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ON CERTAIN TRANSFORMATIONS OF SETS OF POSITIVE MEASURE

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In a recent paper K. C. Ray [2] has proved some theorems on sets of positive measures in R_N (N-dimensional Euclidean space of points or vectors) under certain transformations. In doing so he has considered transformations of the form:

$$x_i' = \sum_{j=1}^{N} a_{ij} x_j + a_{i,\,N+1}; \qquad i = 1,2, \ldots, N,$$

where the coefficients a_{ij} satisfy

$$\left.\begin{aligned} &1 \leqslant a_{ij} < 1 + \frac{\delta}{(M+1)\,N}, \quad && i=j \\ &0 \leqslant a_{ij} < \frac{\delta}{(M+1)\,N}, && i \neq j \end{aligned}\right\} \qquad i, j = 1,2, \ldots, N$$

for suitable positive numbers δ and M. Here the coefficients a_{ij} have been chosen only from suitable right neighbourhoods of 1 (if $i=j$) and 0 (if $i \neq j$).

In this paper we extend some theorems of [2]. In fact, we show that the conclusions of those theorems remain true even if we choose a_{ij} from suitable unrestricted neighbourhoods of 1 (if $i=j$) and 0 ($i \neq j$). The methods of proofs adopted here differ considerably from those of Ray. Moreover, we have given functional representations of our results.

Now we state a well-known result [3], viz, if C is a Lebesgue measurable set in R_N and T any non-singular linear transformation in R_N then $T(C)$ (the set of points of R_N which are the transformed points of C under T) is also Lebesgue measurable and its measure $m(T(C)) = |D|\, m(C)$, where $|D|$ is the absolute value of the determinant of the transformation.

Notations. (i) Lebesgue measure of any measurable set X will be denoted by $|X|$. (ii) $S[c, \rho]$ will stand for the closed sphere in R_N with centre c and radius ρ. (iii) A/B will mean the set of those points of the set A which do not belong to the set B.

Theorem 1. *Let A and B be two closed bounded sets having positive measures and p be any positive integer. Then we can find numbers $M\,(>0)$, $\delta\,(>0)$ and p vectors*

$$u_k = (\alpha^k_{1,\,N+1},\ \alpha^k_{2,\,N+1},\ \ldots,\ \alpha^k_{N,\,N+1})$$

such that if $T_k\ (k=1, 2, \ldots, p)$ be any linear transformations given by

$$x_i' = \sum_{j=1}^{N} a^k_{ij}\, x_j + \alpha^k_{i,\,N+1}, \qquad i = 1, 2, \ldots, N,$$

where

$$\left.\begin{aligned} &(1)\quad 1 - \frac{\delta}{(M+1)N} < a^k_{ij} < 1 + \frac{\delta}{(M+1)N}, && i=j \\ &(2)\quad -\frac{\delta}{(M+1)N} < a^k_{ij} < \frac{\delta}{(M+1)N}, && i\neq j \end{aligned}\right\} \quad i, j = 1, 2, \ldots, N$$

then the points ξ such that $\xi \in A$ and $T_k^{-1}\,\xi \in B$ $(k=1, 2, \ldots, p)$ form a closed set of positive measure.

Proof. Since A and B are of positive measures, there exist two spheres $S_1 = S[a, r]$ and $S_2 = S[b, s]$, where $s = \left(\frac{p}{p+1}\right)^{\frac{1}{N}} r$, such that $|S_1/A| < \varepsilon\, |S_1|$, $|S_2/B| < \varepsilon\, |S_2|$ and $0 < \varepsilon < \frac{1}{2p^2 + 2p + 1}$.

Let $a - b = c$. Since A and B are bounded, there exists a sphere $S[0, M]$ which contains both A and B.

Let $T_1{}^k$ be given by

$$T_1{}^k : \bar{x}_i = x_i + b^k_{i,\,N+1} \qquad (i = 1, 2, \ldots, N; \quad k = 1, 2, \ldots, p),$$

where $(b^k_{1,\,N+1},\ b^k_{2,\,N+1},\ \ldots,\ b^k_{N,\,N+1}) \in S\left[c, \frac{r-s}{2}\right]$ and $T_2{}^k$ be given by

$$T_2{}^k : x_i' = \sum_{j=1}^{N} a^k_{ij}\, \bar{x}_j$$

satisfying (1) and (2), δ being replaced by δ_1, where $0 < \delta_1 < \frac{r-s}{2+r}$.

