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ON AN APPROXIMATION OF FUNCTION AND ITS DERIVATIVES

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Let $w_n(x)$ denote the Jacobi polynomials with the weight function

$$\rho(x)=(1-x)^{-1/2}(1+x)^{-1/2}.$$

If $\hat{w}_n(x)$ denote the corresponding normalized Jacobi polynomials then it is well-known that

$$\hat{w}_n(x)=\left[\frac{2n\,\Gamma(n+1)\,\Gamma(n)}{\Gamma(n+1/2)\,\Gamma(n+1/2)}\right]^{1/2} w_n(x). \tag{1.1}$$

Now let

$$S_n(x)=\sum_{k=0}^{n} a_k\,\hat{w}_k(x)$$

be the n^{th} partial sum of the Fourier series of Jacobi polynomials of a function $f(x)$. Natanson [1] proved the following:

Theorem 1 ([1]). *Let p be a positive integer which is greater than or equal to 1. Then on the interval $[-1, 1]$ every function $f(x)$ with a continuous p^{th} derivative can be expanded in a uniformly convergent Fourier series of Jacobi polynomials $\hat{w}_n(x)$.*

As far as we know this is the latest result on this topic. In this note we improve Natanson's result by establishing the following:

Theorem 2. *If $f(x)$ has p continuous derivatives on $[-1, 1]$ and $f^{(p)}(x)\in$ Lip μ, $(0<\mu<1)$ then for $p+\mu\geqslant 1/2$,*

$$|f(x)-S_n(x)|\leqslant\frac{c_1}{n^{p+\mu-(1/2)}},\quad x\in[-1,\ 1] \tag{1.2}$$

and for $p+\mu\geqslant 2r+(1/2)$; $r\geqslant 1$,

$$|f^{(r)}(x)-S_n^{(r)}(x)|\leqslant\frac{c_2}{n^{p+\mu-2r-(1/2)}},\quad x\in[-1,\ 1], \tag{1.3}$$

where c_1 and c_2 are positive constants.

2. In order to prove the Theorem 2 we shall require the following well known results.

From [1] we have for $\gamma > -1$ and $\lambda > -1$,

$$\frac{\Gamma(n+\gamma+\lambda+1)}{\Gamma(n+\gamma+1)} < d_1 n^{\lambda} \tag{2.1}$$

where d_1 is a positive constant. Now making use of (2.1) one can easily see that

$$\frac{\Gamma(n+1)\,\Gamma(n)}{\Gamma(n+1/2)\,\Gamma(n+1/2)} < d_2. \tag{2.2}$$

Also from [3] we have for $-1 \leqslant x \leqslant 1$,

$$|w_n(x)| \leqslant d_3 n^{-1/2}. \tag{2.3}$$

Then from (1.1), (2.2) and (2.3) it follows that for $-1 \leqslant x \leqslant 1$,

$$|\hat{w}_n(x)| \leqslant d_4, \tag{2.4}$$

where d_i, $i=2, 3, 4$, are all positive constants. Further upon applying Markov's inequality [1] to (2.4) we obtain

$$|\hat{w}_n^{(r)}(x)| \leqslant d_5 n^{2r}, \tag{2.5}$$

where d_5 is a positive constant.

Some Lemmas. In order to prove Theorem 2 we need the following lemmas.

Lemma 1. *If* $-1 \leqslant x \leqslant 1$, *then*

$$\int_{-1}^{1} (1-t^2)^{-1/2} \left| \sum_{k=0}^{n} \hat{w}_k(x)\, \hat{w}_k(t) \right| dt \leqslant c_3 n^{1/2} \tag{3.1}$$

and

$$\int_{-1}^{1} (1-t^2)^{-1/2} \left| \sum_{k=0}^{n} \hat{w}_k^{(r)}(x)\, \hat{w}_k(t) \right| dt \leqslant c_4 n^{2r+(1/2)}, \tag{3.2}$$

where c_3 *and* c_4 *are positive constants.*

Proof. We give here the proof for (3.1) only. The proof of (3.2) can be given on the same lines using (2.5). Making use of (2.4) we obtain

$$\int_{-1}^{1} (1-t^2)^{-1/2} \left[\sum_{k=0}^{n} \hat{w}_k(t)\, \hat{w}_k(x) \right]^2 dt = \sum_{k=0}^{n} |\hat{w}_k(x)|^2 \leqslant c_5 n. \tag{3.3}$$

Consequently using Cauchy's inequality and (3.3) we get

$$\int_{-1}^{1} (1-t^2)^{-1/2} \left| \sum_{k=0}^{n} \hat{w}_k(t)\, \hat{w}_k(x) \right| dt \leqslant$$

$$\leqslant \left[\int_{-1}^{1} (1-t^2)^{-1/2} \left\{ \sum_{k=0}^{n} \hat{w}_k(t)\, \hat{w}_k(x) \right\}^2 dt \right]^{1/2} \left[\int_{-1}^{1} (1-t^2)^{-1/2}\, dt \right]^{1/2} \leqslant c_6 n^{1/2},$$

from which (3.1) follows.

