^{(\leq n)} = 2^n - 2^{\left[\frac{n+1}{2}\right]}$.

2. $$N_2^{(\leq n)} = \begin{cases} 2^{2n} + 3 \cdot 2^{\frac{3n}{2}} - (n+2)\,2^{n+1} - (2n-5)\,2^{\frac{n}{2}} + \frac{1}{4}(n-10)(n+2), \\ \text{if } n \text{ is e en} \\ 2^{2n} + 3 \cdot 2^{\frac{3n+1}{2}} - (4n+9)\,2^{n-1} - (2n-1)\,2^{\frac{n+1}{2}} + \frac{1}{4}(n^2-n-4), \\ \text{if } n \text{ is odd.} \end{cases}$$

3. $$N_3^{(\leq n)} = \begin{cases} (n+2)^2\,2^{n-2} + n(n+2)\,2^{\frac{n-2}{2}} - \frac{3}{8}n(n+2)(n+4), \text{ if } n \text{ is even} \\ (n+1)^2\left(2^{n-2} + 2^{\frac{n-1}{2}}\right) - \frac{1}{8}(3n^3 + 19n^2 + 25n + 1), \\ \text{if } n \text{ is odd.} \end{cases}$$

4. $$N_4^{(\leq n)} = \begin{cases} \frac{1}{4}(n-1)(n^2-4), \text{ if } n \text{ is even} \\ \frac{1}{4}(n-1)^2(n+2), \text{ if } n \text{ is odd.} \end{cases}$$

For example, if $n = 3$, the expression N_3^n gives us two S-bases consisting from 3 functions:

$$\{0;\quad 1 + x_1 + x_2 + x_3 = S_{0,2}^3;\quad S_{2,3}^3\}$$

and

$$\{1;\quad 1 + x_1 + x_2 + x_3 = S_{0,2}^3;\quad S_{2,3}^3\}.$$

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