Also, let $C_k = T_k(S_2 \cap B)$ and $C = S_1 \cap A$, where $T_k = T_2{}^k\, T_1{}^k$.

Let $X = C \cap C_1 \cap C_2 \cap \cdots \cap C_p$.
We shall show that $|X| > 0$.

If x' be the corresponding point of x under $T_2{}^k$ then from the conditions imposed on the coefficients a^k_{ij} it follows that $|x - x'| < \delta_1$.

So, $C_k \subset S_1 \quad (k = 1, 2, \ldots, p)$.

Again, if

$$\Phi(x_{11}, x_{12}, \ldots, x_{1N};\ x_{21}, x_{22}, \ldots, x_{2N}; \ldots; x_{N1}, x_{N2}, \ldots, x_{NN}) = \begin{vmatrix} 1+x_{11} & x_{12}\cdots x_{1N} \\ x_{21} & 1+x_{22}\cdots x_{2N} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ x_{N1} & x_{N2}\cdots 1+x_{NN} \end{vmatrix}$$

then $\Phi(x_{11}, x_{12}, \ldots, x_{1N};\ x_{21}, x_{22}, \ldots, x_{2N}; \ldots; x_{N1}, x_{N2}, \ldots, x_{NN})$ is a continuous function of N^2 variables and

$$\Phi(0, 0, \ldots, 0; \ldots; 0, 0, \ldots, 0) = 1.$$

So, there exists $\delta_2 > 0$ such that

$$\Phi(x_{11}, x_{12}, \ldots, x_{1N};\ x_{21}, x_{22}, \ldots, x_{2N}; \ldots; x_{N1}, x_{N2}, \ldots, x_{NN}) > 1 - \frac{1}{2p^2}$$

for $|x_{ij}| < \delta_2$. Let $\delta = \min\{\delta, \delta_2\}$. If D_k be the determinant of T_k, then

$$D_k > 1 - \frac{1}{2p^2}, \qquad k = 1, 2, \ldots, p.$$

Now, $|X| \geqslant |S_1| - [|C_1'| + |C_2'| + \cdots + |C_p'| + |C'|]$, dashes denote complements with respect to S_1.

But $|C'| = |S_1/A|$ and

$$|C_k'| = |S_1| - D_k|S_2| + D_k|S_2/B| < |S_1| - \left(1 - \frac{1}{2p^2}\right)[|S_2| - |S_2/B|].$$

So, $$|X| > |S_1| - \left[p|S_1| - p|S_2| + \frac{1}{2p}|S_2| + \left(p - \frac{1}{2p}\right)|S_2/B| + |S_1/A|\right] =$$

$$= |S_1| - \left[|S_2| + \frac{1}{2p}|S_2| + \left(p - \frac{1}{2p}\right)|S_2/B| + |S_1/A|\right]$$

$$> |S_1| - \left[|S_2| + \frac{1}{2p}|S_2| + \left(p - \frac{1}{2p}\right)\varepsilon|S_2| + \varepsilon|S_1|\right]$$

$$> 0, \quad \text{since} \quad 0 < \varepsilon < \frac{1}{2p^2 + 2p + 1}.$$

Thus, if $\xi \in X$, then $\xi \in S_1 \cap A$ and $\xi \in T_k(S_2 \cap B)$. i. e., $\xi \in A$ and $T_k^{-1}\xi \in B$, $k = 1, 2, \ldots, p$. This completes the proof.

T h e o r e m 2. *Let A and B be two closed bounded sets of positive measures. There exist a positive number M and linear transformation*

$$T_{\delta_k} : x_i' = \sum_{j=1}^{N} a_{ij}^k x_j + a_{i,N+1}^k, \qquad i = 1, 2, \ldots, N;\ k = 1, 2 \ldots$$

where the coefficients a_{ij}^k satisfy the relations (1), (2) *replacing δ by δ_k such that if $\{\lambda_k\}$, $\lambda_k > 0$ be any null sequence, there exists a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_k\}$ and a point $\xi \in A$ such that $T_{\lambda_{n_k}}^{-1}\xi \in B$, $k = 1, 2, \ldots$*

This theorem can be easily proved by applying Theorem 1.