Lemma 3.2 *If* $-1 \leqslant x \leqslant 1$ *and* $p+\mu \geqslant 0$, *then*

$$\int_{-1}^{1} (1-t^2)^{(p+\mu)/2} \left[(1-t^2)^{-1/2} \left| \sum_{k=0}^{n} \hat{w}_k(x)\, \hat{w}_k(t) \right| \right] dt \leqslant c_7\, n^{1/2} \tag{3.4}$$

and

$$\int_{-1}^{1} (1-t^2)^{(p+\mu)/2} \left[(1-t^2)^{-1/2} \left| \sum_{k=r}^{n} \hat{w}_k^{(r)}(x)\, \hat{w}_k(t) \right| \right] dt \leqslant c_8\, n^{2r+(1/2)}. \tag{3.5}$$

Proof. Since for $p+\mu \geqslant 0$

$$\int_{-1}^{1} (1-t^2)^{(p+\mu)/2} \left[(1-t^2)^{-1/2} \left| \sum_{k=0}^{n} \hat{w}_k(x)\, \hat{w}_k(t) \right| \right] dt$$

$$\leqslant \int_{-1}^{1} (1-t^2)^{-1/2} \left| \sum_{k=0}^{n} \hat{w}_k(x)\, \hat{w}_k(t) \right| dt$$

and hence (3.1) yields the required result. By similar argument (3.2) yields (3.5).

Lemma 3 ([2]). *Let* $f^{(q)}(x) \in \mathrm{Lip}\,\alpha, (0<\alpha<1)$ *in* $[-1, 1]$ *then there is a polynomial* $Q_n(x)$ *of degree at most n possessing the following properties*:

$$|f(x)-Q_n(x)| \leqslant \frac{c_9}{n^{q+\alpha}} \left[\left(\sqrt{(1-x^2)}\right)^{q+\alpha} + \frac{1}{n^{q+\alpha}} \right] \tag{3.6}$$

and

$$|f^{(r)}(x)-Q_n^{(r)}(x)| \leqslant \frac{c_{10}}{n^{q+\alpha-r}} \left[\left(\sqrt{(1-x^2)}\right)^{q+\alpha-r} + \frac{1}{n^{q+\alpha-r}} \right] \tag{3.7}$$

uniformly in $[-1, 1]$ *and* $r=1,2,\ldots,q$.

4. Proof of Theorem. We shall confine ourselves to proving (1.2). The proof of (1.3) can be given on the same lines.

We write for $-1 \leqslant x \leqslant 1$,

$$|f(x)-S_n(x)| \leqslant |f(x)-\pi_n(x)| + |\pi_n(x)-S_n(x)| \leqslant |f(x)-\pi_n(x)| + \tag{4.1}$$

$$+ \int_{-1}^{1} (1-t^2)^{-1/2} |\pi_n(t)-f(t)| \left| \sum_{k=0}^{n} \hat{w}_k(t)\, \hat{w}_k(x) \right| dt = I_1 + I_2,$$

where $\pi_n(x)$ is given by lemma 3.

Now from (3.6) it follows that for $-1 \leqslant x \leqslant 1$,

$$I_1 = |f(x)-\pi_n(x)| \leqslant \frac{c_9}{n^{p+\mu}} \left[\left(\sqrt{(1-x^2)}\right)^{p+\mu} + \frac{1}{n^{p+\mu}} \right] \leqslant c_{11}\, n^{-(p+\mu)}. \tag{4.2}$$

Further using lemma 3 we obtain

$$(4.3)\quad I_2 \leqslant \frac{c_9}{n^{p+\mu}} \int_{-1}^{1} \left[(1-t^2)^{(p+\mu)/2} + \frac{1}{n^{p+\mu}}\right](1-t^2)^{-1/2} \left|\sum_{k=0}^{n} \hat{w}_k(t)\,\hat{w}_k(x)\right| dt \leqslant$$

$$\leqslant \frac{c_9}{n^{p+\mu}} \int_{-1}^{1} (1-t^2)^{(p+\mu)/2} \left[(1-t^2)^{-1/2} \left|\sum_{k=0}^{n} \hat{w}_k(t)\,\hat{w}_k(x)\right| dt + \right.$$

$$+ \frac{c_9}{n^{2(p+\mu)}} \int_{-1}^{1} (1-t^2)^{-1/2} \left|\sum_{k=0}^{n} \hat{w}_k(t)\,\hat{w}_k(x)\right| dt = r_1^* + r_2^*, \text{ (say)}.$$

Making use of lemma 3.2, we have that

$$(4.4)\quad r_1^* = \frac{c_9}{n^{p+\mu}} \int_{-1}^{1} (1-t^2)^{(p+\mu)/2} \left[(1-t^2)^{-1/2} \left|\sum_{k=0}^{n} \hat{w}_k(t)\,\hat{w}_k(x)\right|\right] dt \leqslant c_{12}\, n^{-p-\mu+(1/2)}.$$

Finally, with the help of lemma 3.1, we obtain

$$(4.5)\quad r_2^* = \frac{c_9}{n^{2(p+\mu)}} \int_{-1}^{1} (1-t^2)^{-1/2} \left|\sum_{k=0}^{n} \hat{w}_k(t)\,\hat{w}_k(x)\right| dt \leqslant c_{13}\, n^{-2(p+\mu)+(1/2)}.$$

Consequently from (4.3), (4.4) and (4.5) we get

$$(4.6)\quad I_2 \leqslant c_{14}\, n^{-p-\mu+(1/2)}, \text{ for } p+\mu \geqslant 1/2.$$

Hence (4.1), (4.2) and (4.6) yield for $-1 \leqslant x \leqslant 1$,

$$(4.7)\quad |f(x) - S_n(x)| \leqslant c_{15}\, n^{-p-\mu+(1/2)},$$

from which (1.2) follows.

Remark. If $E_n(f)$ is the best approximation of $f(x)$ by polynomials from H_n, where H_n is the class of all polynomials of degree $\leqslant n$, then one can very easily see from (4.7) that

$$E_n(f) \leqslant \frac{c_{15}}{n^{p+\mu-(1/2)}}, \quad \text{for } p+\mu \geqslant 1/2.$$

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