Theorem 3. *Let $A, A_1, A_2, \ldots, A_{m-1} (m>1)$ be closed bounded sets of positive measures. Then we can find a positive number M, a number $\delta (>0)$ and vectors $(a^k_{1, N+1}, \ldots, a^k_{N, N+1})$ such that if T^k_δ be any linear transformation given by*

$$T^k_\delta : x_i' = \sum_{j=1}^{N} a_{ij} x_j + a^k_{i, N+1}, \quad i=1, 2, \ldots, N; \; k=1, 2, \ldots, m-1$$

and satisfying (1) and (2), then the set of points ξ such that $\xi \in A$ and $T^{k^{-1}}_\delta \xi \in A_k (k=1, 2, \ldots, m-1)$ is a closed set of positive measure.

Proof. Since A is a set of positive measure, there exists a sphere $\Gamma = S[a, r]$ such that

$$|\Gamma \cap A| > \left(1 - \frac{1}{4(m-1)}\right) \nu \quad \text{where} \quad |\Gamma| = \nu.$$

Similarly, there exist spheres $\Gamma_k = S[a_k, s]$ such that

$$|\Gamma_k \cap A_k| > \left(1 - \frac{1}{4(m-1)}\right) \nu_k,$$

where

$$|\Gamma_k| = \nu_k (k=1, 2, \ldots, m-1),$$

and

$$s = \left(1 - \frac{1}{2m}\right)^{\frac{1}{N}} r.$$

Since the sets are bounded, there exists a sphere $S[O, M]$ which contains all the sets A and $A_i (i=1 \ldots, m-1)$. We choose δ_1 such that $0 < \delta_1 < \frac{r-s}{r+1}$.

Let $c_k = a - a_k$, $k=1, \ldots, m-1$. Let the transformations $T_1{}^k$ and T_2 be given by

$$T_1{}^k : \bar{x}_i = x_i + c_k, \quad k=1, 2, \ldots, m-1$$

$$T_2 : x_i' = \sum_{j=1}^{N} a_{ij} \bar{x}_j,$$

where in (1) and (2) δ is to be replaced by δ_1.

Let $X = \Gamma \cap A$ and $X_k = T^k_{\delta_1} (\Gamma_k \cap A_k)$, $k=1, \ldots, m-1$ where $T_2 T_1{}^k = T^k_{\delta_1}$.

We show that $|Y| > 0$,
where $Y = X \cap X_1 \cap X_2 \cap \cdots \cap X_{m-1}$.

If x' be the corresponding point of x under T_2 then from the conditions imposed on the elements a_{ij}, it follows that $|x - x'| < \delta_1$. So, $X_k \subset \Gamma$, $k=1, 2, \ldots, m-1$.

Again, if

$$\Phi(x_{11}, x_{12}, \ldots, x_{1N}; x_{21}, x_{22}, \ldots, x_{2N}; \ldots; x_{N1}, x_{N2}, \ldots, x_{NN})$$
$$= \begin{vmatrix} 1+x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & 1+x_{22} & \cdots & x_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ x_{N1} & x_{N2} & \cdots & 1+x_{NN} \end{vmatrix}$$

then as in theorem 1, there exists $\delta_2 > 0$ such that

$$\Phi(x_{11}, x_{12}, \ldots, x_{1N};\ x_{21}, x_{22}, \ldots, x_{2N}; \ldots; x_{N1}, x_{N2}, \ldots, x_{NN})$$
$$> 1 - \frac{1}{(4m-5)(2m-1)}$$

for

$$|x_{ij}| < \delta_2.$$

Thus, if we choose $\delta = \min\{\delta_1, \delta_2\}$ then

$$D_\delta > 1 - \frac{1}{(4m-5)(2m-1)}$$

where D_δ is the determinant of $T_\delta{}^k$.

Now $|Y| > \nu - [|X'| + |X_1'| + \cdots + |X_{m-1}'|]$,
where the dashes denote complements with respect to Γ.

But $|X'| < \nu - \left(1 - \frac{1}{4(m-1)}\right)\nu = \frac{1}{4(m-1)}\nu$

and

$$|X_k'| < \nu - D_\delta\left(1 - \frac{1}{4(m-1)}\right)\nu_k,$$
$$k = 1, 2, \ldots, m-1.$$
$$= \nu - D_\delta\left(1 - \frac{1}{4(m-1)}\right)\left(1 - \frac{1}{2m}\right)\nu = \nu\left[1 - D_\delta \frac{(4m-5)(2m-1)}{4(m-1)2m}\right].$$

Therefore, $|X_1'| + |X_2' + \cdots + |X_{m-1}'|$

$$< \nu\left[(m-1) - D_\delta \frac{(4m-5)(2m-1)}{8m}\right] < \nu\left[(m-1) - \frac{(4m-5)(2m-1)}{8m} + \frac{1}{8m}\right].$$

Hence, $|Y| > \nu\left[1 - \frac{1}{4(m-1)} - (m-1) + \frac{(4m-5)(2m-1)}{8m} - \frac{1}{8m}\right]$

$$= \nu\left[\frac{m^2-2}{4m(m-1)}\right] > 0.$$

Thus, if $\xi \in Y$ then $\xi \in \Gamma \cap A$ and $\xi \in T_\delta{}^k(\Gamma_k \cap A_k)$, $k = 1, 2, \ldots, m-1$. So, $\xi \in A$ and $T_\delta^{k-1}\ \xi \in A_k$, $k = 1, 2, \ldots, m-1$.

This completes the proof.

We now give functional representations of the above theorems.

For this, we treat transformations as functions and a neighbourhood is defined for each function making the aggregate of such functions into a topological space and the transformations (functions) are treated as points of this space.

We proceed as follows [1]:

Let T denote tthe linear non-singular transformation, viz.

$$x_i' = \sum_{j=1}^{N} a_{ij} x_j + a_{i,N+1}, \quad i = 1, 2, \ldots, N;\ a_{ij} \text{ real.}$$

Now if we write

$$f=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} & a_{1\,N+1} \\ a_{21} & a_{22} & \cdots & a_{2N} & a_{2\,N+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{N1} & a_{N2} & \cdots & a_{NN} & a_{N\,N+1} \end{pmatrix}$$

then by the function f we shall understand the correspondence between a set $C(\subset R_N)$ and $T(C)$. Since T is non-singular, f transforms R_N onto R_N in a biuniform and bicontinuous manner. We denote by $f(C)$ and $f(\xi)$ the transform of the set $C\subset R_N$ and the point $\xi\in R_N$ respectively by the function f (i. e. transformation T).

Let now ε be any positive number and

$$g=\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1N} & a_{1\,N+1} \\ x_{21} & x_{22} & \cdots & x_{2N} & a_{2\,N+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{N1} & x_{N2} & \cdots & x_{NN} & a_{N\,N+1} \end{pmatrix},$$

where x_{ij}'s are any real numbers such that

$$a_{ij}-\varepsilon<x_{ij}<a_{ij}+\varepsilon, \qquad i=1, 2 \ldots, N: j=1, 2, \ldots, N.$$

Then we say that the set of functions g constitutes a neighbourhood of f. Having given $\varepsilon>0$ we say that the neighbourhood of f is determined by ε.

In the light of the above considerations, Theorem 1 proved above may be restated as follows:

Let A and B be two bounded closed sets of positive measures in R_N and p be any positive integer. Then there exist a $\delta(>0)$ and p vectors $c^{(k)}=(c_1^{(k)}, c_2^{(k)}, \ldots, c_N^{(k)})$, $k=1, 2, \ldots, p$ such that if $f_k\,(k=1, 2, \ldots, p)$ be any function belonging to the neighbourhood of

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & c_1^{(k)} \\ 0 & 1 & 0 & \cdots & 0 & c_2^{(k)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & c_N^{(k)} \end{pmatrix}$$

determined by δ, then the set

$$A\cap f_1(B)\cap f_2(B)\cap\cdots\cap f_p(B) \text{ is of positive measure.}$$

Similarly, in the same light functional representations of Theorem 2 and Theorem 3 will follow.

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